Quantum toroidal $\mathfrak{gl}_1$ and its applications

Andrei Negut

Columbia University

11 / 04 / 2014
Quantum toroidal $\mathfrak{gl}_1$

Quantum toroidal $\mathfrak{gl}_1$ is the $\mathbb{C}(q_1, q_2)$–algebra generated by:

$$e_k^+, e_k^-, p_{k'}, \quad \forall k \in \mathbb{Z}, \; k' \in \mathbb{Z}\setminus 0$$

along with two central elements $c, d$. We denote it by $\mathcal{U}$. 

Andrei Negut
Quantum toroidal $\mathfrak{gl}_1$ and its applications
Quantum toroidal $\mathfrak{gl}_1$

- Quantum toroidal $\mathfrak{gl}_1$ is the $\mathbb{C}(q_1, q_2)$–algebra generated by:
  \[ e_k^+, e_k^-, p_{k'} \quad \forall k \in \mathbb{Z}, \; k' \in \mathbb{Z}\setminus 0 \]
  along with two central elements $c, d$. We denote it by $\mathcal{U}$.

- In order to present the relations, we introduce the series:
  \[
  e^\pm(z) = \sum_{k \in \mathbb{Z}} e_k^\pm z^{-k}, \quad p^\pm(z) = \sum_{k \geq 1} p_{\pm k} z^{\mp k}
  \]
Quantum toroidal $\mathfrak{gl}_1$

- Quantum toroidal $\mathfrak{gl}_1$ is the $\mathbb{C}(q_1, q_2)$–algebra generated by:

$$e^+_k, e^-_k, p_k', \quad \forall k \in \mathbb{Z}, \; k' \in \mathbb{Z} \setminus \{0\}$$

along with two central elements $c, d$. We denote it by $\mathcal{U}$.

- In order to present the relations, we introduce the series:

$$e^{\pm}(z) = \sum_{k \in \mathbb{Z}} e^{\pm}_k z^{-k}, \quad p^{\pm}(z) = \sum_{k \geq 1} p_{\pm k} z^{\mp k}$$

- One imposes the following relations on the generators of $\mathcal{U}$:

$$e^{\pm}(z)e^{\pm}(w) = e^{\pm}(w)e^{\pm}(z) \left[ \frac{(qw - z)(w - q_1 z)(w - q_2 z)}{(w - qz)(q_1 w - z)(q_2 w - z)} \right]^{\pm 1}$$

$$[p_{\pm k}, e^{\pm}(z)] = z^{\pm k} e^{\pm}(z), \quad [p_{\mp k}, e^{\pm}(z)] = -(zc)^{\mp k} e^{\pm}(z)$$
Quantum toroidal $\mathfrak{gl}_1$

- We also have the relations $[p_k, p_l] = \frac{\delta_{k+l}^0}{\alpha_k} \left( c^{-k} - c^k \right)$ and:

\[
[e^+(z), e^-(w)] = \frac{1}{\alpha_1} \left[ d^{-1} \delta \left( \frac{cz}{w} \right) h^-(w) - d \delta \left( \frac{cw}{z} \right) h^+(z) \right]
\]
We also have the relations $[p_k, p_l] = \frac{\delta_{k+l}^0}{\alpha_k} (c^{-k} - c^k)$ and:

$$[e^+(z), e^-(w)] = \frac{1}{\alpha_1} \left[ d^{-1} \delta \left( \frac{cz}{w} \right) h^-(w) - d \delta \left( \frac{cw}{z} \right) h^+(z) \right]$$

These latter two relations partially explain the name of $\mathcal{U}$, since it is an affinization of the quantum Heisenberg algebra.
Quantum toroidal $gl_1$

- We also have the relations $[p_k, p_l] = \frac{\delta_{k+l}^0}{\alpha_k} (c^{-k} - c^k)$ and:

  $$[e^+(z), e^-(w)] = \frac{1}{\alpha_1} \left[ d^{-1} \delta \left( \frac{cz}{w} \right) h^-(w) - d\delta \left( \frac{cw}{z} \right) h^+(z) \right]$$

- These latter two relations partially explain the name of $\mathcal{U}$, since it is an affinization of the quantum Heisenberg algebra.

- It contains infinitely many quantum Heisenberg subalgebras, though only one is visible in the above picture (namely $\langle p_k \rangle$).
Quantum toroidal $gl_1$

- We also have the relations $[p_k, p_l] = \frac{\delta_{k+l}^0}{\alpha_k} (c^{-k} - c^k)$ and:

\[
[e^+(z), e^-(w)] = \frac{1}{\alpha_1} \left[ d^{-1} \delta \left( \frac{cz}{w} \right) h^-(w) - d \delta \left( \frac{cw}{z} \right) h^+(z) \right]
\]

- These latter two relations partially explain the name of $\mathcal{U}$, since it is an affinization of the quantum Heisenberg algebra.

- It contains infinitely many quantum Heisenberg subalgebras, though only one is visible in the above picture (namely $\langle p_k \rangle$).

- A shadow of another quantum Heisenberg subalgebra is the relation:

\[
[e_0^+, e_0^-] = \frac{d^{-1} - d}{\alpha_1}
\]

with $e_0^+$ and $e_0^-$ playing the role of the first creation and annihilation operators, respectively.
The elliptic Hall algebra

The elliptic Hall algebra $\mathcal{A}$ has generators $p_{m,n}$ and $\kappa_{m,n}$ for every lattice vector $(m, n) \in \mathbb{Z}^2 \setminus \{0,0\}$, modulo relations:

$$\kappa_{m,n} \cdot \kappa_{m',n'} = \kappa_{m+m',n+n'}, \quad \kappa_{m,n} \text{ central}$$

$$[p_{m,n}, p_{m',n'}] = \delta_{m+m'}^0 \cdot \frac{\kappa_{m,n}^{-1} - \kappa_{m,n}}{\alpha g}$$

if $(m, n)$ and $(m', n')$ are collinear, where $g = \gcd(m, n)$, and:

$$\sum_{k=0}^{\infty} h_{km,n} z^{-k} = \exp \left( \sum_{k=1}^{\infty} \alpha_k p_{km,n} z^{-k} \right)$$
The elliptic Hall algebra $\mathcal{A}$ has generators $p_{m,n}$ and $\kappa_{m,n}$ for every lattice vector $(m,n) \in \mathbb{Z}^2 \setminus \{0,0\}$, modulo relations:

$$\kappa_{m,n} \cdot \kappa_{m',n'} = \kappa_{m+m',n+n'}, \quad \kappa_{m,n} \text{ central}$$

$$[p_{m,n}, p_{m',n'}] = \delta_{m+m'}^0 \cdot \frac{\kappa_{m,n}^{-1} - \kappa_{m,n}}{\alpha_g}$$

if $(m,n)$ and $(m',n')$ are collinear, where $g = \gcd(m,n)$, and:

$$[p_{m,n}, p_{m',n'}] = \kappa^* \cdot \frac{h_{m+m',n+n'}}{\alpha_1}$$

if the clockwise triangle $(0,0), (m,n), (m+m',n+n')$ contains no inside lattice points, where for $\gcd(m,n) = 1$ we set:

$$\sum_{k=0}^{\infty} h_{km,nz-k} := \exp \left( \sum_{k=1}^{\infty} \alpha_k p_{km,nz-k} \right)$$
The elliptic Hall algebra $\mathcal{A}$ has generators $p_{m,n}$ and $\kappa_{m,n}$ for every lattice vector $(m, n) \in \mathbb{Z}^2 \setminus \{0, 0\}$, modulo relations:

$\kappa_{m,n} \cdot \kappa_{m',n'} = \kappa_{m+m',n+n'}$, \hspace{1cm} $\kappa_{m,n}$ central

$[p_{m,n}, p_{m',n'}] = \delta_{m+m'}^0 \cdot \frac{\kappa_{m,n}^{-1} - \kappa_{m,n}}{\alpha_g}$

if $(m, n)$ and $(m', n')$ are collinear, where $g = \gcd(m, n)$, and:

$[p_{m,n}, p_{m',n'}] = \kappa^* \cdot \frac{h_{m+m',n+n'}}{\alpha_1}$

if the clockwise triangle $(0, 0), (m, n), (m+m', n+n')$ contains no inside lattice points, where for $\gcd(m, n) = 1$ we set:

$\sum_{k=0}^{\infty} h_{km,kn}z^{-k} := \exp \left( \sum_{k=1}^{\infty} \alpha_k p_{km,kn}z^{-k} \right)$
The isomorphism

- **Theorem** (Schiffmann-Vasserot) There exists an isomorphism:

  \[ \mathcal{U} \xrightarrow{\cong} \mathcal{A}, \quad e_k^\pm \mapsto p_{\pm 1,k}, \quad p_k \mapsto p_{0,k} \]

  and \( c \mapsto \kappa_{(0,1)}, \quad d \mapsto \kappa_{(1,0)}. \)
The isomorphism

- **Theorem** (Schiffmann-Vasserot) There exists an isomorphism:

\[ \mathcal{U} \xrightarrow{\cong} \mathcal{A}, \quad e_{k}^{\pm} \longrightarrow p_{\pm 1,k}, \quad p_{k} \longrightarrow p_{0,k} \]

and \( c \longrightarrow \kappa^{(0,1)}, \quad d \longrightarrow \kappa^{(1,0)}. \)

- This implies that \( \mathcal{U} \) contains a quantum Heisenberg subalgebra corresponding to any rational number \( \frac{m}{n} \in \mathbb{Q} \cup \{\infty\}: \)

\[ \{p_{km,kn}\}_{k \in \mathbb{Z}} \subset \mathcal{A} \cong \mathcal{U} \]

It also has an \( SL_{2}(\mathbb{Z}) \) symmetry by algebra automorphisms.
The isomorphism

**Theorem** (Schiffmann-Vasserot) There exists an isomorphism:

\[
\mathcal{U} \xrightarrow{\sim} \mathcal{A}, \quad e_k^\pm \mapsto p_{\pm 1,k}, \quad p_k \mapsto p_{0,k}
\]

and \( c \mapsto \kappa^{(0,1)} \), \( d \mapsto \kappa^{(1,0)} \).

This implies that \( \mathcal{U} \) contains a quantum Heisenberg subalgebra corresponding to any rational number \( \frac{m}{n} \in \mathbb{Q} \cup \{ \infty \} \):

\[
\{ p_{km,kn} \}_{k \in \mathbb{Z}} \subset \mathcal{A} \cong \mathcal{U}
\]

It also has an \( SL_2(\mathbb{Z}) \) symmetry by algebra automorphisms.

Using this picture, we can write a formula for the universal \( R \)-matrix of the quantum toroidal algebra \( \mathcal{R} \in \mathcal{U} \hat{\otimes} \mathcal{U} \):

\[
\mathcal{R} = \prod_{\frac{m}{n} \in \mathbb{Q} \cup \{ \infty \}} \exp \left( \sum_{k=1}^{\infty} p_{km,kn} \otimes p_{-km,-kn} \alpha_k \right)
\]
Another incarnation of the algebra $\mathcal{U} \cong \mathcal{A}$ is a deformation of the $\mathcal{W}_{1+\infty}$ algebra. It acts on the bosonic Fock space:

$$F = \mathbb{C}(q_1, q_2)[p_1, p_2, \ldots]$$
Another incarnation of the algebra $\mathcal{U} \cong \mathcal{A}$ is a deformation of the $\mathcal{W}_{1+\infty}$ algebra. It acts on the bosonic Fock space:

$$F = \mathbb{C}(q_1, q_2)[p_1, p_2, ...]$$

by vertex operators:

$$e^+(z) = \exp \left( \sum_{k \geq 1} \frac{p_k z^{-k}}{k} \right) \exp \left( \sum_{k \geq 1} \frac{p_k^* z^k}{k} \right)$$
Another incarnation of the algebra $\mathcal{U} \cong \mathcal{A}$ is a deformation of the $\mathcal{W}_{1+\infty}$ algebra. It acts on the bosonic Fock space:

$$F = \mathbb{C}(q_1, q_2)[p_1, p_2, \ldots]$$

by vertex operators:

$$e^+(z) = \exp \left( \sum_{k \geq 1} \frac{p_k z^{-k}}{k} \right) \exp \left( \sum_{k \geq 1} \frac{p^*_k z^k}{k} \right)$$

The $p_{\pm k}$ act by multiplication with $p_k$ and its adjoint $p^*_k$, and the central charges are $c = q^{\frac{1}{2}}$ and $d = 1$. 
Another incarnation of the algebra $\mathcal{U} \cong \mathcal{A}$ is a deformation of the $\mathcal{W}_{1+\infty}$ algebra. It acts on the bosonic Fock space:

$$F = \mathbb{C}(q_1, q_2)[p_1, p_2, \ldots]$$

by vertex operators:

$$e^+(z) = \exp \left( \sum_{k \geq 1} \frac{p_k z^{-k}}{k} \right) \exp \left( \sum_{k \geq 1} \frac{p_k^* z^k}{k} \right)$$

The $p_{\pm k}$ act by multiplication with $p_k$ and its adjoint $p_k^*$, and the central charges are $c = q^{1/2}$ and $d = 1$.

Composing $n$ vertex operators means that any $r \in \mathcal{U}$ acts by:

$$\int R(z_1, \ldots, z_n) \exp \left( \sum_{k \geq 1} \frac{p_k \sum_{i=1}^n z_i^{-k}}{k} \right) \exp \left( \sum_{k \geq 1} \frac{p_k^* \sum_{i=1}^n z_i^k}{k} \right)$$

for some rational function $R(z_1, \ldots, z_n)$. 

Andrei Negut
Quantum toroidal $\mathfrak{gl}_1$ and its applications
Understanding the $p_{m,n}$

- The assignment $r \rightarrow R(z_1, ..., z_n)$ gives rise to the shuffle algebra interpretation of (half) the algebra $\mathcal{U} \cong \mathcal{A}$. 

Andrei Negut  
Quantum toroidal $\mathfrak{gl}_1$ and its applications
Understanding the $p_{m,n}$

- The assignment $r \rightarrow R(z_1, \ldots, z_n)$ gives rise to the **shuffle algebra** interpretation of (half) the algebra $\mathcal{U} \cong \mathcal{A}$.

- **Theorem (N)** Assume $\gcd(m, n) = 1$ for simplicity. The rational function that corresponds to $p_{m,n}$ is:

$$P_{m,n} = \frac{\prod_{i=1}^{n} z_i^{\left\lfloor \frac{mi}{n} \right\rfloor - \left\lfloor \frac{m(i-1)}{n} \right\rfloor}}{(1 - \frac{z_2q}{z_1}) \ldots (1 - \frac{z_nq}{z_{n-1}}) \prod_{i<j} \frac{(z_i - z_j)(z_i - zjq)}{(z_i - zjq_1)(z_i - zjq_2)}$$
Understanding the $p_{m,n}$

- The assignment $r \to R(z_1, ..., z_n)$ gives rise to the **shuffle algebra** interpretation of (half) the algebra $\mathcal{U} \cong A$.

- **Theorem (N)** Assume $\gcd(m, n) = 1$ for simplicity. The rational function that corresponds to $p_{m,n}$ is:

$$P_{m,n} = \frac{\prod_{i=1}^{n} z_i^{\left\lfloor\frac{mi}{n}\right\rfloor - \left\lfloor\frac{m(i-1)}{n}\right\rfloor}}{(1 - \frac{z_2 q}{z_1}) \cdots (1 - \frac{z_n q}{z_{n-1}}) \prod_{i<j} (z_i - z_j)(z_i - z_j q)(z_i - z_j q_1)(z_i - z_j q_2)} \prod_{i<j} (z_i - z_j)(z_i - z_j q)(z_i - z_j q_1)(z_i - z_j q_2)$$

- Similar formulas exist in the non co-prime case, e.g. we have:

$$P_{0,n} = \frac{1 + \frac{z_n q}{z_{n-1}} + \cdots + \frac{z_n q^{n-1}}{z_1}}{(1 - \frac{z_2 q}{z_1}) \cdots (1 - \frac{z_n q}{z_{n-1}}) \prod_{i<j} (z_i - z_j)(z_i - z_j q)(z_i - z_j q_1)(z_i - z_j q_2)} \prod_{i<j} (z_i - z_j)(z_i - z_j q)(z_i - z_j q_1)(z_i - z_j q_2)$$
Knot invariants

- The algebra $\mathcal{U} \cong \mathcal{A}$ is isomorphic to (a stabilization) of the spherical Cherednik DAHA of type $A$. 

- In a joint work with Eugene Gorsky, this allowed us to identify: 

\[ \langle v(a) | p_m, n | 0 \rangle \]

with the superpolynomial $P_{m, n}(a, q_1, q_2)$. 

- This is a three variable invariant of the $(m, n)$ torus knot, studied by many authors, notably Aganagic-Shakirov (refined Chern-Simons theory) and Cherednik (DAHA knot invariants). 

- This allows us to obtain the following formula:

\[ P_{m, n} = \int \prod_{n i = 1} z^{\lfloor m_i n \rfloor - \lfloor m(i-1) n \rfloor} \prod_{i < j} (z_i - z_j)(z_i - z_j q_1)(z_i - z_j q_2) \]

Andrei Negut

Quantum toroidal $gl_1$ and its applications
Knot invariants

- The algebra $\mathcal{U} \cong \mathcal{A}$ is isomorphic to (a stabilization) of the spherical Cherednik DAHA of type $A$.

- In a joint work with Eugene Gorsky, this allowed us to identify:

$$\langle \nu(a) | \rho_{m,n} | 0 \rangle$$

with the superpolynomial $\mathcal{P}_{m,n}(a, q_1, q_2)$. 

Andrei Negut
Quantum toroidal $\mathfrak{gl}_1$ and its applications
Knot invariants

- The algebra $\mathcal{U} \cong \mathcal{A}$ is isomorphic to (a stabilization) of the spherical Cherednik DAHA of type $A$.

- In a joint work with Eugene Gorsky, this allowed us to identify:

$$\langle \nu(a) | p_{m,n} | 0 \rangle$$

with the superpolynomial $P_{m,n}(a, q_1, q_2)$.

- This is a three variable invariant of the $(m, n)$ torus knot, studied by many authors, notably Aganagic-Shakirov (refined Chern-Simons theory) and Cherednik (DAHA knot invariants).
Knot invariants

- The algebra $\mathcal{U} \cong \mathcal{A}$ is isomorphic to (a stabilization) of the spherical Cherednik DAHA of type $A$.

- In a joint work with Eugene Gorsky, this allowed us to identify:

$$\langle v(a) | p_{m,n} | 0 \rangle$$

with the **superpolynomial** $\mathcal{P}_{m,n}(a, q_1, q_2)$.

- This is a three variable invariant of the $(m, n)$ torus knot, studied by many authors, notably Aganagic-Shakirov (refined Chern-Simons theory) and Cherednik (DAHA knot invariants).

- This allows us to obtain the following formula:

$$\mathcal{P}_{m,n} = \int \frac{\prod_{i=1}^{n} z_i^{\left\lfloor \frac{mi}{n} \right\rfloor - \left\lfloor \frac{(i-1)m}{n} \right\rfloor} \left(1 - a z_i\right)}{\left(1 - \frac{z_2 q}{z_1}\right) \cdots \left(1 - \frac{z_n q}{z_{n-1}}\right)} \prod_{i<j} \frac{(z_i - z_j)(z_i - z_j q)}{(z_i - z_j q_1)(z_i - z_j q_2)}$$
Macdonald $q$–difference (symmetric Dunkl) operators

- Fock space coincides with the space of symmetric functions:

\[ F = \mathbb{C}(q_1, q_2)[x_1, x_2, ...]^{\text{Sym}} \]
Macdonald $q-$difference (symmetric Dunkl) operators

- Fock space coincides with the space of symmetric functions:

$$F = \mathbb{C}(q_1, q_2)[x_1, x_2, ...]^{\text{Sym}}$$

- Macdonald introduced certain operators $\Delta_1, \Delta_2, ...$ on $F$, whose eigenvectors are the famous Macdonald polynomials:

$$\Delta_n \cdot M_\lambda = \left( \sum_{\square=(i,j) \in \lambda} q_1^{ni} q_2^{nj} \right) \cdot M_\lambda, \quad \forall \lambda \text{ partition}$$
Macdonald $q$–difference (symmetric Dunkl) operators

- Fock space coincides with the space of symmetric functions:
  \[ F = \mathbb{C}(q_1, q_2)[x_1, x_2, \ldots]^\text{Sym} \]

- Macdonald introduced certain operators $\Delta_1, \Delta_2, \ldots$ on $F$, whose eigenvectors are the famous Macdonald polynomials:
  \[ \Delta_n \cdot M_\lambda = \left( \sum_{\square=(i,j)\in\lambda} q_1^{n_i} q_2^{n_j} \right) \cdot M_\lambda, \quad \forall \lambda \text{ partition} \]

- It turns out that $\Delta_n = p_{0,n} \in \mathcal{A}$, and since $\mathcal{A} \cong \mathcal{U}$ acts on the space $F$, we obtain:
  \[ \Delta_n = \int \frac{1 + \frac{z_n q}{z_{n-1}} + \ldots + \frac{z_n q^{n-1}}{z_1}}{(1 - \frac{z_2 q}{z_1}) \ldots (1 - \frac{z_n q}{z_{n-1}})} \prod_{i<j} \frac{(z_i - z_j)(z_i - z_j q)}{(z_i - z_j q_1)(z_i - z_j q_2)} \]
  \[ \exp \left( \sum_{k \geq 1} \frac{p_k}{k} (z_1^{-k} + \ldots + z_n^{-k}) \right) \exp \left( \sum_{k \geq 1} \frac{p_k^*}{k} (z_1^{k} + \ldots + z_n^{k}) \right) \]
The second module we will study for the algebra $\mathcal{A}$ comes about geometrically, via the **Hilbert scheme** of points on $\mathbb{C}^2$. This is the variety $\text{Hilb}_d$ which parametrizes colength $d$ ideals: $I \subset \mathbb{C}[x, y]$. It is smooth and quasi-projective of dimension $2d$. It is acted on by the torus $T = \mathbb{C}^* \times \mathbb{C}^*$, which scales the two coordinate directions of $\mathbb{C}^2$. Hence we may study the $T$-equivariant $K$-theory groups: $K = \bigoplus_{d=0}^{\infty} K_T(\text{Hilb}_d)$. The idea of studying these groups together for all $d$ goes back to the work of Nakajima and Grojnowski in cohomology.
The Hilbert Scheme

- The second module we will study for the algebra $\mathcal{A}$ comes about geometrically, via the **Hilbert scheme** of points on $\mathbb{C}^2$
- This is the variety $\text{Hilb}_d$ which parametrizes colength $d$ ideals:

$$I \subset \mathbb{C}[x, y]$$
The Hilbert Scheme

- The second module we will study for the algebra \( \mathcal{A} \) comes about geometrically, via the **Hilbert scheme** of points on \( \mathbb{C}^2 \)
- This is the variety \( \text{Hilb}_d \) which parametrizes colength \( d \) ideals:
  \[
  I \subset \mathbb{C}[x, y]
  \]
- It is smooth and quasi-projective of dimension \( 2d \)
The second module we will study for the algebra $\mathcal{A}$ comes about geometrically, via the **Hilbert scheme** of points on $\mathbb{C}^2$

This is the variety $\text{Hilb}_d$ which parametrizes colength $d$ ideals:

$$I \subset \mathbb{C}[x, y]$$

It is smooth and quasi-projective of dimension $2d$

It is acted on by the torus $T = \mathbb{C}^* \times \mathbb{C}^*$, which scales the two coordinate directions of $\mathbb{C}^2$
The second module we will study for the algebra $\mathcal{A}$ comes about geometrically, via the **Hilbert scheme** of points on $\mathbb{C}^2$

This is the variety $\text{Hilb}_d$ which parametrizes colength $d$ ideals:

$$I \subset \mathbb{C}[x, y]$$

It is smooth and quasi-projective of dimension $2d$

It is acted on by the torus $T = \mathbb{C}^* \times \mathbb{C}^*$, which scales the two coordinate directions of $\mathbb{C}^2$

Hence we may study the $T$–equivariant $K$–theory groups:

$$K = \bigoplus_{d=0}^{\infty} K_T(\text{Hilb}_d)$$
The second module we will study for the algebra $A$ comes about geometrically, via the Hilbert scheme of points on $\mathbb{C}^2$.

This is the variety $\text{Hilb}_d$ which parametrizes colength $d$ ideals:

$$I \subset \mathbb{C}[x, y]$$

It is smooth and quasi-projective of dimension $2d$.

It is acted on by the torus $T = \mathbb{C}^* \times \mathbb{C}^*$, which scales the two coordinate directions of $\mathbb{C}^2$.

Hence we may study the $T$–equivariant $K$–theory groups:

$$K = \bigoplus_{d=0}^{\infty} K_T(\text{Hilb}_d)$$

The idea of studying these groups together for all $d$ goes back to the work of Nakajima and Grojnowski in cohomology.
Geometry of the Hilbert Scheme

- We may consider the tautological rank $d$ bundle $\text{Taut}_d$ on $\text{Hilb}_d$, whose fibers are given by:

\[
\text{Taut}_d|_I = \mathbb{C}[x, y]/I
\]
We may consider the tautological rank $d$ bundle $\text{Taut}_d$ on $\text{Hilb}_d$, whose fibers are given by:

$$\text{Taut}_d|_I = \mathbb{C}[x, y]/I$$

We also have the simple Nakajima correspondences:

$$\text{Hilb}_d \xleftarrow{\pi^-} \text{Hilb}_{d,d+1} = \{(I \supset I')\} \xrightarrow{\pi^+} \text{Hilb}_{d+1}$$
We may consider the tautological rank $d$ bundle $\text{Taut}_d$ on $\text{Hilb}_d$, whose fibers are given by:

$$\text{Taut}_d|_I = \mathbb{C}[x, y]/I$$

We also have the simple Nakajima correspondences:

$$\text{Hilb}_d \overset{\pi^-}{\leftarrow} \text{Hilb}_{d,d+1} = \{(I \supset I')\} \overset{\pi^+}{\rightarrow} \text{Hilb}_{d+1}$$

and the line bundle $\mathcal{L}$ on $\text{Hilb}_{d,d+1}$ whose fiber over a pair $(I \supset I')$ is the one-dimensional quotient $I/I'$
We may consider the tautological rank $d$ bundle $\text{Taut}_d$ on $\text{Hilb}_d$, whose fibers are given by:

$$\text{Taut}_d|_I = \mathbb{C}[x, y]/I$$

We also have the simple Nakajima correspondences:

$$\text{Hilb}_d \xleftarrow{\pi^-} \text{Hilb}_{d,d+1} = \{(I \supset I')\} \xrightarrow{\pi^+} \text{Hilb}_{d+1}$$

and the line bundle $\mathcal{L}$ on $\text{Hilb}_{d,d+1}$ whose fiber over a pair $(I \supset I')$ is the one-dimensional quotient $I/I'$

These constructions give rise to operators on $K$–theory:

$$e_n^\pm : K \to K, \quad e_n^\pm(\alpha) = \pi_\pm^* ([\mathcal{L}]^n \cdot \pi_{\mp*}(\alpha))$$

which shift the degree $d$ by $\pm 1$
We are now ready to give the second module structure for $\mathcal{U}$.
The action on $K$-theory

- We are now ready to give the second module structure for $\mathcal{U}$

- **Theorem** (Feigin-Tsymbaliuk, Schiffmann-Vasserot) There is an action of $\mathcal{U}$ on $K$, given by the above operators $e_n^\pm$ and:

  \[ p_n(\alpha) = \alpha \cdot [\text{Taut}]^n, \quad p_{-n}(\alpha) = \alpha \cdot [\text{Taut}^\vee]^n \]

- One may then ask about the action of the generators $p_m^0$ on $K$, for example the particular case of $p_m^0$.

- In cohomology, Nakajima realized $p_m^0$ via the correspondence:

  \[ \text{Hilb}_d \leftarrow \text{Hilb}_{d + m} \rightarrow \text{Hilb}_{d + m} \]

- This locus doesn't work in $K$-theory, because it is badly behaved (it's not even lci) and has too little structure.
The action on $K$–theory

- We are now ready to give the second module structure for $\mathcal{U}$

- **Theorem** (Feigin-Tsymbaliuk, Schiffmann-Vasserot) There is an action of $\mathcal{U}$ on $K$, given by the above operators $e_n^\pm$ and:

  $$p_n(\alpha) = \alpha \cdot [\text{Taut}]^n, \quad p_{-n}(\alpha) = \alpha \cdot [\text{Taut}^\vee]^n$$

- One may then ask about the action of the generators $p_{m,n}$ on $K$, for example the particular case of $p_{m,0}$
The action on $K$–theory

- We are now ready to give the second module structure for $U$

- **Theorem** (Feigin-Tsymbaliuk, Schiffmann-Vasserot) There is an action of $U$ on $K$, given by the above operators $e_n^\pm$ and:

  $$p_n(\alpha) = \alpha \cdot [\text{Taut}]^n, \quad p_{-n}(\alpha) = \alpha \cdot [\text{Taut}^\vee]^n$$

- One may then ask about the action of the generators $p_{m,n}$ on $K$, for example the particular case of $p_{m,0}$

- In cohomology, Nakajima realized $p_{m,0}$ via the correspondence:

  $$\text{Hilb}_d \leftarrow \text{Hilb}_{d,d+m} = \{(I \supset I')\} \rightarrow \text{Hilb}_{d+m}$$
The action on $K$–theory

- We are now ready to give the second module structure for $\mathcal{U}$

- **Theorem** (Feigin-Tsymbaliuk, Schiffmann-Vasserot) There is an action of $\mathcal{U}$ on $K$, given by the above operators $e_n^{\pm}$ and:

$$p_n(\alpha) = \alpha \cdot [\text{Taut}]^n, \quad p_{-n}(\alpha) = \alpha \cdot [\text{Taut}^\vee]^n$$

- One may then ask about the action of the generators $p_{m,n}$ on $K$, for example the particular case of $p_{m,0}$

- In cohomology, Nakajima realized $p_{m,0}$ via the correspondence:

$$\text{Hilb}_d \leftarrow \text{Hilb}_{d,d+m} = \{(I \supset I')\} \rightarrow \text{Hilb}_{d+m}$$

- This locus doesn’t work in $K$–theory, because it is badly behaved (it’s not even lci) and has too little structure
We resolve the above locus by the flag Hilbert scheme:

\[ \text{Hilb}_{d,d+m} = \{(I_d \supset ... \supset I_{d+m})\} \subset \text{Hilb}_d \times \text{Hilb}_{d+1} \times ... \times \text{Hilb}_{d+m} \]

where all the support points are required to coincide.
The flag Hilbert scheme

- We resolve the above locus by the **flag Hilbert scheme**: 
  \[ \text{Hilb}_{d,d+m} = \{(I_d \supset \ldots \supset I_{d+m})\} \subset \text{Hilb}_d \times \text{Hilb}_{d+1} \times \ldots \times \text{Hilb}_{d+m} \]
  where all the support points are required to coincide

- Note that this locus is dimension \( m - 1 \) greater than the composition of \( m \) simple Nakajima correspondences
We resolve the above locus by the *flag Hilbert scheme*:

\[ \text{Hilb}_{d,d+m} = \{(I_d \supset \ldots \supset I_{d+m})\} \subset \text{Hilb}_d \times \text{Hilb}_{d+1} \times \ldots \times \text{Hilb}_{d+m} \]

where all the support points are required to coincide.

Note that this locus is dimension \( m - 1 \) greater than the composition of \( m \) simple Nakajima correspondences.

The flag Hilbert schemes are quite rich in line bundles:

\[ \mathcal{L}_1, \ldots, \mathcal{L}_m \quad \text{whose fibers are} \quad \mathcal{L}_i = I_{d+m-i}/I_{d+m-i-1} \]
The flag Hilbert scheme

- We resolve the above locus by the flag Hilbert scheme:

\[ \text{Hilb}_{d,d+m} = \{(I_d \supset \ldots \supset I_{d+m})\} \subset \text{Hilb}_d \times \text{Hilb}_{d+1} \times \ldots \times \text{Hilb}_{d+m} \]

where all the support points are required to coincide

- Note that this locus is dimension \( m - 1 \) greater than the composition of \( m \) simple Nakajima correspondences

- The flag Hilbert schemes are quite rich in line bundles:

\[ \mathcal{L}_1, \ldots, \mathcal{L}_m \quad \text{whose fibers are} \quad \mathcal{L}_i = I_{d+m-i}/I_{d+m-i-1} \]

- **Theorem** (N) the operator \( p_{m,n} \in \mathcal{A} \cong \mathcal{U} \) acts on \( K \) via:

\[
p_{m,n}(\alpha) = \pi^+_* \left( \prod_{i=1}^{m} [\mathcal{L}_i]^{\left\lfloor \frac{ni}{m} \right\rfloor - \left\lfloor \frac{n(i-1)}{m} \right\rfloor} \cdot \pi^{-*}(\alpha) \right)
\]