GROUP ACTIONS AND GROUP REPRESENTATIONS IN LOW DIMENSIONAL TOPOLOGY

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The classification of surfaces theorem can give the misleading impression that the topology of surfaces is a finished project. While the surfaces themselves have been classified and their basic topological invariants are perfectly understood, the study of *maps between surfaces* is a very rich subject that has deep connections to topology, group theory, dynamics, algebraic geometry, representation theory, and more. This project will explore some aspects of this, focusing on the setting of surfaces X equipped with a symmetry group G. This sets up an arena for the representation theory of G to interact with the topology of X via the action of G on the homology $H_1(X)$. Replacing the surface X with a graph, there is a corresponding theory which is equally rich and still connected to deep phenomena in topology and beyond.

The main question we are interested in concerns the "primitive homology" of a G-regular covering $X \to Y$. This is the subspace $H_1^{pr}(X) \leqslant H_1(X)$ spanned by components of lifts of simple closed curves on Y. It is easy to see that $H_1^{pr}(X)$ is a G-subrepresentation of $H_1(X)$, and the major question is simply *when is there an equality* $H_1^{pr}(X) = H_1(X)$*?* This apparently naïve question turns out to be intimately connected to some deep problems at the border between topology and algebraic geometry. As a first approximation to an answer, one would like to know when every irrep of G appears in the decomposition of $H_1^{pr}(X)$. It is now known that there are examples of covers for which $H_1^{pr}(X)$ is a strict subrepresentation of $H_1(X)$, but our understanding of this phenomenon is far from complete. In our project, we will explore this area with the aim of proving some new theorems. Most likely a certain portion of the project will involve writing some computer code.

Minimum required background.

- A first course in abstract algebra (group theory).

Desirable but non-mandatory experience.

- Experience with the basics of representation theory (of finite groups).
- Experience with mathematical computing (Mathematica, Sage) or basic familiarity with Python.
- We will develop the necessary topology as we go, but some exposure to concepts like H_1 and π_1 for surfaces wouldn't hurt.

The ideal student will be excited at the prospect of drawing lots of pictures, computing group characters, and asking their computers to do their dirty work.