Homework 1

Additive number theory seminar

Due February 28, 2023 by 11:40 AM

Choose **one** of the following problems and write a clear and readable solution.

Problem 1. The Fibonacci numbers are defined by the recurrence $a_0 = a_1 = 1$, $a_n = a_{n-1} + a_{n-2}$ for $n \geq 2$. Consider the "modified Fibonacci numbers" defined by $b_0 = b_1 = 1$ but $b_n = b_{n-1} + 2b_{n-2}$ for $n \geq 2$ (so the first few terms are $1, 1, 3, 5, 11, 21, 43, \ldots$). Using generating functions, find an exact formula for b_n.

Problem 2. For any integer $k \geq 2$, let

$$q = \left\lfloor \left(\frac{3}{2}\right)^k \right\rfloor.$$

Prove that $g(k) \geq 2^k + q - 2$, i.e. there exists a positive integer which cannot be written as the sum of $2^k + q - 3$ nonnegative kth powers. (Hint: consider the number $N = q2^k - 1$.)

Problem 3. For any integer $k \geq 2$, show that the number of integers less than or equal to x that can be written as the sum of k nonnegative kth powers is at most $\frac{x}{k!} + O(x^{1-\frac{1}{k}})$. (Hint: if $n \leq x$ is the sum of k nonnegative kth powers $n = a_1^k + \cdots + a_k^k$, then (ordering the a_i increasing) we can associate to this representation a tuple of integers $0 \leq a_1 \leq \cdots \leq a_k \leq n^{1/k} \leq x^{1/k}$, and the number of such tuples is given by a binomial coefficient.)

Using this bound, conclude that $G(k) \geq k+1$ for all $k \geq 2$, i.e. there are infinitely many positive integers which *cannot* be written as a sum of k nonnegative kth powers.