# A class of globally solvable systems of BSDE and applications

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# (Forward) SDE

**The equation:**

$$X_0 = x, \quad dX_t = \mu(X_t)\,dt + \sigma(X_t)\,dB_t,\ t \in [0,T].$$

**Causality principle(s)**:

$$X_t = F(t, \{B_s\}_{s\in[0,t]}) \qquad \text{(strong)}$$
$$\{X_s\}_{s\in[0,t]} \perp\!\!\!\perp \{B_s - B_t\}_{s\in[t,T]} \qquad \text{(weak)}$$

**Solution by simulation** (Euler scheme):

$$1)\ X_0 = x, \quad 2)\ X_{t+\Delta t} \approx X_t + \mu(X_t)\,\Delta t + \sigma(X_t)\,\Delta\zeta,$$

where we draw $\Delta\zeta = B_{t+\Delta t} - B_t$ from $N(0, \sqrt{\Delta t})$.

# BSDE ARE NOT BACKWARD SDE

**The equation:**

$$dX_t = \mu(X_t)\,dt + \sigma(X_t)\,dB_t,\ t \in [0,T], \qquad X_T = \xi.$$

**Backwards solution by simulation:**

1) $X_T = \xi$, 2) $X_{t-\Delta t} \approx X_t - \mu(X_{t-\Delta t})\,\Delta t - \sigma(X_{t-\Delta t})(B_t - B_{t-\Delta t})$

The solution is no longer defined, or, at best, no longer adapted:

e.g., if $dX_t = dB_t,\ X_T = 0$ then $X_t = B_t - B_T$.

**Fix**: to restore adaptivity, make $Z_t = \sigma(X_t)$ a part of the solution

$$dX_t = \mu(X_t)\,dt + Z_t\,dB_t, \quad X_T = \xi.$$

**MRT:** for $\mu \equiv 0$ we get the martingale representation problem:

$$dX_t = Z_t\,dB_t, \quad X_T = \xi.$$

# Backward SDE

**A change of notation:**

$$dY_t = -f(Y_t, Z_t)\,dt + Z_t\,dB_t,\ t \in [0,T], \qquad Y_T = \xi.$$

A *solution* is a pair $(Y, Z)$. The function $f$ is called the *driver*.

Time- and uncertainty-dependence is often added:

$$dY_t = -f(t, \omega, Y_t, Z_t)\,dt + Z_t\,dB_t,\ t \in [0,T], \qquad Y_T = \xi(\omega),$$

and the $\omega$-dependence factored through a (forward) diffusion

$$\begin{aligned} X_0 = x,\ dX_t &= \mu(t, X_t)\,dt + \sigma(t, X_t)\,dB_t \\ dY_t &= -f(t, X_t, Y_t, Z_t)\,dt + Z_t\,dB_t,\ Y_T = g(X_t). \end{aligned}$$

# Existing theory - dimension 1

**Linear:** Bismut '73, (or even Wentzel, Kunita-Watanabe or Itô)

**Lipschitz:** Pardoux-Peng '90

**Linear-growth:** Lepeltier-San Martin '97

**With reflection:** El Karoui et al '95, Cvitanić-Karatzas '96

**Constrained:** Buckdahn-Hu '98, Cvitanić-Karatzas-Soner '98

**Quadratic:** Kobylanski '00

**Superquadratic:** Delbaen-Hu-Bao '11 - mostly negative

# Existing theory - systems

**Lipschitz drivers:** Pardoux-Peng '90

**Smallness:** Tevzadze '08

**Quadratic global existence:** Peng '99 - open problem

**Non-existence:** Frei - dos Reis '11

**Quadratic global existence - special cases**:

Tan '03, Jamneshan-Kupper-Luo '14,
Cheridito-Nam '15, Hu-Tang '15

# A PDE connection

**A single equation:** under regularity conditions, the pair $(Y, Z)$ is a Markovian solution, i.e., $Y = v(t, B_t)$, to

$$dY_t = -f(Y_t, Z_t)\,dt + Z_t\,dB_t,\; Y_T = g(B_T)$$

if and only if $v$ is a viscosity solution to

$$v_t + \tfrac{1}{2}\Delta v + f(v, Dv) = 0,\; v(T, \cdot) = g.$$

**Systems:** no such characterization (“if” direction when the PDE system admits a smooth solution).

no maximum principle $\to$ no notion of a viscosity solution.

# The approach of Kobylanski

**Approximation:** approximate both the driver $f$ and the terminal condition $g$ by Lipschitz functions; ensure monotonicity.

**Monotone convergence:** use the comparison (maximum) principle to get monotonicity of solutions

**BMO-bounds**: use the quadratic growth of $f$ to get uniform bounds on the approximations to $Z$; exponential transforms

$$H_t^\alpha = \exp(\alpha Y_t) \text{ is a submartingale for large enough } \alpha,$$

since

$$dH_t^\alpha = \alpha H_t^\alpha Z_t \, dB_t + \alpha H_t^\alpha \left( \tfrac{1}{2} \alpha Z_t^2 - f(Y_t, Z_t) \right) dt$$

**Unfortunately:** this will not work for systems for two reasons:

1. There is no comparison principle for systems
2. The exponential transform no longer works.

# Our Setup

**The driving diffusion:** let $X$ be a uniformly-elliptic inhomogeneous diffusion on $\mathbb{R}^d$ with (globally) Lipschitz and bounded coefficients.

**Markovian solutions:** a pair $\boldsymbol{v} : [0,T] \times \mathbb{R}^d \to \mathbb{R}^N$, $\boldsymbol{w} : [0,T] \times \mathbb{R}^{N\times d}$ of Borel functions such that $\boldsymbol{Y} := \boldsymbol{v}(\cdot, X)$ is a continuous semimartingale, and

$$g(X_T) = Y_t - \int_t^T \boldsymbol{f}(s, X_s, \boldsymbol{Y}_s, \boldsymbol{Z}_s)\, ds + \int_t^T \boldsymbol{Z}_s\, dB_s,$$

where $\boldsymbol{Z} := \boldsymbol{w}(\cdot, X)$.

**Variants:** *bounded* or *(locally) Hölderian* solutions (when $\boldsymbol{v}$ has that property) or a *bmo-solution* (when $\boldsymbol{w}(t, X_t)$ is in bmo).

# A substitute for the exponential transform

Set $\langle \boldsymbol{z}, \boldsymbol{z} \rangle_{a(t,x)} = \boldsymbol{z} a(t,x) \boldsymbol{z}^T$, where $a = \sigma\sigma^T$ (double the coefficient matrix for the second-order part of the generator of $X$):

**Definition:** Given a constant $c > 0$, a function $h \in C^2(\mathbb{R}^N)$ is called a *c-Lyapunuov function* for $\boldsymbol{f}$ if $h(\mathbf{0}) = 0, Dh(\mathbf{0}) = \mathbf{0}$, and there exists a constant $k$ such that

$$\frac{1}{2} D^2 h(\boldsymbol{y}) : \langle \boldsymbol{z}, \boldsymbol{z} \rangle_{a(t,x)} - Dh(\boldsymbol{y}) \boldsymbol{f}(t, x, \boldsymbol{y}, \boldsymbol{z}) \geq |\boldsymbol{z}|^2 - k \qquad (1)$$

for all $(t, x, \boldsymbol{y}, \boldsymbol{z}) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^N \times \mathbb{R}^{N \times d}$, with $|\boldsymbol{y}| \leq c$.

**Intuitively:** $h(Y_t) + kt$ must be a 'very strict' submartingale, whenever $Y$ is a solution. As mentioned before, for $N = 1$, $h(y) = e^{\alpha y}$, for large-enough $\alpha$.

# The main result

**Theorem.** Let $X$ be a uniformly elliptic diffusion with bounded, Lipschitz coefficients, and $\boldsymbol{f}$ be a continuous driver of (at-most) quadratic growth in $\boldsymbol{z}$. Suppose that there exits a constant $c > 0$ such that

- $\boldsymbol{g}$ is bounded and in $C^{\alpha}$,
- $\boldsymbol{f}$ admits a $c$-Lyapunov function, and
- $\boldsymbol{Y}$ is "a-priori bounded" by $c$.

Then the BSDE system

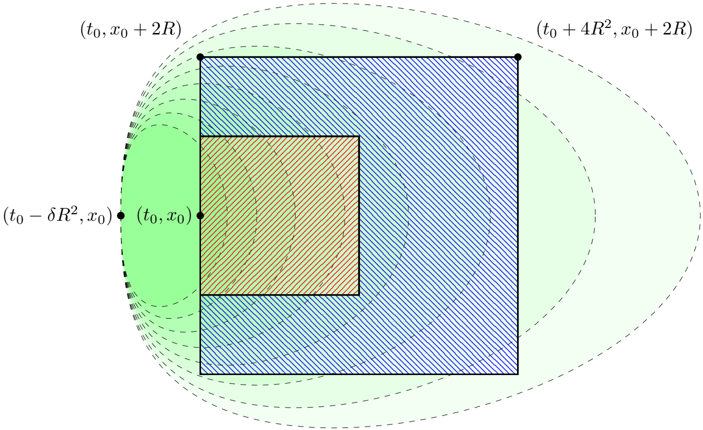
$$d\boldsymbol{Y}_t = -\boldsymbol{f}(t, X_t, \boldsymbol{Y}_t, \boldsymbol{Z}_t)\,dt + \boldsymbol{Z}_t\,dB_t, \quad \boldsymbol{Y}_T = \boldsymbol{g}(B_T),$$

has a Hölderian solution $(\boldsymbol{v}, \boldsymbol{w})$, with $\int \boldsymbol{Z}\,dB$ a BMO-martingale and $\boldsymbol{w} = D\boldsymbol{v}$, in the distributional sense on $(0, T) \times \mathbb{R}^d$.

This solution is, moreover, unique in the class of all Markovian solutions if $\boldsymbol{f}$ is $\boldsymbol{y}$-independent and

$$|\boldsymbol{f}(t, x, \boldsymbol{z}_2) - \boldsymbol{f}(t, x, \boldsymbol{z}_1)| \leq C(|\boldsymbol{z}_1| + |\boldsymbol{z}_2|)\,|\boldsymbol{z}_2 - \boldsymbol{z}_1|\,.$$

# A peek into the Proof




$$\iint_{\text{red}} |Dv|^2 \leq C \iint_{\text{blue}} |Dv|^2 + R^{2\alpha}$$

We use the “hole filling” method (Widman ’76) and its variants (Struwe ’81, Bensoussan-Frehse, ’02) - and apply it to get Campanato (and therefore Hölder) a-priori estimates.

# The Bensoussan-Frehse (BF) condition

**Proposition.** If $f$ admits a decomposition

$$f(t,x,y,z) = \operatorname{diag}(z\,\mathsf{l}(t,x,y,z)) + \mathsf{q}(t,x,y,z) + \mathsf{s}(t,x,y,z) + \mathsf{k}(t,x),$$

with

$$\begin{aligned}
&|\mathsf{l}(t,x,y,z)| \leq C(1+|z|), && \text{(quadratic-linear)}\\
&\left|\mathsf{q}^i(t,x,y,z)\right| \leq C\Big(1+\textstyle\sum_{j=1}^i \left|z^j\right|^2\Big), && \text{(quadratic-triangular)}\\
&|\mathsf{s}(t,x,y,z)| \leq \kappa(|z|),\ \lim_{z\to\infty} \tfrac{\kappa(z)}{z^2} = 0, && \text{(subquadratic)}\\
&\mathsf{k} \in \mathbb{L}^\infty([0,T]\times\mathbb{R}^d), && \text{($z$-independent),}
\end{aligned}$$

Then a $c$-Lyapunov function exists for each $c > 0$.

**Extensions:** an approximate decomposition will do, as well. To the best of our knowledge, all systems solved in the literature satisfy the (BF) condition (in $z$-dependence).

# Stochastic Equilibria in Incomplete Markets

**Setup:** $\{\mathcal{F}_t\}_{t\in[0,T]}$ generated by two independent BMs $B$ and $W$

**Price:** $dS^\lambda_t = \lambda_t\,dt + \sigma_t\,dB_t + \boxed{0\,dW_t}$ ( WLOG $\sigma_t \equiv 1$! )

**Agents:** $U^i(x) = -\exp(-x/\delta^i),\ E^i \in \mathbb{L}^0(\mathcal{F}_T),\ i = 1,\dots,I$

**Demand:** $\hat{\pi}^{\lambda,i} := \operatorname{argmax}_{\pi\in\mathcal{A}^\lambda} \mathbb{E}\left[U^i\Big(\int_0^T \pi_u\,dS^\lambda_u + E^i\Big)\right].$

**Goal:** Is there an *equilibrium market price of risk* $\lambda$, i.e., does there exist a process $\lambda$ such that the *clearing conditions*

$$\sum_{i=1}^I \hat{\pi}^{\lambda,i} = 0$$

hold?

# Stochastic Equilibria in Incomplete Markets

**A characterization:** Kardaras, Xing and Ž, '15, give the following characterization: a process $\lambda \in \text{bmo}$ is an equilibrium market price of risk *if and only if* it admits a representation of the form

$$A[\boldsymbol{\mu}] := \sum_{i=1}^{N} \alpha_i \mu^i,$$

for some solution $(\boldsymbol{\mu}, \boldsymbol{\nu}, \boldsymbol{Y}) \in \text{bmo} \times \text{bmo} \times \mathcal{S}^\infty$ of

$$\begin{cases} dY_t^i = \mu_t^i \, dB_t + \nu_t^i \, dW_t + \left( \frac{1}{2} (\nu_t^i)^2 - \frac{1}{2} A[\boldsymbol{\mu}]_t^2 + A[\boldsymbol{\mu}] \mu_t^i \right) dt, \\ Y_T^i = G^i, \qquad i = 1, \dots, I, \end{cases}$$

where $\alpha^i = \delta^i / (\sum_j \delta^j)$, $G^i = E^i / \delta^i$.

**Theorem (**Xing, Ž.**)** If there exists a regular enough function $\boldsymbol{g}$ and a diffusion $X$ such that $G^i = g^i(X_T)$, for all $i$, then a stochastic equilibrium exists and is unique in the class of Markovian solutions.

# Martingales on Manifolds

**$\Gamma$-martingales:** Let $M$ be an $N$-dimensional differentiable manifold endowed with an affine connection $\Gamma$. A continuous semimartingale $\boldsymbol{Y}$ on $M$ is called a $\Gamma$-*martingale* if

$$f(\boldsymbol{Y}_t) - \frac{1}{2}\int_0^t \mathrm{Hess} f(d\boldsymbol{Y}_s, d\boldsymbol{Y}_s),\ t \in [0, T],$$

is a local martingale for each smooth $f : M \to \mathbb{R}$, where

$$(\mathrm{Hess} f)_{ij}(\boldsymbol{y}) = D_{ij} f(\boldsymbol{y}) - \textstyle\sum_{k=1}^N \Gamma_{ij}^k(\boldsymbol{y}) D_k f(\boldsymbol{y}).$$

**A coordinate representation:** By Itô's formula, $\boldsymbol{Y}$ is a $\Gamma$-martingale if and only if its coordinate representation has the following form

$$dY_t^k = -\boldsymbol{f}^k(\boldsymbol{Y}_t, \boldsymbol{Z}_t)\, dt + \boldsymbol{Z}_t^k\, dW_t$$

where $\boldsymbol{f}^k(\boldsymbol{y}, \boldsymbol{z}) = \frac{1}{2}\sum_{i,j=1}^d \Gamma_{ij}^k(\boldsymbol{y})(\boldsymbol{z}^i)^\top \boldsymbol{z}^j$.

# Martingales on Manifolds

**A Problem:** Given an $N$-dimensional Brownian motion $B$ and an $M$-valued random variable $\xi$, construct a $\Gamma$-martingale $\boldsymbol{Y}$ with $\boldsymbol{Y}_T = \xi$.

**Solution:** Easy in the Euclidean case - we filter $\boldsymbol{Y}_t = \mathbb{E}[\xi|\mathcal{F}_t]$. In general, solution may not exist. Under various conditions, such processes were constructed by Darling '95 and Blache '05, '06.

**Our contribution:** Taken together, the existence of a Lyapunov function and a-priori boundedness are (essentially) equivalent to the existence of a so-called doubly-convex function $h$ on a neighborhood of a support of $\xi$.

Conversely, this sheds new light on the meaning of $c$-Lyapunov functions: loosely speaking - they play the role of convex functions, but in the geometry dictated by $\boldsymbol{f}$.

# Sretan rođendan, Yannis!