Homework 2 solutions

Additive number theory seminar

Due March 27, 2023 by 11:40 AM

Problem 1. Let $k \geq 2$ be an integer, and say that a positive integer n is k-free if it is not divisible by any kth power (so for example 2-free numbers are the squarefree numbers). Show that the number of k-free numbers less than or equal to x is $\frac{1}{\zeta(k)}x + O(x^{1/k})$, where $\zeta(s)$ is the Riemann zeta function. You can use without proof the fact that

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right),$$

and the fact that

$$\sum_{d^k|n} \mu(d) = \begin{cases} 1 & n \text{ is } k\text{-free} \\ 0 & \text{otherwise} \end{cases}$$

Solution. Via the second fact, the number of k-free numbers less than or equal to x is

$$\begin{aligned}
\sum_{n \leq x} \sum_{d^k|n} \mu(d) &= \sum_{d \leq x^{1/k}} \mu(d) \sum_{\substack{n \leq x \\ d^k|n}} 1 \\
&= \sum_{d \leq x^{1/k}} \mu(d) \left\lfloor \frac{x}{d^k} \right\rfloor \\
&= \sum_{d \leq x^{1/k}} \mu(d) \left(\frac{x}{d^k} + O(1)\right) \\
&= x \sum_{d \leq x^{1/k}} \frac{\mu(d)}{d^k} + \sum_{d \leq x^{1/k}} O(1) \\
&= \frac{x}{\zeta(k)} + x \sum_{d > x^{1/k}} \frac{\mu(d)}{d^k} + O(x^{1/k}).
\end{aligned}$$

We have

$$\left| \sum_{d > x^{1/k}} \frac{\mu(d)}{d^k} \right| \leq \sum_{d > x^{1/k}} \frac{1}{d^k} \leq \int_{x^{1/k}}^{\infty} \frac{1}{t^k}\, dt = \frac{1}{k-1} x^{\frac{1}{k}-1},$$

so it is smaller than the error term $O(x^{1/k})$ and so can safely be ignored; so we conclude that the number of k-free numbers up to x is $\frac{1}{\zeta(k)}x + O(x^{1/k})$ as desired.

Problem 2. Treating the probability of an integer n being divisible by another integer q as a random event with probability $\frac{1}{q}$ and assuming that these events are independent, derive the following heuristics, neglecting any error terms:

(a) The probability of a positive integer n being squarefree is $\prod_p \left(1 - \frac{1}{p^2}\right)$. (This product is $\frac{6}{\pi^2} = \frac{1}{\zeta(2)}$, so this guess is confirmed by our calculation from class.)

(b) Using Mertens's theorem, the probability of a positive integer n being prime is $\frac{2e^{-\gamma}}{\log n}$, where γ is the Euler–Mascheroni constant. (This is contradicted by the prime number theorem, which can be interpreted as saying the probability of a random positive integer n being prime is $\frac{1}{\log n}$; since $1 \neq 2e^{-\gamma} \approx 1.1229$, this means this sort of heuristic is not always correct! This is because in fact being divisible by different primes is not quite independent, but related in complicated ways.)

Solution.

(a) An integer n is squarefree if and only if it is not divisible by the square of any prime, so since we assume the events are independent and each has probability $1 - \frac{1}{p^2}$ we get the expected product.

(b) An integer n is prime if it is not divisible by any $p \leq \sqrt{n}$, so the probability is

$$\prod_{p \leq \sqrt{n}} \left(1 - \frac{1}{p}\right).$$

By Mertens's theorem this is $\frac{e^{-\gamma}}{\log \sqrt{n}} = \frac{2e^{-\gamma}}{\log n}$.

Problem 3. Apply Brun's sieve to show that the number of *triplet* primes up to x, i.e. numbers $n \leq x$ such that n, $n+2$, and $n+6$ are all prime, is $O(\frac{x(\log \log x)^3}{(\log x)^3})$. (If we'd instead asked for n, $n+2$, and $n+4$ to all be prime, the only example is $n = 3$.) The error bounding is the same as for twin primes after taking $z = e^{\frac{1}{\gamma \log \log x}}$, so omit it and focus on the main term.

Solution. Here $\omega(p)$ is 1 for $p = 2$, 2 for $p = 3$, and 3 for $p \geq 5$, so

$$W(z) = \frac{1}{2} \cdot \frac{1}{3} \cdot \prod_{5 \leq p \leq z} \left(1 - \frac{3}{z}\right) = O\left(\prod_{p \leq z} \left(1 - \frac{1}{z}\right)^3\right) = O\left(\frac{1}{\log(z)^3}\right).$$

Therefore taking $z = e^{\frac{1}{\gamma \log \log x}}$, $\log z = \frac{\log x}{\gamma \log \log x}$ and so the main term is

$$xW(z) = O\left(\frac{x}{(\log z)^3}\right) = O\left(\frac{x(\log \log x)^3}{(\log x)^3}\right).$$