THE TRIPLE PRODUCT L-FUNCTION

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In number theory it is often the case that understanding the behavior of certain so-called *L-functions* at specific values leads to arithmetic results. For example, a celebrated result of Dirichlet states that for m and n relatively prime integers, there are infinitely many prime numbers among the sequence

$$m, m+n, m+2n, m+3n, m+4n, \ldots.$$

This fact can be deduced from knowing that for every non-trivial Dirichlet character χ modulo n, the L-function $L(\chi, s)$ is holomorphic at $s = 1$, whereas the Riemann zeta function $\zeta(s)$ has a pole at $s = 1$.

In modern parlance, the character χ can be interpreted as an *automorphic representation* of GL_1. An automorphic representation π of GL_2 corresponds to another important object for number theorists: namely, a modular form. In slightly more generality, given a quaternion algebra B and three autormorphic representations π_1, π_2 and π_3 of B, one can form the *triple product* $\Pi := \pi_1 \otimes \pi_2 \otimes \pi_3$. An important formula of Ichino roughly states that for $\varphi \in \Pi$,

$$\int_{[B]} \varphi(b)\, db = C \cdot L(\Pi, \tfrac{1}{2}) \cdot \prod_{v \in S} I_v(\varphi),$$

where C is constant and S is a finite set (which depends on Π and φ)[1]. For number theoretic applications, it is usually necessary to understand well the *local factors* I_v which have been studied in many—but not all—cases.

In this project we intend to make sense of and understand Ichino's formula. In particular, we will look at various choices of Π and calculate the local factors I_v for some cases that have not been previously considered. To do so will require learning about and applying the representation theory of various groups (both real and p-adic). Depending on time and interest, we will also explore possible applications of our formulas.

Interested students must have already taken Intro to Modern Algebra I/II (or an equivalent), meaning that they have studied finite groups and are comfortable working with various field extensions of the rational numbers including number fields and their various completions. Having taken courses in Algebraic Number Theory, Representation Theory of Finite Groups, Complex Analysis and/or Analytic Number Theory would be a plus. It is not necessary, but it would be beneficial, to have familiarity with modular forms. Note that we intend to break up the project into subparts which can be more or less analytic/algebraic, but please indicate in your application whether or not you are particularly interested in one or the other of analytic and algebraic number theory.

[1] Don't worry if all or most of this paragraph is gibberish. Learning what it means is part of the project.