Homework 1 solutions

Additive number theory seminar

Due February 28, 2023 by 11:40 AM

Problem 1. The Fibonacci numbers are defined by the recurrence $a_0 = a_1 = 1$, $a_n = a_{n-1} + a_{n-2}$ for $n \geq 2$. Consider the "modified Fibonacci numbers" defined by $b_0 = b_1 = 1$ but $b_n = b_{n-1} + 2b_{n-2}$ for $n \geq 2$ (so the first few terms are $1, 1, 3, 5, 11, 21, 43, \ldots$). Using generating functions, find an exact formula for b_n.

Solution. Let

$$F(x) = \sum_{n=0}^{\infty} b_n x^n.$$

We have $b_n = b_{n-1} + 2b_{n-2}$ for $n \geq 2$ and $b_0 = b_1 = 1$, so

$$\begin{aligned}
F(x) &= \sum_{n=0}^{\infty} b_n x^n \\
&= 1 + x + \sum_{n=2}^{\infty} (b_{n-1} + 2b_{n-2}) x^n \\
&= 1 + x + \sum_{n=2}^{\infty} b_{n-1} x^n + 2 \sum_{n=2}^{\infty} b_{n-2} x^n \\
&= 1 + x + \sum_{n=1}^{\infty} b_n x^{n+1} + 2 \sum_{n=0}^{\infty} b_n x^{n+2} \\
&= 1 + x + x(F(x) - 1) + 2x^2 F(x) \\
&= 1 + xF(x) + 2x^2 F(x).
\end{aligned}$$

Collecting terms of $F(x)$ gives

$$F(x)(1 - x - 2x^2) = 1,$$

i.e.

$$F(x) = \frac{1}{1 - x - 2x^2}.$$

The polynomial $1 - x - 2x^2 = (1 - 2x)(1 + x)$ has zeros at $x = \frac{1}{2}$ and $x = 1$, and by partial fractions we can write

$$F(x) = \frac{1}{1 - x - 2x^2} = \frac{A}{1 - 2x} + \frac{B}{1 + x} = \frac{A(1 + x) + B(1 - 2x)}{(1 - 2x)(1 + x)} = \frac{A + B + (A - 2B)x}{1 - x - 2x^2}$$

so $A + B = 1$, $A - 2B = 0$, i.e. $A = \frac{3}{3}$, $B = \frac{1}{3}$, so

$$F(x) = \frac{1}{3}\left(\frac{2}{1 - 2x} + \frac{1}{1 + x}\right).$$

From geometric series we know that

$$\frac{2}{1-2x} = \sum_{n=0}^{\infty} 2 \cdot 2^n \cdot x^n, \qquad \frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n \cdot x^n,$$

so

$$F(x) = \sum_{n=0}^{\infty} \frac{2^{n+1} + (-1)^n}{3} x^n.$$

By comparing coefficients, we conclude that

$$b_n = \frac{2^{n+1} + (-1)^n}{3}.$$

For example, $b_0 = \frac{2+1}{3} = 1$, $b_1 = \frac{4-1}{3} = 1$, $b_2 = \frac{8+1}{2} = 3$, $b_3 = \frac{16-1}{3} = 5$, etc.

Problem 2. For any integer $k \geq 2$, let

$$q = \left\lfloor \left(\frac{3}{2}\right)^k \right\rfloor.$$

Prove that $g(k) \geq 2^k + q - 2$, i.e. there exists a positive integer which cannot be written as the sum of $2^k + q - 3$ nonnegative kth powers. (Hint: consider the number $N = q2^k - 1$.)

Solution. First, observe that by definition $0 \leq \left(\frac{3}{2}\right)^k - q < 1$, so if $N = q2^k - 1$ then $3^k - N - 1 = 2^k \left(\left(\frac{3}{2}\right)^k - q\right)$ and so $0 \leq 3^k - N - 1 < 2^k$, i.e. $3^k - 2^k - 1 < N \leq 3^k - 1$.

Choose a minimal representation of N as a sum of kth powers, i.e. $N = a_1^k + \cdots + a_s^k$ with $a_i \geq 0$ such that there is no other representation of N as a sum of fewer than s nonnegative kth powers. Since $N < 3^k$, none of the a_i can be greater than or equal to 3, so all a_i are either 1 or 2 (if any were 0, we could drop them to get a smaller representation). Say that there are α 1's and β 2's, so

$$N = \alpha + \beta 2^k.$$

Since this representation is minimal, it follows that $\alpha < 2^k$, since if there were 2^k copies of 1^k in the representation we could bundle them all together into a single 2^k, thus reducing the number of nonzero summands required. On the other hand since $\alpha \equiv \alpha + \beta 2^k = N = q2^k - 1 \equiv -1 \pmod{2^k}$, we can write $\alpha = m2^k - 1$ for some integer m, so since $\alpha < 2^k$ we must have $m = 1$, i.e. $\alpha = 2^k - 1$, so

$$N = (\beta + 1)2^k - 1,$$

so $\beta = q - 1$. This tells us that our representation must consist of $q - 1$ copies of 2^k and $2^k - 1$ copies of $1^k = 1$, for a total of $q - 1 + 2^k - 1 = 2^k + q - 2$ terms, so the smallest possible value of s such that this N can be written as a sum of s nonnegative kth powers is $2^k + q - 2$. Thus $g(k) \geq 2^k + q - 2$.

We can compute that for $k = 2$, $q = 2$; for $k = 3$, $q = 3$; and for $k = 4$, $q = 5$. Thus we deduce $g(2) \geq 2^2 + 2 - 2 = 4$, $g(3) \geq 2^3 + 3 - 2 = 9$, and $g(4) \geq 2^4 + 5 - 2 = 19$. These are all actually equal to the true values, as we've seen in the $k = 2$ and $k = 3$ cases. One might conjecture that for all k one in fact has

$$g(k) = 2^k + q - 2 = 2^k + \left\lfloor \left(\frac{3}{2}\right)^k \right\rfloor - 2,$$

and this is believed to be true for all k; it is known that there are only finitely many exceptions, and that they must be for $k \geq 471600000$.

Problem 3. For any integer $k \geq 2$, show that the number of integers less than or equal to x that can be written as the sum of k nonnegative kth powers is at most $\frac{x}{k!} + O(x^{1-\frac{1}{k}})$. (Hint: if $n \leq x$ is the sum of k nonnegative kth powers $n = a_1^k + \cdots + a_k^k$, then (ordering the a_i increasing) we can associate to this representation a tuple of integers $0 \leq a_1 \leq \cdots \leq a_k \leq n^{1/k} \leq x^{1/k}$, and the number of such tuples is given by a binomial coefficient.)

Using this bound, conclude that $G(k) \geq k+1$ for all $k \geq 2$, i.e. there are infinitely many positive integers which *cannot* be written as a sum of k nonnegative kth powers.

Solution. By the hint, finding a representation of n as a sum of k nonnegative kth powers gives numbers $0 \leq a_1 \leq \cdots \leq a_k \leq n^{1/k} \leq x^{1/k}$. Choosing such numbers is equivalent to choosing k elements out of the first $x^{1/k}$ integers; there are $\binom{\lfloor x^{1/k} \rfloor + 1}{k}$ ways of making this choice, and so there are at most $\binom{\lfloor x^{1/k} \rfloor + 1}{k}$ integers less than or equal to x which can be represented in this way. We have

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n \cdot (n-1) \cdots (n-k+1)}{k!} = \frac{n^k + O(n^{k-1})}{k!} = \frac{n^k}{k!} + O(n^{k-1}),$$

so $\binom{\lfloor x^{1/k} \rfloor + 1}{k} = \frac{(\lfloor x^{1/k} \rfloor + 1)^k}{k!} + O((\lfloor x^{1/k} \rfloor + 1)^{k-1}) = \frac{x}{k!} + O(x^{1-\frac{1}{k}})$, since $\lfloor x^{1/k} \rfloor + 1 = x + O(1)$; therefore the first part of the result follows.

To conclude, since $k \geq 2$ we have $k! \geq 2! = 2$, so there are at most $\frac{x}{2} + O(x^{1-1/k})$ integers which cannot be written in this form, which means there are $x - \frac{x}{2} + O(x^{1-1/k}) = \frac{x}{2} + O(x^{1-1/k})$ integers which cannot be written in this form; since the main term has higher order than the error term, this goes to infinity as $x \to \infty$, so there are infinitely many such integers, so it cannot be true that $G(k) \leq k$, i.e. $G(k) \geq k + 1$. Note that in the case $k = 3$, this immediately gives $G(3) \geq 4$, which is already the best known lower bound, recovering a result from Keila's talk.