Stable bases of Hilbert schemes

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We will study the Hilbert scheme:

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which is a smooth variety of dimension \(2d\).
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There is an action of the torus $T = \mathbb{C}^*_{q_1} \times \mathbb{C}^*_{q_2}$ on $\text{Hilb}_d$, where the two factors simply scale the coordinate directions.

In fact, the fixed points of this action are just monomial ideals:

$$I_\lambda = (x^{\lambda_0}, x^{\lambda_1}y, x^{\lambda_2}y^2, \ldots) \in \text{Hilb}_d$$

for any partition $\lambda = (\lambda_0 \geq \lambda_1 \geq \lambda_2 \geq \ldots)$ of $d$.

The classes of the fixed points form a basis in the (localized) $K$-theory groups:

$$K^* = \bigoplus_{d=0}^{\infty} K^*_{ Tart } (\text{Hilb}_d)$$

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The Hilbert scheme

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- The classes of the fixed points form a basis in the (localized) \( K \)-theory groups:

\[ K = \bigoplus_{d=0}^{\infty} K_T^*(\text{Hilb}_d) \]
The $K$–theory

- In fact, the Bridgeland-King-Reid-Haiman equivalence:

$$D^b_T \text{Coh}(\text{Hilb}_d) \cong D^b_T \text{Coh}(\mathbb{C}^{2d})^{S(d)}$$

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where $\Lambda = \mathbb{C}(q_1, q_2)[x_1, x_2, ...]^{\text{Sym}}$ denotes the ring of symmetric functions in infinitely many variables.
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Under this isomorphism, the classes of the fixed points \([l_\lambda]\) go to the modified Macdonald polynomials \( P_\lambda \), defined as:

\[ P_\lambda = \sum_{\mu \leq \lambda} m_\mu \cdot \text{constant} \]

and the \( P_\lambda \) are orthogonal with respect to a certain inner product, where \( \leq \) is the dominance ordering on partitions.
Maulik-Okounkov introduced a certain basis of $K$ for any rational number $\frac{m}{n}$, called the **stable basis**:

$$s_{\lambda}^{m/n} = \sum_{\mu \leq \lambda} P_{\mu} \cdot c_{\lambda}^{\mu}$$
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where the constants $c_{\lambda}^{\mu} \in \mathbb{Z}[q_1^{\pm 1}, q_2^{\pm 1}]$ satisfy:

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and if $c_{\lambda}^{\mu} = \sum \pm q_1^x q_2^y$, then the only $x - y$ which appear lie in:

$$\frac{m}{n} (o_{\mu} - o_{\lambda}) + [\min_{\mu}, \max_{\mu}) \quad \forall \mu < \lambda$$
Maulik-Okounkov introduced a certain basis of $K$ for any rational number $\frac{m}{n}$, called the **stable basis**:

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where $o_\mu = O(1)|_{l_\mu} = \sum_{\Box \in \mu} a(\Box) - l(\Box)$ and:

$$\min_\mu = -|\mu| - \sum_{\Box \in \mu} a(\Box), \quad \max_\mu = \sum_{\Box \in \mu} l(\Box)$$
It is easy to see that:

\[ s^\infty_\lambda = P_\lambda \]
Bases of symmetric functions

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are the usual Schur functions.
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- So in general, the basis \( \{ s_\lambda^{m/n} \} \) interpolates between the bases of Schur functions and Macdonald polynomials.
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- So in general, the basis \( \{s_\lambda^{m/n}\} \) interpolates between the bases of Schur functions and Macdonald polynomials.

- The change of basis between:
  \[ \{s_\lambda^{m/n-\text{small number}}\} \text{ and } \{s_\lambda^{m/n+\text{small number}}\} \]

is the half monodromy matrix of the quantum difference equation of the Hilbert scheme (Bezrukavnikov-Okounkov)
Toward operators

- The particular bases $P_\lambda$ and $s_\lambda$ are nice with respect to two important families of operators:

$$D_k \cdot P_\lambda = P_\lambda e_k \left( \{ q_1^x q_2^y \} \square = (x,y) \in \lambda \right)$$

where $D_k$ are the well-known $q-$difference operators,

- These operators seem to behave differently, but there exists a family of operators $e_m/n_k$ which interpolates between them.

- Moreover, we will show that $e_m/n_k$ act nicely in the basis $s_m/n_\lambda$. 
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and the Pieri rules for multiplication by elementary symmetric functions:

$$e_k \cdot s_\lambda = \sum_{\mu=\lambda+k \ added \ boxes \ no \ two \ next \ to \ each \ other} s_\mu$$
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These operators seem to behave differently, but there exists a family of operators $e_k^{m/n}$ which interpolates between them

Moreover, we will show that $e_k^{m/n}$ act nicely in the basis $s_\lambda^{m/n}$
The elliptic Hall algebra

The **elliptic Hall algebra** \( \mathcal{A} \) has generators \( p_{m,n} \) and \( \kappa_{m,n} \) for every lattice vector \((m, n) \in \mathbb{Z}^2 \setminus \{0, 0\}\), modulo relations:

\[
\kappa_{m,n} \cdot \kappa_{m',n'} = \kappa_{m+m',n+n'} \quad \text{\( \kappa_{m,n} \) central}
\]

\[
[p_{m,n}, p_{m',n'}] = \delta_{m+m'}^0 \frac{g}{[g]_{q_1,q_2}} (\kappa_{m,n} - \kappa_{m,n}^{-1})
\]

if \((m, n)\) and \((m', n')\) are collinear, where \( g = \gcd(m, n) \), and:

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\sum_{k=0}^\infty h_{km,n} z^{-k} = \exp \left( \sum_{k=1}^\infty p_{km,n} z^{-k} \right)
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The elliptic Hall algebra $\mathcal{A}$ has generators $p_{m,n}$ and $\kappa_{m,n}$ for every lattice vector $(m, n) \in \mathbb{Z}^2 \setminus \{0, 0\}$, modulo relations:

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$\kappa_{m,n}$ central

$$[p_{m,n}, p_{m',n'}] = \delta_{m+m'}^0 \frac{g}{[g]_{q_1,q_2}} \left( \kappa_{m,n} - \kappa_{m,n}^{-1} \right)$$

if $(m, n)$ and $(m', n')$ are collinear, where $g = \gcd(m,n)$, and:

$$[p_{m,n}, p_{m',n'}] = \frac{h_{m+m',n+n'}}{[1]_{q_1,q_2}}$$

if the clockwise triangle $(0, 0), (m, n), (m + m', n + n')$ contains no inside lattice points, where for $\gcd(m,n) = 1$ we set:

$$\sum_{k=0}^{\infty} h_{km,n} = \exp\left( \sum_{k=1}^{\infty} p_{km,n} \right)$$

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\[ \sum_{k=0}^{\infty} h_{km,kn} z^{-k} := \exp \left( \sum_{k=1}^{\infty} \frac{p_{km,kn} z^{-k}}{k} [k]_{q_1,q_2} \right) \]
For any \( \gcd(m, n) = 1 \), we set:

\[
p^m/n_k = p_{km, kn} \cdot (1 - q_1^{-k})
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and write \( e^m/n_k \) for the elementary symmetric functions corresponding to the power-sum functions \( p^m/n_k \).
For any $\gcd(m, n) = 1$, we set:

$$p_k^{m/n} = p_{km,kn} \cdot (1 - q_1^{-k})$$

and write $e_k^{m/n}$ for the elementary symmetric functions corresponding to the power-sum functions $p_k^{m/n}$.

It was proved that the algebra $A$ acts on the vector space $K \cong \Lambda$ (Schiffmann-Vasserot, Feigin-Tsymbaliuk).
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The generators \( e_0^k \) of \( \mathcal{A} \) act on \( K \cong \Lambda \) as multiplication by usual elementary symmetric functions,

while the generators \( e_\infty^k \) act as the \( q \)-difference operators \( D_k \).
Action on $K$

- For any $\gcd(m, n) = 1$, we set:
  
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- The generators $e_k^0$ of $\mathcal{A}$ act on $K \cong \Lambda$ as multiplication by usual elementary symmetric functions,

- while the generators $e_k^\infty$ act as the $q$–difference operators $D_k$.

- In general, the family of operators $e_k^{m/n} \in \text{End}(K)$ 
  interpolates between these two extremes.
Theorem (N) the $\frac{m}{n}$ Pieri rule: for any rational number $m/n$, any $k \in \mathbb{N}$ and any partition $\lambda$, we have:

$$e_k^{m/n} \cdot s^{m/n}_\lambda = \sum_{\text{no two next to each other}} s^{m/n}_\mu \prod_{i=1}^k (-1)^{\text{ht } R_i} \chi_{m}(R_i)$$

where for a $n$-ribbon $R$ with boxes $(x_1, y_1), \ldots, (x_n, y_n)$ ordered from northwest to southeast we write:

$$\chi_{m}(R) = \prod_{j=1}^n (q^{x_j y_j} - 1)^{\lfloor \frac{mj}{n} \rfloor - \lfloor \frac{m(j-1)}{n} \rfloor}$$
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$n = 5$, $k = 2$

$\lambda = (1)$

$\mu = (4, 4, 3)$
Theorem (N) the $\frac{m}{n}$ Pieri rule: for any rational number $m/n$, any $k \in \mathbb{N}$ and any partition $\lambda$, we have:

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Example:

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In particular, the vector $e_{1}^{m/n} \cdot 1 \in K \cong \Lambda$ is important in geometry and representation theory.
Application

- In particular, the vector \( e_1^{m/n} \cdot 1 \in K \cong \Lambda \) is important in geometry and representation theory.
- It is conjectured to correspond to the unique finite dimensional irreducible module of the rational Cherednik algebra, under the Gordon-Stafford functor.
- According to the Pieri rule on the previous slide, this vector equals:

\[
\sum_{i=1}^{n} s_{m/n}(i, 1, \ldots, 1) \cdot q^{i-1+r_{m/n}(i-1)} (-q^2)^{r_{m/n}(n-i)}
\]

where \( r_{m/n}(k) = \lfloor \frac{m}{n} \rfloor + \cdots + \lfloor \frac{mk}{n} \rfloor \).
- This also gives another interpretation of the rational shuffle conjecture for symmetric functions.
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