Probabilistic derivation of higher equations of motion in Liouville CFT

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Abstract

We rigorously derive two simple cases of the higher equations of motion for Liouville conformal field theory. These equations were predicted in physics by Zamolodchikov by identifying the scaling dimension of certain primary operators of the CFT. We work in the probabilistic framework of Liouville theory first introduced by David-Kupiainen-Rhodes-Vargas on the Riemann sphere which allows to perform rigorous computations. In a future work we plan to use these equations to compute the so-called correlation numbers of minimal Liouville gravity.

1 Introduction and main results

Liouville CFT is a fundamental ingredient of the quantization of 2d gravity introduced by Alexander Polyakov [15] where in the formal summation over Riemannian metric tensors it governs the behavior of the conformal factor of the metric.

Let (Σ, g) be a Riemann surface equipped with a Riemannian metric tensor g. Locally the metric g can be written in conformal coordinates as

$$g = e^{\sigma} \mathrm{d}z\overline{\mathrm{d}z} \tag{1}$$

where σ is known as the conformal factor. If g is of constant curvature K, then σ satisfies the Liouville equation:

$$\partial\overline{\partial}\sigma = -\frac{K}{2}e^{\sigma}. \tag{2}$$

This equation leads to an infinite series of differential equations for σ with remarkable relations to the representation theory of the Virasoro algebra. The structure of the equations is as follows. For an integer $n \geq 1$, consider the quantity $e^{(1-n)\sigma/2}$. Let also

$$T = -\frac{1}{4}(\partial\sigma)^2 + \frac{1}{2}\partial^2\sigma \tag{3}$$

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be the classical energy-momentum tensor for the equation (2). It is believed [17] that there exist a "classical BPZ operator"

$$D_n = D_n(\partial, T, \partial T, \ldots, \partial^n T), \tag{4}$$

where D_n is a degree n homogeneous polynomial in its arguments, and where by convention $\partial^k T$ is defined to have degree $k+2$ and ∂ degree 1. This operator, in partiular, annihilates $e^{(1-n)\sigma/2}$:

$$D_n e^{(1-n)\sigma/2} = 0. \tag{5}$$

For example, $D_1 = \partial$ and $D_2 = \partial^2 - e^{\sigma/2}(\partial^2 e^{-\sigma/2}) = \partial^2 + T$.

Now define $\overline{D}_n = D_n(\overline{\partial}, \overline{\partial}T, \ldots, \overline{\partial}^n T)$, the same operator as D_n but with anti-holomorphic derivatives. Then the classical higher equations of motion are conjectured to be

$$D_n \overline{D_n} \sigma e^{(1-n)\sigma/2} = B_n e^{(1+n)\sigma/2}, \tag{6}$$

where $B_n = 2(-1)^{n+1} n!(n-1)!(1/2)^n$. In particular, for $n = 1$ the equation (6) is just the usual Liouille equation (2).

Alexei Zamolodchikov [17] generalized these equations of motion to the Liouville CFT that can be thought of as a quantization of the theory of the conformal factor σ. This quantization depends on a parameter γ (or $b = \gamma/2$ in a different convention [16]) that is related to the central charge of the theory. Observables in the LCFT are denoted as $V_\alpha(z) =: e^{\alpha\phi(z)}:$, where the Liouville field ϕ is a quantization of σ. Correlation functions of LCFT then correspond to the expectation of the product of such observables denoted by:

$$\langle V_{\alpha_1}(z_1), \ldots, V_{\alpha_n}(z_n)\rangle_\Sigma. \tag{7}$$

Mathematically these correlation functions have been defined for certain range of parameters [11] using probability theory, see below in the section 3.

Each correlator is acted upon by the Virasoro algebra with generators $\{L_n\}_{n\in\mathbb{Z}}$ by the formulas which on the Riemann sphere take the form:

$$L_{-1} = \partial_z,\ L_{-n} = \sum_{i\neq 1}\left[\frac{(n-1)\Delta_i}{(z-z_i)^n} - \frac{\partial_i}{(z-z_i)^{n-1}}\right], \quad n > 1. \tag{8}$$

Here one has $\Delta_i = \frac{\alpha_i}{2}(Q - \frac{\alpha_i}{2})$. The Virasoro algebra is a quantization (central extension to be more precise) of the Witt algebra $\{l_n\}_{n\in\mathbb{Z}}$, $l_n = -z^{n+1}\partial_z$ of holomorphic vector fields on $\mathbb{C}^*$. Its commutation relations are given by:

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{24}n(n^2-1)\delta_{n,-m}, \tag{9}$$

where the central charge is related to the parameter γ of the Liouville theory $c = 1 + 6Q^2$, $Q = \gamma/2 + 2/\gamma$.

One can use the Virasoro algebra to construct differential operators acting on all the correlation functions. Recall that the universal enveloping algebra of a Lie algebra is an algebra generated by tensor products of the elements of the Lie algebra with the relation that the commutator is equal to the Lie bracket. Equations we are interested in such as the BPZ equations [5] or the higher equations of motion are written in terms of the universal enveloping of the Virasoro algebra. Recall that Virasoro algebra acts on all correlation functions by the formulas (8). When we say an observable satisfies an equation written in terms of Virasoro algebra it actually means that all correlation functions with this observable satisfy a differential equation produced by the Virasoro action (8) on the correlators.

There exist special observables $V_{m,n}(z) := V_{\alpha_{m,n}}(z)$ indexed by a pair of natural numbers that are called degenerate fields. Here

$$\alpha_{m,n} = Q - m\frac{2}{\gamma} + n\frac{\gamma}{2}. \tag{10}$$

These degenerate fields satisfy BPZ equations [5, 16],

$$D_{m,n}V_{m,n}(z) = 0. \tag{11}$$

These fields correspond to reducible Verma modules of the Virasoro algebra [5]. The BPZ equations were proved for $m = 1$ or $n = 1$ infinite series in the probabilistic setup by Tunan Zhu [19]. Classical BPZ equations can be obtained as a limit of the quantum ones for $D_{n,1}$. Other equations do not have a good classical limit and are due to the well-known duality $\gamma/2 \to 2/\gamma$ in the Liouville CFT.

The classical equations of motion (6) also have an analogue in the quantum side known as the higher equations of motion of Liouville CFT. They were derived in the physics framework by Alexei Zamolodchikov and take the form:

$$D_{m,n}\overline{D_{m,n}}\phi(z)\, V_{m,n}(z) = B_{m,n}\, V_{\alpha_{m,n}+2mn}(z). \tag{12}$$

These equations have a similar form to (6) and the latter ones are believed to be obtained by taking the classical limit of the $(n, 1)$-series of (12).

In this note we investigate the equations of motion in the two simplest cases $(m, n) = (1, 1)$ and $(m, n) = (2, 1)$ using the rigorous probabilistic framework of Liouville CFT on Riemann sphere developed in [11, 14]. In particular, we show that equations (12) are correct in the case when background metric is flat near the insertion point z: $\log(\hat{g}) = 0$ and when z is apart from the other insertions in the correlator (7)

The precise statements we get is the following

Theorem 1. *Let* $\mathbf{z} = \{z_i\}_{i=1}^{n}$ *be such that* $z_i \neq z_j \neq z$ *for all* i, j *and* $\{\alpha_i\}$ *satisfy the condition* $\forall i\ \alpha_i < Q$. *Let also* $|z| < 1$ *so that the background metric* (18) *is flat around* z. *Then we have the following equalities in the distributional sense: Let* $\gamma + \sum_i \alpha_i < 2Q$, *then*

$$\partial_{\overline{z}}\partial_z \langle \phi(z)\, V_{\alpha_1}(z_1)\dots V_{\alpha_n}(z_n)\rangle = -\frac{\pi\mu\gamma}{2}\langle V_\gamma(z)\, V_{\alpha_1}(z_1)\dots V_{\alpha_n}(z_n)\rangle. \tag{13}$$

Let $3\gamma/2 + \sum_i \alpha_i < 2Q$ *and*

$$D_{2,1} = L_{-1}^2 - \frac{\gamma^2}{4}L_{-2}, \tag{14}$$

then

$$D_{2,1}\overline{D_{2,1}}\langle \phi(z)V_{-\gamma/2}(z)\, V_{\alpha_1}(z_1)\dots V_{\alpha_n}(z_n)\rangle = -\frac{\mu^2\pi^2\gamma^3}{4}\langle V_{3\gamma/2}(z)\, V_{\alpha_1}(z_1)\dots V_{\alpha_n}(z_n)\rangle, \tag{15}$$

where the correlators and their derivatives are defined as a limit $\epsilon \to 0$, *see section 3.*

In this note we consider the flat metric, in general additional metric terms will appear. In particular, such terms inevitably appear in the integration over moduli, see discussion below. We plan to investigate these terms later.

Liouville Gravity Higher equations of motion are interesting on their own accord as equations relating logarithmic correlation functions including degenerate Liouville fields and non-logarithmic correlators with non-degenerate fields. However, one of the main reasons why they were introduced [17] is their applications to the theory of Liouville Gravity (LG) or integration over moduli.

Quantum gravity is a theory of integrating over all possible metrics. On a Riemann surface this can be done by integrating over conformal classes and over the conformal factor in each conformal class. The latter can be realized by a Liouville CFT which being a QFT of the conformal factor reproduces integration over all possible conformal factors. Using LCFT correlation functions we can construct differential forms on the moduli space of Riemann surfaces. Then integration over conformal classes is performed by integrating these differential forms over the moduli space of Riemann surfaces (Deligne-Mumford spaces $\mathcal{M}_{g,n}$).

For example, a general Liouville Gravity correlation number on a sphere can be schematically written as

$$\int_{\mathcal{M}_{0,n}} \langle V_{\alpha_1}(z_1)\dots V_{\alpha_n}(z_n)\rangle \cdot \langle \Phi_{\alpha_1}(z_1)\dots \Phi_{\alpha_n}(z_n)\rangle^M, \tag{16}$$

where

$$\langle \Phi_{\alpha_1}(z_1)\dots \Phi_{\alpha_n}(z_n)\rangle^M \tag{17}$$

is a matter (+ghosts) correlation function whose transformation properties are determined by the fact that when multiplied by the LCFT correlation it becomes a differential form on $\mathcal{M}_{0,n}$. In particular, the following relation between conformal dimensions of the fields should hold $\Delta(\Phi_{\alpha_i}) + \Delta(V_{\alpha_i}) = 1$. Matter correlation function must satisfy conformal Ward identities and BPZ equations similar to LCFT correlation functions. For example, one could take the correlation function to be a Minimal Model correlation function [5]. We remark, that apriory the correlation function at colliding insertions are not defined, and the integral is taken over a non-compact moduli space $\mathcal{M}_{0,n}$ without the boundary.

The idea of [17, 7] is to use the higher equations of motion to simplify the differential form in (16). The form of the HEM (12) is suggestive that the integrand can be represented as a sum of exact terms plus contributions from the metric "curvature terms" and the boundary points when $z \to z_i$. The latter terms are much simpler to compute and it is expected that their careful computation will allow to reduce the correlation numbers to purely cohomological or recursive relations.

Another reason that such cohomological/recursive formulas are expected to hold is due to the conjectured relation between Liouville Gravity and other models of 2d quantum gravity: topological gravity and Matrix Models (see, e.g. [12]). One of the most known results in this area is the KPZ exponents [13]. This relation was extensively studied in the 90s by physicists but remains mostly unsolved due to the fact that computations in LCFT are very complicated.

It is noteworthy that some important partial progress was made after the HEM were introduced: [7] managed to compute several 4-point correlation numbers on a sphere which constitute the first case with nontrivial integration over moduli since $\mathcal{M}_{0,4} \simeq S^2 \backslash \{0, 1, \infty\}$. In this case the integral (16) can be also performed numerically [18, 1] using conformal bootstrap [16, 5] for LCFT and Minimal Models correlation functions. Computations of [7] were used in [6] to provide the relation between these 4-point correlation numbers and those in matrix models for the Lee-Yang series of Minimal Models in the matter sector that correspond to $\gamma = 2\sqrt{2/(2p + 1)}$. For other Minimal Models progress was made using Frobenius manifolds: [8] and then in [10], [4]. One of the issues that stops this method is that even general 4-point correlation numbers on the sphere remain a mystery because sometimes direct HEM application in the physical framework fails for (probably) analytic reasons.

It was partly investigated by one of the authors in [1]. There are also some results for 1-point correlation numbers on the torus [9] and on the disk [3, 2].

One of the motivations of the current paper is to set up a framework to compute the correlation numbers using a mathematically rigorous form of the HEM which we expect to produce new results and shed light on the problems of the previous methods.

2 Classical equations of motion

Here we demonstrate the classical version of the first nontrivial Higher Equation of Motion. Even though the computation is simple it is instructive for the quantum case in the section 6 below since part of the computation technique is the same.

The conformal factor obeys the Liouville equation:

$$\Delta\sigma = -\frac{K}{2}e^{\sigma}.$$

In the classical case, $e^{-\frac{\sigma}{2}}$ plays the role of the degenerate field. We compute the second derivative of this field:

$$\partial^2(e^{-\frac{\sigma}{2}}) = \partial\left(-\frac{1}{2}(\partial\sigma)e^{-\frac{\sigma}{2}}\right) = \left(\frac{1}{4}(\partial\sigma)^2 - \frac{1}{2}\partial^2\sigma\right)e^{-\frac{\sigma}{2}}.$$

Therefore we obtain the equation:

$$\left(\partial^2 - \frac{1}{4}(\partial\sigma)^2 + \frac{1}{2}\partial^2\sigma\right)e^{-\frac{\sigma}{2}} = 0.$$

This should be viewed as the classical second order BPZ equation with the operators $L_{-1} = \partial$ and $L_{-2} = -\frac{1}{4}(\partial\sigma)^2 + \frac{1}{2}\partial^2\sigma$. Here the operator L_{-2} is also the classical stress-energy tensor T. The following notations are related to the analogous notations in the quantum case below:

$$Q_1 = -\frac{1}{2}(\partial\sigma)e^{-\frac{\sigma}{2}}, \quad Q_{1,1} = \frac{1}{4}(\partial\sigma)^2e^{-\frac{\sigma}{2}}, \quad Q_2 = -\frac{1}{2}(\partial^2\sigma)e^{-\frac{\sigma}{2}}.$$

Lets now move to the classical higher equations of motion. The analogue of the logarithmic field is $\sigma e^{-\frac{\sigma}{2}}$. Let $D_2 = L_{-1}^2 + T$. We compute:

$$\begin{aligned}\overline{D}_2D_2(\sigma e^{-\frac{\sigma}{2}}) &= \overline{D}_2((\partial^2\sigma) + 2(\partial\sigma)\partial)e^{-\frac{\sigma}{2}} + \overline{D}_2\sigma D_2e^{-\frac{\sigma}{2}}\\ &= \overline{D}_2((\partial^2\sigma) - (\partial\sigma)^2)e^{-\frac{\sigma}{2}}.\end{aligned}$$

Next we use that $\overline{D}_2$ and ∂^2 commute. We write:

$$2\overline{D}_2\partial^2e^{-\frac{\sigma}{2}} = 2\partial^2\overline{D}_2e^{-\frac{\sigma}{2}} = 0.$$

This implies:

$$\overline{D}_2\left(\frac{1}{2}(\partial\sigma)^2 - (\partial^2\sigma)\right)e^{-\frac{\sigma}{2}} = 0.$$

Putting these last steps together we get that:

$$\overline{D}_2D_2(\sigma e^{-\frac{\sigma}{2}}) = -\frac{1}{2}\overline{D}_2(\partial\sigma)^2e^{-\frac{\sigma}{2}} = -2\overline{D}_2Q_{1,1}.$$

We also have the equation:

$$-2\overline{D}_2L_{-1}e^{-\frac{\sigma}{2}} = 0,$$

which implies:

$$\left((\bar\partial\Delta\sigma) + 2(\Delta\sigma)\bar\partial\right)e^{-\frac{\sigma}{2}} = 0.$$

Now we can compute:

$$\begin{aligned}\overline{D}_2(\partial\sigma)^2e^{-\frac{\sigma}{2}} &= \left(\bar\partial^2(\partial\sigma)^2 + 2\bar\partial(\partial\sigma)^2\bar\partial\right)e^{-\frac{\sigma}{2}}\\ &= \left(2((\bar\partial\Delta\sigma)\partial\sigma + (\Delta\sigma)^2) + 4(\Delta\sigma)(\partial\sigma)\bar\partial\right)e^{-\frac{\sigma}{2}}\\ &= 2(\Delta\sigma)^2e^{-\frac{\sigma}{2}} = \frac{K^2}{2}e^{\frac{3\sigma}{2}}.\end{aligned}$$

In the second to last inequality we have used the previous equation and in the last equality the equation $\Delta\sigma = e^{\sigma}$. The conclusion is thus that:

$$\overline{D}_2D_2(\sigma e^{-\frac{\sigma}{2}}) = -\frac{K^2}{4}e^{\frac{3\sigma}{2}}.$$

3 Probabilistic definition of Liouville CFT on the Riemann sphere

We follow the probabilistic framework to Liouville CFT first introduced in [11] and further developed in [14]. We work on the Riemann sphere $\mathbb{S}^2 = \mathbb{C} \cup \{\infty\}$ viewed as the complex plane with a point at infinity, which is equipped with a Riemmannian metric g. For simplicity we will work with the metric:

$$g(z) = |z|_+^{-4} \quad \text{with} \quad |z|_+ := \max(1, |z|). \tag{18}$$

This metric is flat in the unit disk. We next definition gives the covariance of the Gaussian free field on $\mathbb{S}^2$.

Definition 3.1. *(Gaussian free field) The Gaussian free field X is the centered Gaussian process on $\mathbb{C}$ with covariance given by, for $x, y \in \mathbb{C}$:*

$$\mathbb{E}[X(x)X(y)] = \log \frac{1}{|x-y|} + \log |x|_+ + \log |y|_+. \tag{19}$$

Since the variance at each point is infinite, X is not defined pointwise and exists as a random distribution. It also satisfies:

$$\int_0^{2\pi} X(e^{i\theta}) d\theta = 0. \tag{20}$$

Next we introduce a regularization X_ϵ of our field X, which depends on a small parameter $\epsilon > 0$. Define $\eta_\epsilon = \frac{1}{\epsilon^2}\eta(\frac{|x|^2}{\epsilon^2})$ where η is a non-negative smooth function defined on $\mathbb{R}_+$ with compact support in $[\frac{1}{2}, 1]$ that satisfies $\pi \int_0^\infty \eta(t)dt = 1$. We then define the smooth field by the convolution $X_\epsilon := X * \eta_\epsilon$. We now define the associated Gaussian multiplicative chaos (GMC) measure.

Definition 3.2. *(Gaussian multiplicative chaos) Fix a $\gamma \in (0, 2)$. The Gaussian multiplicative chaos measure associated to the field X is defined by the following limit,*

$$e^{\gamma X(x)} d^2x = \lim_{\epsilon \to 0} e^{\gamma X_\epsilon(x) - \frac{\gamma^2}{2}\mathbb{E}[X_\epsilon(x)^2]} d^2x, \tag{21}$$

where the convergence is in probability and in the sense of weak convergence of measures on $\mathbb{C}$. More precisely, for a continuous compactly supported function f on $\mathbb{C}$, the following convergence holds in probability:

$$\int_{\mathbb{R}} f(x) e^{\gamma X(x)} d^2x = \lim_{\epsilon \to 0} \int_{\mathbb{C}} f(x) e^{\gamma X_\epsilon(x) - \frac{\gamma^2}{2}\mathbb{E}[X_\epsilon(x)^2]} d^2x. \tag{22}$$

For convenience we introduce the following shorthand notation for the Liouville field ϕ on $\mathbb{C}$,

$$\phi(z) = X(x) + \frac{Q}{2} \log g(z) + c, \tag{23}$$

and for the regularized vertex operators

$$V_{\alpha,\epsilon}(z) = e^{\alpha(X_\epsilon(z)+c) - \frac{\alpha^2}{2}\mathbb{E}[X_\epsilon(z)^2]} g(z)^{\Delta_\alpha},$$

where again $\Delta_\alpha = \frac{\alpha}{2}(Q - \frac{\alpha}{2})$.

Definition 3.3. *(Probabilistic definition of the LCFT correlation on $\mathbb{S}^2$) Consider disjoint points $z_k \in \mathbb{C}$ with associated weights $\alpha_k \in \mathbb{R}$ satisfying $\alpha_k < Q$ and $\sum_k \alpha_k > 2Q$. Then one can defined:*

$$\langle \prod_{k=1}^N V_{\alpha_k}(z_k) \rangle := \lim_{\epsilon \to 0} \int_{\mathbb{R}} e^{-2Qc} \mathbb{E}\left[\prod_{k=1}^N V_{\alpha_k,\epsilon}(z_k) e^{-\mu e^{\gamma c} \int_{\mathbb{C}} V_{\gamma,\epsilon}(x) d^2x} \right] dc.$$

The higher equations of motions will be expressed on correlations where one insertion plays a special role. We will denote this insertion by z_0 or simply z. It will have an associated weight denoted by α_0. Furthermore we need will consider correlation which can at this special point $z_0 = z$ a so-called logarithmic field. We give the definition below.

Definition 3.4. *(Probabilistic definition of the LCFT correlation on $\mathbb{S}^2$ with logarithmic fields) Consider a special point $z = z_0$ with weight $\alpha_0 < Q$ and N disjoint points $z_k \in \mathbb{C}$ with associated weights $\alpha_k \in \mathbb{R}$ satisfying $\alpha_k < Q$ and $\sum_{k=0}^N \alpha_k > 2Q$. Then one can define:*

$$\langle \phi(z) \prod_{k=0}^N V_{\alpha_k}(z_k) \rangle := \lim_{\epsilon \to 0} \int_{\mathbb{R}} e^{-2Qc} \mathbb{E}\left[\phi_\epsilon(z) \prod_{k=0}^N V_{\alpha_k,\epsilon}(z_k) e^{-\mu \int_{\mathbb{C}} V_{\gamma,\epsilon}(x) d^2x} \right] dc.$$

Throughout the proof we will use the following shorthand notations for correlation functions with or without the logarithmic field:

$$\langle z, \mathbf{z} \rangle_{\mathsf{log}} := \langle \phi(z) \prod_{k=0}^N V_{\alpha_k}(z_k) \rangle, \quad \langle z, \mathbf{z} \rangle := \langle \prod_{k=0}^N V_{\alpha_k}(z_k) \rangle. \tag{24}$$

We finish this section by giving the following lemma which tells us that the Laplacian of the regularized logarithm converge up to a constant to a delta function as the regularization ϵ is sent to 0.

Lemma 3.1. *Let $|z| \leq 1$. Let $\mathcal{A}_\epsilon(x, z)$ be a continuous function such that $\lim_{\epsilon \to 0} \mathcal{A}_\epsilon(x, z) = \mathcal{A}(x, z)$ and such that the integral*

$$I_\epsilon = \int_{\mathbb{C}} \mathrm{d}x\, \partial_z \partial_{\bar{z}} (\mathbb{E}[X_\epsilon(x) X_\epsilon(z)]) \cdot \mathcal{A}_\epsilon(x, z) \tag{25}$$

is absolutely convergent. Then one has:

$$\lim_{\epsilon \to 0} I_\epsilon = \pi \mathcal{A}(z, z). \tag{26}$$

Proof. Recall that

$$\frac{1}{(z)_\epsilon} = \int_{\mathbb{C}} \mathrm{d}^2 x_1 \int_{\mathbb{C}} \mathrm{d}^2 x_2 \, \frac{1}{z - x_1 + x_2} \eta_\epsilon(x_1) \eta_\epsilon(x_2), \tag{27}$$

where $\eta_\epsilon(x) = \epsilon^{-2} \eta(|x|^2/\epsilon^2)$, and η is a smooth function with compact support separated from 0 such that $\pi \int_0^\infty \eta(x) \mathrm{d}x = 1$. In particular $\int_{\mathbb{C}} \mathrm{d}^2 x \, \eta_\epsilon(x) = 1$. Let $\psi(z)$ be a smooth test function with compact support. Using first Stokes and then Fubini theorems we can write

$$\langle \psi(x), \partial_{\bar{z}} 1/(z)_\epsilon \rangle = -\langle \partial_{\bar{z}} \psi(z), 1/(z)_\epsilon \rangle = -\int_{\mathbb{C}^2} \mathrm{d}^2 x_1 \mathrm{d}^2 x_2 \int_{\mathbb{C}} \mathrm{d}^2 z \, \frac{\partial_{\bar{z}} \psi(z)}{z - x_1 + x_2} \eta_\epsilon(x_1) \eta_\epsilon(x_2)$$

The interior integral is equal to

$$\begin{aligned} &\int_{\mathbb{C}} \mathrm{d}^2 z \, \frac{\partial_{\bar{z}} \psi(z)}{z - x_1 + x_2} \eta_\epsilon(x_1) \eta_\epsilon(x_2) \\ &= -\frac{1}{2i} \oint_{|z| = \delta} \mathrm{d}z \, \frac{\psi(z)}{z - x_1 + x_2} \eta_\epsilon(x_1) \eta_\epsilon(x_2) = -\pi \psi(x_1 - x_2) \eta_\epsilon(x_1) \eta_\epsilon(x_2), \end{aligned}$$

where we used the Stokes formula and $\mathrm{d}^2 y = -\mathrm{d}y \mathrm{d}\bar{y}/2i$. Then taking the limit we obtain:

$$\lim_{\epsilon \to 0} \langle \psi(x), \partial_{\bar{z}} 1/(z)_\epsilon \rangle = \pi \psi(0). \tag{28}$$

□

4 A simple case: the $(1,1)$ equation.

Recall the notation $\phi = X + \frac{Q}{2}\log g + c$ given in equation (23). The quantity that will obey the $(1,1)$ equation is given by the limit:

$$\langle\phi(z)\prod_{k=1}^{N}V_{\alpha_k}(z_k)\rangle := \lim_{\epsilon\to 0}\int_{\mathbb{R}} e^{-2Qc}\mathbb{E}\left[\phi_\epsilon(z)\prod_{k=1}^{N}V_{\alpha_k,\epsilon}(z_k)e^{-\mu\int_{\mathbb{C}}V_{\gamma,\epsilon}(x)d^2x}\right]dc.$$

In the case applying the operator of the higher equation of motion amounts to computing $\partial_{\bar z}\partial_z\langle\phi_\epsilon(z)\prod_{k=1}^{N}V_{\alpha_k,\epsilon}(z_k)\rangle$. By using the Cameron-Martin formula given in Lemma A.1, note that one can write:

$$\begin{aligned}
&\int_{\mathbb{R}} e^{-2Qc}\mathbb{E}\left[X_\epsilon(z)\prod_{k=1}^{N}V_{\alpha_k,\epsilon}(z_k)e^{-\mu\int_{\mathbb{C}}V_{\gamma,\epsilon}(x)d^2x}\right]dc\\
&=\frac{d}{dt}_{|t=0}\int_{\mathbb{R}} e^{-2Qc}\mathbb{E}\left[e^{tX_\epsilon(z)-\frac{t^2}{2}\mathbb{E}[X_\epsilon(z)^2]}\prod_{k=1}^{N}V_{\alpha_k,\epsilon}(z_k)e^{-\mu\int_{\mathbb{C}}V_{\gamma,\epsilon}(x)d^2x}\right]dc\\
&=\frac{d}{dt}_{|t=0}\int_{\mathbb{R}} e^{-2Qc}\mathbb{E}\left[\prod_{k=1}^{N}V_{\alpha_k,\epsilon}(z_k)e^{\alpha_k t\mathbb{E}[X_\epsilon(z)X_\epsilon(z_k)]}e^{-\mu\int_{\mathbb{C}}V_{\gamma,\epsilon}(x)e^{\gamma t\mathbb{E}[X_\epsilon(z)X_\epsilon(x)]}d^2x}\right]dc.
\end{aligned}$$

By applying this formula to the GFF part of the field ϕ one obtains:

$$\begin{aligned}
\langle\phi_\epsilon(z)\prod_{k=1}^{N}V_{\alpha_k,\epsilon}(z_k)\rangle &= \int_{\mathbb{R}} e^{-2Qc}\mathbb{E}\left[\left(\frac{Q}{2}\log g(z)+c\right)\prod_{k=1}^{N}V_{\alpha_k,\epsilon}(z_k)e^{-\mu\int_{\mathbb{C}}V_{\gamma,\epsilon}(x)d^2x}\right]dc\\
&+\sum_{k=1}^{N}\alpha_k\mathbb{E}[X_\epsilon(z)X_\epsilon(z_i)]\int_{\mathbb{R}} e^{-2Qc}\mathbb{E}\left[\prod_{k=1}^{N}V_{\alpha_k,\epsilon}(z_k)e^{-\mu\int_{\mathbb{C}}V_{\gamma,\epsilon}(x)d^2x}\right]dc\\
&-\mu\gamma\int_{\mathbb{C}}d^2x_1\mathbb{E}[X_\epsilon(z)X_\epsilon(x_1)]\int_{\mathbb{R}} e^{-2Qc}\mathbb{E}\left[V_{\gamma,\epsilon}(x_1)\prod_{k=1}^{N}V_{\alpha_k,\epsilon}(z_k)e^{-\mu\int_{\mathbb{C}}V_{\gamma,\epsilon}(x)d^2x}\right]dc.
\end{aligned}$$

Now applying the operator $\partial_z\partial_{\bar z}$ to the final line we get that:

$$\begin{aligned}
\partial_{\bar z}\partial_z\langle\phi_\epsilon(z)\prod_{k=1}^{N}V_{\alpha_k,\epsilon}(z_k)\rangle &= \left(\frac{Q}{2}\partial_{\bar z}\partial_z\log g(z)+\sum_{k=1}^{N}\alpha_k\partial_{\bar z}\partial_z\mathbb{E}[X_\epsilon(z)X_\epsilon(z_i)]\right)\langle\prod_{k=1}^{N}V_{\alpha_k}(z_k)\rangle\\
&-\mu\gamma\int_{\mathbb{C}}d^2x_1\partial_{\bar z}\partial_z(\mathbb{E}[X_\epsilon(z)X_\epsilon(x_1)])\langle V_\gamma(x_1)\prod_{k=1}^{N}V_{\alpha_k}(z_k)\rangle.
\end{aligned}$$

In the line above the first term vanishes because $|z|<1$ implies that $\log g(z)=0$ and the second term vanishes since $\partial_z\partial_{\bar z}\mathbb{E}[X_\epsilon(z)X_\epsilon(x_1)]=0$. The third term is computed using the lemma 3.1, so that we get

$$\partial_{\bar z}\partial_z\langle\phi(z)\prod_{k=1}^{N}V_{\alpha_k}(z_k)\rangle = -\pi\mu\gamma/2\langle V_\gamma(z)\prod_{k=1}^{N}V_{\alpha_k}(z_k)\rangle. \tag{29}$$

This completes the proof of equation (13).

5 Derivative rules

In this section we list the key lemmas that will be required to compute the derivatives of the correlation function required by the higher equations of motion beyond the simple case of the $(1,1)$ equation. We will extensively use the shorthand notations (24) for correlation functions. We also will use the notations

$$P_n = \sum_{j=1}^{N} \frac{\alpha_j}{2(z_j - z)_\epsilon^n}, \; P_{\mathbf{n}} = \prod_i P_{n_i}, \tag{30}$$

$$Q_{\mathbf{q}} = \left(\frac{\mu\gamma}{2}\right)^p \int_{\mathbb{C}^p} \prod_{j=1}^{p} \frac{1}{(y_j - z)_\epsilon^{q_j}} \langle z, \mathbf{z}, \mathbf{y} \rangle_\epsilon \mathrm{d}^2 y, \tag{31}$$

and the same expression but with the logarithmic field:

$$Q_{\mathbf{q},\mathsf{log}} = \left(\frac{\mu\gamma}{2}\right)^p \int_{\mathbb{C}^p} \prod_{j=1}^{p} \frac{1}{(y_j - z)_\epsilon^{q_j}} \langle z, \mathbf{z}, \mathbf{y} \rangle_{\mathsf{log},\epsilon} \mathrm{d}^2 y. \tag{32}$$

We start by the following simple lemma which will be used to cancel the background metric dependent terms in the upcoming computations of the derivatives.

Lemma 5.1. *One has the identity:*

$$\mu\gamma \int_{\mathbb{C}} \langle z; \mathbf{z}, y \rangle_{\mathsf{log},\epsilon} d^2 y = \left(\sum_{l=0}^{N} \alpha_l - 2Q \right) \langle z; \mathbf{z} \rangle_{\mathsf{log},\epsilon} + \langle z; \mathbf{z} \rangle_\epsilon.$$

Proof. Starting from the expression

$$\langle z; \mathbf{z} \rangle_{\mathsf{log},\epsilon} = \int_{\mathbb{R}} e^{-2Qc} \mathbb{E} \left[\phi_\epsilon(z) V_{-\chi,\epsilon}(z) \prod_{k=1}^{N} V_{\alpha_k,\epsilon}(z_k) e^{-\mu \int_{\mathbb{C}} V_{\gamma,\epsilon}(x) d^2 x} \right] dc,$$

we perform the change of variable c to $c + \frac{1}{\gamma} \log \frac{1}{\mu}$ to obtain the expression:

$$\mu^{\frac{1}{\gamma}(2Q+\chi-\sum_{k=1}^{N}\alpha_k)} \lim_{\epsilon\to 0} \int_{\mathbb{R}} e^{-2Qc} \mathbb{E} \left[\phi_\epsilon(z) V_{-\chi,\epsilon}(z) \prod_{k=1}^{N} V_{\alpha_k,\epsilon}(z_k) e^{-\int_{\mathbb{C}} V_{\gamma,\epsilon}(x) d^2 x} \right] dc$$
$$- \frac{1}{\gamma} (\log \mu) \mu^{\frac{1}{\gamma}(2Q+\chi-\sum_{k=1}^{N}\alpha_k)} \lim_{\epsilon\to 0} \int_{\mathbb{R}} e^{-2Qc} \mathbb{E} \left[V_{-\chi,\epsilon}(z) \prod_{k=1}^{N} V_{\alpha_k,\epsilon}(z_k) e^{-\int_{\mathbb{C}} V_{\gamma,\epsilon}(x) d^2 x} \right] dc.$$

We now obtain take a derivative:

$$- \int_{\mathbb{C}} \langle z; \mathbf{z}, y \rangle_{\mathsf{log},\epsilon} d^2 y$$
$$= \frac{1}{\gamma} (2Q + \chi - \sum_{k=1}^{N} \alpha_k) \mu^{\frac{1}{\gamma}(2Q+\chi-\sum_{k=1}^{N}\alpha_k)-1} \int_{\mathbb{R}} e^{-2Qc} \mathbb{E} \left[\phi_\epsilon(z) V_{-\chi,\epsilon}(z) \prod_{k=1}^{N} V_{\alpha_k,\epsilon}(z_k) e^{-\int_{\mathbb{C}} V_{\gamma,\epsilon}(x) d^2 x} \right] dc$$
$$- \frac{1}{\gamma} (2Q + \chi - \sum_{k=1}^{N} \alpha_k) (\log \mu) \mu^{\frac{1}{\gamma}(2Q+\chi-\sum_{k=1}^{N}\alpha_k)-1} \int_{\mathbb{R}} e^{-2Qc} \mathbb{E} \left[V_{-\chi,\epsilon}(z) \prod_{k=1}^{N} V_{\alpha_k,\epsilon}(z_k) e^{-\int_{\mathbb{C}} V_{\gamma,\epsilon}(x) d^2 x} \right] dc$$
$$- \frac{1}{\gamma} \mu^{\frac{1}{\gamma}(2Q+\chi-\sum_{k=1}^{N}\alpha_k)-1} \int_{\mathbb{R}} e^{-2Qc} \mathbb{E} \left[V_{-\chi,\epsilon}(z) \prod_{k=1}^{N} V_{\alpha_k,\epsilon}(z_k) e^{-\int_{\mathbb{C}} V_{\gamma,\epsilon}(x) d^2 x} \right] dc$$
$$= \frac{1}{\gamma\mu} \left(2Q - \sum_{l=0}^{N} \alpha_l \right) \langle z; \mathbf{z} \rangle_{\mathsf{log},\epsilon} - \frac{1}{\gamma\mu} \langle z; \mathbf{z} \rangle_\epsilon.$$

This implies the claim of the lemma. □

Next we state two lemmas that give the derivatives of the correlation functions with or without the logarithmic insertion. The first of these lemmas has been proved in [19].

Lemma 5.2. *(Derivative rule for ordinary correlations) The following formula holds*

$$\partial_{z_i}\langle V_{-\chi,\epsilon}(z)\prod_{l=1}^N V_{\alpha_l,\epsilon}(z_l)\rangle$$
$$= -\frac{\alpha_i\chi}{2(z-z_i)_\epsilon}\langle z,\mathbf{z}\rangle_\epsilon + \sum_{j\neq i}\frac{\alpha_i\alpha_j}{2(z_j-z_i)_\epsilon}\langle z,\mathbf{z}\rangle_\epsilon - \frac{\mu\gamma\alpha_i}{2}\int_{\mathbb{C}}\frac{1}{(y-z_i)_\epsilon}\langle z,\mathbf{z},y\rangle_\epsilon \mathrm{d}^2y,$$

and for the z derivative:

$$\partial_z\langle V_{-\chi,\epsilon}(z)\prod_{l=1}^N V_{\alpha_l,\epsilon}(z_l)\rangle = -\sum_{j=1}^N\frac{\chi\alpha_j}{2(z_j-z)_\epsilon}\langle z,\mathbf{z}\rangle_\epsilon + \frac{\mu\gamma\chi}{2}\int_{\mathbb{C}}\frac{1}{(y-z)_\epsilon}\langle z,\mathbf{z},y\rangle_\epsilon \mathrm{d}^2y.$$

Now the analogue result for a correlation with the logarithmic insertion.

Lemma 5.3. *(Derivative rule for logarithmic correlations) The following formula holds*

$$\partial_{z_i}\langle \phi_\epsilon(z)V_{-\chi,\epsilon}(z)\prod_{l=1}^N V_{\alpha_l,\epsilon}(z_l)\rangle$$
$$= \frac{\alpha_i}{2(z-z_i)_\epsilon}\langle z,\mathbf{z}\rangle_\epsilon - \frac{\alpha_i\chi}{2(z-z_i)_\epsilon}\langle z,\mathbf{z}\rangle_{\mathsf{log},\epsilon} + \sum_{j\neq i}\frac{\alpha_i\alpha_j}{2(z_j-z_i)_\epsilon}\langle z,\mathbf{z}\rangle_{\mathsf{log},\epsilon} - \frac{\mu\gamma\alpha_i}{2}\int_{\mathbb{C}}\frac{1}{(y-z_i)_\epsilon}\langle z,\mathbf{z},y\rangle_{\mathsf{log},\epsilon}\mathrm{d}^2y,$$

and for the z derivative:

$$\partial_z\langle \phi_\epsilon(z)V_{-\chi,\epsilon}(z)\prod_{l=1}^N V_{\alpha_l,\epsilon}(z_l)\rangle = -\sum_{j=1}^N\frac{\chi\alpha_j}{2(z_j-z)_\epsilon}\langle z,\mathbf{z}\rangle_{\mathsf{log},\epsilon} + \frac{\mu\gamma\chi}{2}\int_{\mathbb{C}}\frac{1}{(y-z)_\epsilon}\langle z,\mathbf{z},y\rangle_{\mathsf{log},\epsilon}\mathrm{d}^2y$$
$$+\sum_{j=1}^N\frac{\alpha_j}{2(z_j-z)_\epsilon}\langle z,\mathbf{z}\rangle_\epsilon - \frac{\mu\gamma}{2}\int_{\mathbb{C}}\frac{1}{(y-z)_\epsilon}\langle z,\mathbf{z},y\rangle_\epsilon\mathrm{d}^2y.$$

Proof. Let us start with the first identity. When differentiating we get the quantity:

$$\alpha_i\langle\phi_\epsilon(z)\partial_{z_i}(X_\epsilon(z_i)+\frac{Q}{2}\log g(z_i))V_{-\chi,\epsilon}(z)\prod_{l=1}^N V_{\alpha_l,\epsilon}(z_l)\rangle.$$

The first step is to apply the Gaussian integration by parts formula of Lemma A.1 to the term containing the $\partial_{z_i}X_\epsilon(z_i)$. One obtains:

$$\langle\phi_\epsilon(z)\partial_{z_i}X_\epsilon(z_i)V_{-\chi,\epsilon}(z)\prod_{l=1}^N V_{\alpha_l,\epsilon}(z_l)\rangle$$
$$= \partial_{z_i}\mathbb{E}[\phi_\epsilon(z)X_\epsilon(z_i)]\langle z,\mathbf{z}\rangle_\epsilon - \chi\partial_{z_i}\mathbb{E}[X_\epsilon(z_i)\phi_\epsilon(z)]\langle z,\mathbf{z}\rangle_{\mathsf{log},\epsilon}$$
$$+\sum_{j\neq i}\alpha_j\partial_{z_i}\mathbb{E}[X_\epsilon(z_i)\phi_\epsilon(z_j)]\langle z,\mathbf{z}\rangle_{\mathsf{log},\epsilon} - \mu\gamma\int_{\mathbb{C}}\partial_{z_i}\mathbb{E}[X_\epsilon(z_i)\phi_\epsilon(y)]\langle z,\mathbf{z},y\rangle_{\mathsf{log},\epsilon}d^2y.$$

Now the above expression reduces to the expression claimed in the lemma up to the following terms:

$$-\frac{1}{4}\partial_{z_i}g(z_i)\left(\langle z,\mathbf{z}\rangle_\epsilon + (\sum_j\alpha_j-\chi-2Q)\langle z,\mathbf{z}\rangle_{\mathsf{log},\epsilon} - \mu\gamma\int_{\mathbb{C}}\langle z,\mathbf{z},y\rangle_{\mathsf{log},\epsilon}d^2y\right).$$

These terms then cancels thanks to the results of Lemma 5.1.

Let us now move to the expression for the ∂_z derivative. The computation is analogous except that this time there is a terms that will not be canceled by the identity of Lemma 5.1. By computing the derivative we get:

$$\partial_z \langle \phi_\epsilon(z) V_{-\chi,\epsilon}(z) \prod_{l=1}^N V_{\alpha_l,\epsilon}(z_l) \rangle_\delta = \frac{Q}{2} \partial_z \log g(z) \langle z, \mathbf{z} \rangle_\epsilon$$
$$+ \langle \partial_z X_\epsilon(z) V_{-\chi,\epsilon}(z) \prod_{l=1}^N V_{\alpha_l,\epsilon}(z_l) \rangle - \chi \langle \phi_\epsilon(z) \partial_z X_\epsilon(z) V_{-\chi,\epsilon}(z) \prod_{l=1}^N V_{\alpha_l,\epsilon}(z_l) \rangle.$$

Integrating by parts the first term of the last line:

$$\langle \partial_z X_\epsilon(z) V_{-\chi,\epsilon}(z) \prod_{l=1}^N V_{\alpha_l,\epsilon}(z_l) \rangle = -\chi \partial_z \mathbb{E}[X_\epsilon(z)\phi_\epsilon(z)] \langle z, \mathbf{z} \rangle_\epsilon + \sum_j \alpha_j \partial_z \mathbb{E}[X_\epsilon(z)\phi_\epsilon(z_j)] \langle z, \mathbf{z} \rangle_\epsilon$$
$$- \mu\gamma \int_{\mathbb{C}} \partial_z \mathbb{E}[X_\epsilon(z)\phi_\epsilon(y)] \langle z, \mathbf{z}, y \rangle_\epsilon d^2y.$$

By integrating by parts the last term of the last line:

$$-\chi \langle \phi_\epsilon(z) \partial_z X_\epsilon(z) V_{-\chi,\epsilon}(z) \prod_{l=1}^N V_{\alpha_l,\epsilon}(z_l) \rangle = -\chi \partial_z \mathbb{E}[\phi_\epsilon(z) X_\epsilon(z)] \langle z, \mathbf{z} \rangle$$
$$+ \chi^2 \partial_z \mathbb{E}[X_\epsilon(z)\phi_\epsilon(z)] \langle z, \mathbf{z} \rangle_{\mathsf{log}} - \chi \sum_j \alpha_j \partial_z \mathbb{E}[X_\epsilon(z)\phi_\epsilon(z_j)] \langle z, \mathbf{z} \rangle_{\mathsf{log}}$$
$$+ \chi\mu\gamma \int_{\mathbb{C}} \partial_z \mathbb{E}[X_\epsilon(z)\phi_\epsilon(y)] \langle z, \mathbf{z}, y \rangle_{\mathsf{log}} d^2y.$$

Here we assume $z \in \mathbb{D}$ where the metric is flat, namely $g = 1$ in a small ball surrounding z. Therefore the extra metric dependent term vanishes. □

Recall the operators from equation (8):

$$L_{-1} = \partial_z, \; L_{-2} = \sum_{l=1}^N \left(\frac{\Delta_{\alpha_l}}{(z_l - z)^2} - \frac{1}{(z_l - z)} \partial_{z_l} \right). \tag{33}$$

We now give the two lemmas that explicitly compute the action of the operators L_{-1}^2 and L_{-2}.

Lemma 5.4. *The following relation holds:*

$$L_{-1}^2 Q_{0,\mathsf{log}} = (\chi^2 P_1^2 - \chi P_2) Q_{0,\mathsf{log}} + \chi(1 - \frac{\chi\gamma}{2}) Q_{2,\mathsf{log}} - 2\chi^2 P_1 Q_{1,\mathsf{log}} + \chi^2 Q_{1,1,\mathsf{log}}$$
$$+ P_2 Q_0 - Q_2 - 2\chi P_1^2 Q_0 + 4\chi P_1 Q_1 + \chi\gamma Q_2 - 2\chi Q_{1,1}.$$

Proof. Starting from the expression for the z derivative given in Lemma 5.3

$$\partial_z Q_{0,\mathsf{log}} = -\sum_{j=1}^N \frac{\chi\alpha_j}{2(z_j - z)_\epsilon} \langle z, \mathbf{z} \rangle_{\mathsf{log},\epsilon} + \frac{\mu\gamma\chi}{2} \int_{\mathbb{C}} \frac{1}{(y - z)_\epsilon} \langle z, \mathbf{z}, y \rangle_{\mathsf{log},\epsilon} \mathrm{d}^2 y$$
$$+ \sum_{j=1}^N \frac{\alpha_j}{2(z_j - z)_\epsilon} \langle z, \mathbf{z} \rangle_\epsilon - \frac{\mu\gamma}{2} \int_{\mathbb{C}} \frac{1}{(y - z)_\epsilon} \langle z, \mathbf{z}, y \rangle_\epsilon \mathrm{d}^2 y,$$

we take another derivative in z to get:

$$\begin{aligned}\partial_{zz}Q_{0,\mathsf{log}} = &-\sum_{j=1}^{N}\frac{\chi\alpha_j}{2(z_j-z)^2}\langle z,\mathbf{z}\rangle_{\mathsf{log},\epsilon}+\frac{\mu\gamma\chi}{2}\int_{\mathbb{C}}\frac{1}{(y-z)^2}\langle z,\mathbf{z},y\rangle_{\mathsf{log},\epsilon}\mathrm{d}^2y\\ &+\sum_{j=1}^{N}\frac{\alpha_j}{2(z_j-z)^2}\langle z,\mathbf{z}\rangle_{\epsilon}-\frac{\mu\gamma}{2}\int_{\mathbb{C}}\frac{1}{(y-z)^2}\langle z,\mathbf{z},y\rangle_{\epsilon}\mathrm{d}^2y\\ &-\sum_{j=1}^{N}\frac{\chi\alpha_j}{2(z_j-z)_\epsilon}\partial_z\langle z,\mathbf{z}\rangle_{\mathsf{log},\epsilon}+\frac{\mu\gamma\chi}{2}\int_{\mathbb{C}}\frac{1}{(y-z)_\epsilon}\partial_z\langle z,\mathbf{z},y\rangle_{\mathsf{log},\epsilon}\mathrm{d}^2y\\ &+\sum_{j=1}^{N}\frac{\alpha_j}{2(z_j-z)_\epsilon}\partial_z\langle z,\mathbf{z}\rangle_{\epsilon}-\frac{\mu\gamma}{2}\int_{\mathbb{C}}\frac{1}{(y-z)_\epsilon}\partial_z\langle z,\mathbf{z},y\rangle_{\epsilon}\mathrm{d}^2y.\end{aligned}$$

This can be rewritten as:

$$\begin{aligned}&-\chi P_2Q_{0,\mathsf{log}}+\chi Q_{2,\mathsf{log}}+P_2Q_0-Q_2-\chi P_1L_{-1}Q_{0,\mathsf{log}}+P_1L_{-1}Q_0\\ &+\frac{\mu\gamma\chi}{2}\int_{\mathbb{C}}\frac{1}{(y-z)_\epsilon}\partial_z\langle z,\mathbf{z},y\rangle_{\mathsf{log},\epsilon}\mathrm{d}^2y-\frac{\mu\gamma}{2}\int_{\mathbb{C}}\frac{1}{(y-z)_\epsilon}\partial_z\langle z,\mathbf{z},y\rangle_{\epsilon}d^2y.\end{aligned}$$

By treating the extra point y as a spectator insertion of weight γ and applying again Lemma 5.3 one obtains the identities:

$$\begin{aligned}\partial_z\langle z,\mathbf{z},y\rangle_{\mathsf{log},\epsilon}=&-\sum_{j=1}^{N}\frac{\chi\alpha_j}{2(z_j-z)_\epsilon}\langle z,\mathbf{z},y\rangle_{\mathsf{log},\epsilon}-\frac{\chi\gamma}{2(y-z)_\epsilon}\langle z,\mathbf{z},y\rangle_{\mathsf{log},\epsilon}+\frac{\mu\gamma\chi}{2}\int_{\mathbb{C}}\frac{1}{(x-z)_\epsilon}\langle z,\mathbf{z},y,x\rangle_{\mathsf{log},\epsilon}\mathrm{d}^2x\\ &+\sum_{j=1}^{N}\frac{\alpha_j}{2(z_j-z)_\epsilon}\langle z,\mathbf{z},y\rangle_\epsilon+\frac{\gamma}{2(y-z)_\epsilon}\langle z,\mathbf{z},y\rangle_\epsilon-\frac{\mu\gamma}{2}\int_{\mathbb{C}}\frac{1}{(x-z)_\epsilon}\langle z,\mathbf{z},y,x\rangle_\epsilon\mathrm{d}^2x,\\ \partial_z\langle z,\mathbf{z},y\rangle_\epsilon=&-\sum_{j=1}^{N}\frac{\chi\alpha_j}{2(z_j-z)_\epsilon}\langle z,\mathbf{z},y\rangle_\epsilon-\frac{\chi\gamma}{2(y-z)_\epsilon}\langle z,\mathbf{z},y\rangle_\epsilon+\frac{\mu\gamma\chi}{2}\int_{\mathbb{C}}\frac{1}{(x-z)_\epsilon}\langle z,\mathbf{z},y,x\rangle_\epsilon\mathrm{d}^2x.\end{aligned}$$

We thus compute:

$$-\frac{\mu\gamma}{2}\int_{\mathbb{C}}\frac{1}{(y-z)_\epsilon}\partial_z\langle z,\mathbf{z},y\rangle_\epsilon\mathrm{d}^2y=\chi P_1Q_1+\frac{\chi\gamma}{2}Q_2-\chi Q_{1,1},$$

$$\frac{\mu\gamma\chi}{2}\int_{\mathbb{C}}\frac{1}{(y-z)_\epsilon}\partial_z\langle z,\mathbf{z},y\rangle_{\mathsf{log},\epsilon}\mathrm{d}^2y=-\chi^2P_1Q_{1,\mathsf{log}}-\frac{\chi^2\gamma}{2}Q_{2,\mathsf{log}}+\chi^2Q_{1,1,\mathsf{log}}+\chi P_1Q_1+\frac{\chi\gamma}{2}Q_2-\chi Q_{1,1}.$$

Note that one can rewrite the claim of Lemmas 5.2 and 5.3 as:

$$L_{-1}Q_0=-\chi P_1Q_0+\chi Q_1,\quad L_{-1}Q_{0,\mathsf{log}}=-\chi P_1Q_{0,\mathsf{log}}+\chi Q_{1,\mathsf{log}}+P_1Q_0-Q_1.$$

Putting everything together one obtains:

$$\begin{aligned}L_{-1}^2Q_{0,\mathsf{log}}&=-\chi P_2Q_{0,\mathsf{log}}+\chi Q_{2,\mathsf{log}}+P_2Q_0-Q_2-\chi P_1L_{-1}Q_{0,\mathsf{log}}+P_1L_{-1}Q_0\\ &\quad-\chi^2P_1Q_{1,\mathsf{log}}-\frac{\chi^2\gamma}{2}Q_{2,\mathsf{log}}+\chi^2Q_{1,1,\mathsf{log}}+2\chi P_1Q_1+\chi\gamma Q_2-2\chi Q_{1,1}\\ &=(\chi^2P_1^2-\chi P_2)Q_{0,\mathsf{log}}+\chi(1-\frac{\chi\gamma}{2})Q_{2,\mathsf{log}}-2\chi^2P_1Q_{1,\mathsf{log}}+\chi^2Q_{1,1,\mathsf{log}}\\ &\quad+P_2Q_0-Q_2-2\chi P_1^2Q_0+4\chi P_1Q_1+\chi\gamma Q_2-2\chi Q_{1,1}.\end{aligned}$$

□

Lemma 5.5. *The following relation holds:*

$$L_{-2}Q_{0,\mathsf{log}} = (-P_1^2 + \frac{1}{\chi}P_2)Q_{0,\mathsf{log}} + 2P_1Q_{1,\mathsf{log}} + (\chi - \frac{2}{\gamma})Q_{2,\mathsf{log}} - Q_{1,1,\mathsf{log}} + P_2Q_0 - Q_2 + o(\epsilon).$$

Proof. By using the result of Lemma 5.3 we obtain:

$$L_{-2}\langle z,\mathbf{z}\rangle_\epsilon = \left(-\sum_j\sum_{l\neq j}\frac{\alpha_i\alpha_j}{2(z_j-z)(z_l-z_j)_\epsilon} - \sum_j\frac{\chi\alpha_j}{2(z_j-z)(z_j-z)_\epsilon} + \sum_j\frac{\Delta_{\alpha_j}}{(z_j-z)^2}\right)\langle z,\mathbf{z}\rangle_{\mathsf{log},\epsilon} + \sum_j\frac{\mu\gamma\alpha_j}{2(z_j-z)}\int_{\mathbb{C}}\frac{1}{(y-z_j)_\epsilon}\langle z,\mathbf{z},y\rangle_{\mathsf{log},\epsilon} + \sum_j\frac{\alpha_j}{2(z_j-z)^2}\langle z,\mathbf{z}\rangle_\epsilon.$$

To group the terms in $\langle z,\mathbf{z}\rangle_{\mathsf{log},\epsilon}$ in the desired form we will use the identity:

$$\frac{1}{(x_1-x_2)(x_2-z)} - \frac{1}{(x_1-x_2)(x_1-z)} = \frac{1}{(x_1-z)(x_2-z)}.$$

We will now perform an integration by parts on the term with the integration over $\mathbb{C}$. Starting from the term

$$\sum_{i=1}^N\frac{\mu\gamma\alpha_i}{2(z_i-z)}\int_{\mathbb{C}}\frac{1}{(y-z_i)_\epsilon}\langle z,\mathbf{z},y\rangle_{\mathsf{log}}\mathrm{d}^2y,$$

we compute the difference:

$$\sum_{i=1}^N\frac{\mu\gamma\alpha_i}{2(z_i-z)}\int_{\mathbb{C}}\frac{1}{(y-z_i)_\epsilon}\langle z,\mathbf{z},y\rangle_{\mathsf{log}}\mathrm{d}^2y - \sum_{i=1}^N\frac{\mu\gamma\alpha_i}{2}\int_{\mathbb{C}}\frac{1}{(y-z_i)_\epsilon}\frac{1}{(y-z)_\epsilon}\langle z,\mathbf{z},y\rangle_{\mathsf{log}}\mathrm{d}^2y = \sum_j\frac{\mu\gamma\alpha_j}{2(z_j-z)}\int_{\mathbb{C}}\frac{1}{(y-z)}\langle z,\mathbf{z},y\rangle_{\mathsf{log},\epsilon}d^2y + o(\epsilon).$$

We then perform the following integration by parts:

$$\int_{\mathbb{C}}\sum_j\frac{\alpha_j\mu\gamma}{2(y-z_j)_\epsilon(y-z)}\langle z,\mathbf{z},y\rangle_{\epsilon,\mathsf{log}}d^2y = -\frac{2}{\gamma}\int_{\mathbb{C}}\frac{\mu\gamma}{2(y-z)^2}\langle z,\mathbf{z},y\rangle_{\epsilon,\mathsf{log}}d^2y + \chi\int_{\mathbb{C}}\frac{\mu\gamma}{2(y-z)(y-z)_\epsilon}\langle z,\mathbf{z},y\rangle_{\epsilon,\mathsf{log}}d^2y - \int_{\mathbb{C}^2}\frac{\mu^2\gamma^2}{2(x-y)_\epsilon(y-z)}\langle z,\mathbf{z},x,y\rangle_{\epsilon,\mathsf{log}}d^2xd^2y - \frac{\mu\gamma}{2}\int_{\mathbb{C}}\frac{1}{(y_j-z)_\epsilon^2}\langle z,\mathbf{z},\mathbf{y}\rangle_\epsilon\mathrm{d}^2y.$$

By using symmetry on the double integral above and collecting all the terms we obtain the claimed result. □

6 The $(2,1)$ higher equation of motion.

We now move to study the equations of order 2. The operator we wish to apply is thus $(\overline{L}_{-1}^2 + \chi^2\overline{L}_{-2})(L_{-1}^2 + \chi^2L_{-2})$. We will furthermore assume that $\chi = \frac{\gamma}{2}$. We will apply this to:

$$\langle z;\mathbf{z}\rangle_{\mathsf{log}} := \langle\phi(z)V_{-\chi,\epsilon}(z)\prod_{l=1}^N V_{\alpha_l,\epsilon}(z_l)\rangle = 4e^{-2\chi Q^2}\lim_{\epsilon\to0}\int_{\mathbb{R}}e^{-2Qc}\mathbb{E}\left[(\phi_\epsilon(z)+c)\,V_{-\chi,\epsilon}(z)\prod_{k=1}^N V_{\alpha_k,\epsilon}(z_k)e^{-\mu\int_{\mathbb{C}}V_{\gamma,\epsilon}(x)d^2x}\right]dc.$$

Lets first apply the operator $L_{-1}^2 + \chi^2 L_{-2}$. By the second order BPZ equation we know:

$$\left(L_{-1}^2 + \chi^2 L_{-2}\right) \lim_{\epsilon\to 0} \int_{\mathbb{R}} e^{-2Qc} \mathbb{E}\left[V_{-\chi,\epsilon}(z) \prod_{k=1}^{N} V_{\alpha_k,\epsilon}(z_k) e^{-\mu \int_{\mathbb{C}} V_{\gamma,\epsilon}(x) d^2x} \right] dc \to 0.$$

As performed in [19] this cancellation is obtained by the fact that

$$\begin{aligned} L_{-1}^2 Q_0 &= (-\chi P_2 + \chi^2 P_1^2) Q_0 - 2\chi^2 P_1 Q_1 + \chi^2 Q_{1,1} + \chi(-\frac{\gamma\chi}{2} + 1) Q_2, \\ L_{-2} Q_0 &= (-P_1^2 + \frac{1}{\chi} P_2) Q_0 + 2P_1 Q_1 - Q_{1,1} + (\chi - \frac{2}{\gamma}) Q_2 + o(\epsilon), \end{aligned}$$

and where thus the error term $o(\epsilon)$ vanishes as $\epsilon \to 0$. Moving now to the case of the logarithmic insertions, combing the results of Lemmas 5.4 and 5.5:

$$(L_{-1}^2 + \chi^2 L_{-2}) Q_{0,\mathsf{log}} = \chi^2 (P_2 Q_0 - Q_2) + P_2 Q_0 - Q_2 - 2\chi P_1^2 Q_0 + 4\chi P_1 Q_1 + \chi\gamma Q_2 - 2\chi Q_{1,1}.$$

Lets now apply the anti-holomorphic operator to this answer. We notice that applied to Q_0 it gives 0. We get:

$$(\overline{L}_{-1}^2 + \chi^2 \overline{L}_{-2})(L_{-1}^2 + \chi^2 L_{-2}) Q_{0,\mathsf{log}} = (\overline{L}_{-1}^2 + \chi^2 \overline{L}_{-2}) \left((\chi\gamma - 1 - \chi^2) Q_2 + 4\chi P_1 Q_1 - 2\chi Q_{1,1} \right).$$

We use the notation $D_2 = L_{-1}^2 + \chi^2 L_{-2}$ and $\overline{D}_2 = \overline{L}_{-1}^2 + \chi^2 \overline{L}_{-2}$. By commuting the holomorphic and anti-holomorphic operators one obtains that:

$$\overline{D}_2 Q_0 = 0, \quad \overline{D}_2 (P_1 L_{-1} Q_0) = 0, \quad \overline{D}_2 (L_{-2} Q_0) = 0.$$

Combining these three relation implies that:

$$\overline{D}_2 \left(Q_{1,1} + (\frac{1}{\chi} - \frac{\gamma}{2}) Q_2 \right) = 0.$$

From here one can deduce that the expression

$$(\overline{L}_{-1}^2 + \chi^2 \overline{L}_{-2}) \left((\chi\gamma - 1 - \chi^2) Q_2 + 4\chi P_1 Q_1 - 2\chi Q_{1,1} \right)$$

is equal to

$$-\frac{\gamma}{2} \overline{D}_2 Q_{1,1} \tag{34}$$

using the fact that we are assuming that $\chi = \frac{\gamma}{2}$. We thus now need to analyze carefully these two terms using the regularization procedure. Before doing so, let us first record a consequence of the fact that $\overline{D}_2 Q_1 = 0$ (which is itself a consequence of $\overline{D}_2(P_1 L_{-1} Q_0) = 0$). Written out explicitly $\overline{D}_2 Q_1 = 0$ implies:

$$\begin{aligned} \overline{D}_2 Q_1 = &\int_{\mathbb{C}} d^2x \left(\partial_{\overline{z}\overline{z}} \frac{1}{(x-z)_\epsilon} \right) \langle z, \mathbf{z}, x \rangle_\epsilon + \\ &+ 2 \int_{\mathbb{C}} d^2x \left(\partial_{\overline{z}} \frac{1}{(x-z)_\epsilon} \right) \partial_{\overline{z}} \langle z, \mathbf{z}, x \rangle_\epsilon + \int_{\mathbb{C}} d^2x \frac{1}{(x-z)_\epsilon} \overline{D}_2 \langle z, \mathbf{z}, x \rangle_\epsilon = 0. \end{aligned} \tag{35}$$

To compute the last term we can use the following trick. We know that $\overline{D}_2 \langle z, \mathbf{z} \rangle_\epsilon = o(\epsilon)$, which implies by completing the operator with x as a spectator

$$\overline{D}_2 \langle z, \mathbf{z}, x \rangle_\epsilon = -\chi^2 \left(\frac{\Delta_\gamma}{(\overline{x} - \overline{z})_\epsilon^2} - \frac{1}{(\overline{x} - \overline{z})_\epsilon} \partial_{\overline{x}} \right) \langle z, \mathbf{z}, x \rangle_\epsilon + o(\epsilon),$$

where here we have used the notation:

$$\frac{1}{(\overline{x} - \overline{z})_\epsilon^2} := \partial_{\overline{z}} \frac{1}{(\overline{x} - \overline{z})_\epsilon}.$$

Record $\Delta_\gamma = 1$. Therefore:

$$\int_{\mathbb{C}} d^2x \frac{1}{(x-z)_\epsilon} \overline{D}_2 \langle z, \mathbf{z}, x \rangle_\epsilon = -\chi^2 \int_{\mathbb{C}} d^2x \frac{1}{(x-z)_\epsilon} \left(\frac{1}{(\overline{x}-\overline{z})^2_\epsilon} - \frac{1}{(\overline{x}-\overline{z})_\epsilon} \partial_{\overline{x}} \right) \langle z, \mathbf{z}, x \rangle_\epsilon + o(\epsilon)$$
$$= -\chi^2 \int_{\mathbb{C}} d^2x \left(\partial_{\overline{x}} \frac{1}{(x-z)_\epsilon} \right) \frac{1}{(\overline{x}-\overline{z})_\epsilon} \langle z, \mathbf{z}, x \rangle_\epsilon + o(\epsilon).$$

We therefore have the identity:

$$\int_{\mathbb{C}} d^2x \left(\partial_{\overline{z}\overline{z}} \frac{1}{(x-z)_\epsilon} \right) \langle z, \mathbf{z}, x \rangle + 2 \int_{\mathbb{C}} d^2x \left(\partial_{\overline{z}} \frac{1}{(x-z)_\epsilon} \right) \partial_{\overline{z}} \langle z, \mathbf{z}, x \rangle$$
$$- \chi^2 \int_{\mathbb{C}} d^2x \left(\partial_{\overline{x}} \frac{1}{(x-z)_\epsilon} \right) \frac{1}{(\overline{x}-\overline{z})_\epsilon} \langle z, \mathbf{z}, x \rangle = o(\epsilon). \quad (36)$$

We will use it with y of weight γ added as an extra spectator point. Lets now move to computing $\overline{D}_2 Q_{1,1}$. We start by writing:

$$\overline{D}_2 Q_{1,1} = 2 \int_{\mathbb{C}^2} d^2x d^2y \left(\partial_{\overline{z}\overline{z}} \frac{1}{(x-z)_\epsilon} \right) \frac{1}{(y-z)_\epsilon} \langle z, \mathbf{z}, x, y \rangle_\epsilon$$
$$+ 2 \int_{\mathbb{C}^2} d^2x d^2y \left(\partial_{\overline{z}} \frac{1}{(x-z)_\epsilon} \right) \left(\partial_{\overline{z}} \frac{1}{(y-z)_\epsilon} \right) \langle z, \mathbf{z}, x, y \rangle_\epsilon$$
$$+ 4 \int_{\mathbb{C}^2} d^2x d^2y \left(\partial_{\overline{z}} \frac{1}{(x-z)_\epsilon} \right) \frac{1}{(y-z)_\epsilon} \partial_{\overline{z}} \langle z, \mathbf{z}, x, y \rangle_\epsilon$$
$$+ \int_{\mathbb{C}^2} d^2x d^2y \frac{1}{(x-z)_\epsilon} \frac{1}{(y-z)_\epsilon} \overline{D}_2 \langle z, \mathbf{z}, x, y \rangle_\epsilon + o(\epsilon).$$

Again we write that:

$$\int_{\mathbb{C}^2} d^2x d^2y \frac{1}{(x-z)_\epsilon} \frac{1}{(y-z)_\epsilon} \overline{D}_2 \langle z, \mathbf{z}, x, y \rangle_\epsilon$$
$$= -\chi^2 \int_{\mathbb{C}^2} d^2x d^2y \frac{1}{(x-z)_\epsilon} \frac{1}{(y-z)_\epsilon} \left(\frac{1}{(\overline{x}-\overline{z})^2_\epsilon} - \frac{1}{(\overline{x}-\overline{z})_\epsilon} \partial_{\overline{x}} + \frac{1}{(\overline{y}-\overline{z})^2_\epsilon} - \frac{1}{(\overline{y}-\overline{z})_\epsilon} \partial_{\overline{y}} \right) \langle z, \mathbf{z}, x, y \rangle_\epsilon + o(\epsilon)$$
$$= -2\chi^2 \int_{\mathbb{C}^2} d^2x d^2y \left(\partial_{\overline{x}} \frac{1}{(x-z)_\epsilon} \right) \frac{1}{(y-z)_\epsilon} \frac{1}{(\overline{x}-\overline{z})_\epsilon} \langle z, \mathbf{z}, x, y \rangle_\epsilon + o(\epsilon).$$

Using $2 \int_{\mathbb{C}} d^2y \frac{1}{(y-z)_\epsilon}$ times equation (36) with y added as a spectator point we can cancel three of the four terms of in $\overline{D}_2 Q_{1,1}$ and end up with:

$$\overline{D}_2 Q_{1,1} = 2 \left(\frac{\mu\gamma}{2} \right)^2 \int_{\mathbb{C}^2} d^2x d^2y \left(\partial_{\overline{z}} \frac{1}{(x-z)_\epsilon} \right) \left(\partial_{\overline{z}} \frac{1}{(y-z)_\epsilon} \right) \langle z, \mathbf{z}, x, y \rangle_\epsilon + o(\epsilon). \quad (37)$$

Now we send ϵ to 0 and use Lemma 3.1 to evaluate the integrals in x and y with the result:

$$\overline{D}_2 Q_{1,1} = \frac{(\mu\gamma\pi)^2}{2} \langle z, \mathbf{z}, x, y \rangle_{x=y=z}. \quad (38)$$

Using the expression (34) we find

$$\overline{D_{2,1}} D_{2,1} \langle z; \mathbf{z} \rangle = -\frac{\mu^2 \pi^2 \gamma^3}{4} \langle z, \mathbf{z}, x, y \rangle_{x=y=z}. \quad (39)$$

This finishes the proof of equation (15).

A Some useful facts in probability

We state the Cameron-Martin formula also known as the Girsanov theorem.

Lemma A.1. *Let $Y(x)$ be a Gaussian process on the sphere, and Z a Gaussian random variable such that $(X(x), Z)$ is jointly Gaussian. Then for any suitable functional F one has:*

$$\mathbb{E}\left[e^{Z-\frac{1}{2}\mathbb{E}[Z^2]}F(X)\right] = \mathbb{E}\left[F(X+\mathbb{E}[X(\cdot)Z])\right].$$

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