Final project guidelines

Additive number theory seminar

1. Timeline and requirements

The goal of the project is to 1) investigate some problem using the mathematical concepts we've studied in this class and 2) write an expository paper on the topic, i.e. explain it in detail to an audience unfamiliar with it.

This is a fairly open-ended project; you may use any resources you like, and the converse of this is that it is your job to find (and properly cite) references to understand your desired topic. That said, I am happy to help you find resources if you are having difficulty, especially on more obscure topics. (You don't have to notify me about your proposed topic, but if I don't hear from you I'll assume you are on top of finding sources etc.)

Timeline

The deadlines are as follows:

April 17	First draft due
April 24	Peer feedback due
May 1	Final deadline: submit your complete, edited project.

Guidelines

The primary goal of this project is to understand the mathematics of your topic; nearly as important however is clearly communicating that understanding. Imagine that you are trying to explain the concepts you have studied to someone who has a similar amount of background to you, but has not necessarily studied these particular topics.

Like any paper, in addition to the main body of the exposition your paper should include a short introduction, explaining the main ideas, motivation, and background of your paper, as well as a list of sources. (The specific formatting of your sources does not matter so long as it is clear.)

All papers should be typed.[1] I encourage you to use LaTeX[2], and am happy to hold workshops on it if there is interest; however it is not required, and you may use whatever software you prefer.

There is no hard guideline for the length of your papers: they should be the length they need to be in order to concisely and clearly explain your topic in detail to the appropriate

[1] If this is a particular hardship for you, we can discuss alternatives.

[2] LaTeX is a software system for creating documents, especially those involving large numbers of mathematical or scientific symbols, and is probably what virtually all mathematical documents you have encountered at least in college were written in, including this one; there are many editors available, including online ones such as `overleaf.com`.

audience, but in practice probably papers should be at least around three pages (possibly less, but only if they are very well-written).

Grading

Papers will be graded for:

- mathematical correctness;
- clarity and quality of exposition;
- depth, style, creativity.

Like other assignments in this class, they will receive a single overall grade of either E (excellent), S (satisfactory), or N (not yet satisfactory):

- a paper which is clear, mathematically correct, and interesting and of appropriate depth will receive a mark of E;
- a paper which is largely correct and readable but has meaningful errors, is sometimes unclear, or has insufficient scope will receive a mark of S;
- a paper which has essential errors, is not readable, or is meaningfully incomplete will receive a mark of N.

As you'll have an opportunity to receive feedback from me as well as your peers, I expect everyone to be able to meet the high standard for an E.

2. Topic suggestions

Any of these should be taken as a collection of related possible ideas around which to base your project; you do not necessarily need to cover everything mentioned, and might cover aspects not mentioned.

Waring's problem for polynomials. Waring's problem asks: if $f(x) = x^k$ for some integer $k \geq 2$, is there a finite number s such that every positive integer n can be written as $n = f(x_1) + \cdots + f(x_s)$ for some nonnegative integers $x_1, \ldots, x_s$? If so, what is the smallest possible choice $g(k)$ of s for k? If we weaken the requirement to "every sufficiently large integer" instead of "every positive integer," this is instead $G(k)$.

One can also ask the same question for other polynomials $f(x)$. Investigate this question: are $g(f)$ and $G(f)$, defined analogously, still finite? Can one bound them?

Since even Waring's problem is still not answered completely (e.g. we don't know the value of $G(3)$!), you (presumably) won't be able to answer this question completely; but try to say something in some interesting cases. For example, maybe you can prove something for particular examples of f (other than the monomials $f(x) = x^k$ as classically); or maybe you can prove some weak result in general.

The Waring–Goldbach problem. Similarly, we can modify Waring's problem by restricting which numbers we allow. One way to do this, which is related to the Goldbach conjecture, is to require that the x_i be prime: that is, for each $k \geq 1$, can we write every (possibly sufficiently large) integer as a sum of a bounded number of kth powers of prime numbers?

In the case $k = 1$, this is the Goldbach–Schnirelman theorem. In general, this seems quite difficult; see if you can prove some cases, or give a heuristic. You could also do some research into what is known, and give a summary of the ideas involved.

Sums of squares and theta series. The generating function for squares,

$$f(x) = \sum_{n=0}^{\infty} x^{n^2},$$

is closely related to something called the theta function:

$$\theta(x) = \sum_{n=-\infty}^{\infty} x^{n^2}.$$

(Indeed, you can check that $\theta(x) = 2f(x) - 1$.) This has an interesting property: $\theta(e^{2\pi i z})$ is a *modular form of weight* $\frac{1}{2}$ *and level* 4, which essentially just means that

$$\theta\left(e\left(\frac{z}{1-4z}\right)\right) = \sqrt{1-4z}\,\theta(e(z)),$$

where $e(z) = e^{2\pi i z}$.

Using this relationship, one can relate $f(x)^s$ (which recall is the generating function of $r_{2,s}(n)$) to powers of θ, which are modular forms of higher weight. For example,

$$f(x)^2 = \frac{1}{4}(\theta(x)+1)^2 = \frac{1}{4}\theta(x)^2 + \frac{1}{2}\theta(x) + \frac{1}{4},$$

so $f(e(z))^2$ is a sum of modular forms. In particular, it is possible to understand $r_{2,2}(n)$ by understanding the Fourier coefficients of θ^2, a modular form of weight 1. Try to understand how this works and how one can study Fourier coefficients of modular forms. It turns out that θ^2 is equal to an Eisenstein series, and so one can compute its Fourier coefficients explicitly; explain how this works, and give some examples.

Although technically the material here isn't too complicated, it is closely related to some very deep fields (e.g. the right definition of modular forms is already very complicated) so be careful not to get dragged in too deep.

Primes of the form $x^2 + y^4$. It is not too difficult (though still pretty nontrivial) to show that there are infinitely many primes of the form x^2+y^2; on the other hand, it is a major open problem to show that there are infinitely many primes of the form x^2+1. A breakthrough result of Friedlander and Iwaniec is a sort of in-between result: there are infinitely many primes of the form x^2+y^4, i.e. we can restrict from initially requiring the second term to be a square to more strictly asking it to be a fourth power, but not yet all the way to requiring it to be exactly 1. Explain some of the ideas that go into this proof.

Squarefree values of polynomials. We saw how to estimate the proportion of squarefree integers using the Eratosthenes–Legendre sieve. For a polynomial $f(n)$, one can also ask: for what proportion of positive integers n will $f(n)$ be squarefree? For certain polynomials f, e.g. $f(n) = n^2$, the answer is clearly 0; but otherwise (if f is squarefree as a polynomial) one expects that the proportion should converge to some nonzero constant; proving this in general seems quite difficult. Heuristically derive a conjecture for the density of squarefree values of $f(n)$ (either in general or for certain examples), and see if you can prove it in some cases. It might also be interesting to collect some numerical data and see if it seems to agree with your conjectures; this might require some coding.

Sieve methods on algebraic varieties. It is also possible in some cases to use sieve methods to count rational points on algebraic varieties, i.e. solutions to Diophantine equations. Investigate this story and give some examples; possible cases of interest include counting points over finite fields, Browning's work over number fields, Hooley's work on sums of kth powers via sieves, and many others. Although there exist elementary cases of this sort of application, I suspect this topic will be much more approachable if you have some experience with algebraic geometry.

Bounded gaps between primes. The twin prime conjecture states that there are infinitely many pairs of distinct primes $p \neq q$ with $|p - q| = 2$; since the minimal possible distance for distinct primes (other than 2 and 3) is 2, equivalently we could say $|p - q| \leq 2$. A weaker conjecture would be: there exists some constant C such that there are infinitely many distinct primes $p \neq q$ with $|p - q| \leq C$. In 2013, a breakthrough paper by Yitang Zhang proved this latter conjecture, for $C = 70000000$; subsequent work of James Maynard and the Polymath project proved that the constant C can be taken as low as 246. The main tool is the Goldston-Pintz-Yıldırım sieve and various modifications of it; explain some of the ideas that go into the proof.

This is (mostly) "pure sieve theory," and in principle should be accessible using the methods from this class; but it is one of the apogees of the field, so (depending on the depth of your investigation) this would definitely be one of the more ambitious projects.

The parity problem. A pattern in sieve-theoretic results which approach major conjectures is that we can often prove not quite the original conjecture, but a variant where e.g. a prime is replaced with a semiprime (as for Chen's theorems with regard to Goldbach's conjecture or the twin prime conjecture). This is related to a general obstacle in sieve theory, called the parity problem: sieves often have difficulty distinguishing numbers with even numbers of primes and numbers with odd numbers of primes (so for example have difficulty distinguishing primes and semiprimes), especially for lower bounds. Investigate why this problem arises, and give some examples; perhaps also look into various approaches to circumventing the parity problem.

Choose your own. Propose your own topic! It should be related to the material from this class, so Waring's problem, the circle method, sieve theory, Goldbach's conjecture, etc. Otherwise you are free to choose any topic that interests you, using the above as a guide.