Viability, Arbitrage and Preferences

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Thera Stochastics
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The super-replication plays a cruicial role in this work.

Definition 5.3. The *fair price* (or *value*) at $t = 0$ for the ACC (T, f, g) of Definition 4.1, is the number

$$V_0 \triangleq \inf\{x \geq 0;\ \exists(\pi, C) \in \mathscr{H}(T, x)\}. \tag{5.6}$$

Let $x \geq 0$ be any number for which there exists a hedging strategy $(\pi, C) \in \mathscr{H}(T, x)$; the optional sampling theorem applied to the nonnegative supermartingale of (3.14) then gives, in conjunction with properties (5.1) and (5.2),

$$\tilde{E}_T(Q_\tau) = \tilde{E}_T\left[\beta(\tau) f_\tau + \int_0^\tau \beta(s) g_s\, ds\right] \leq \tilde{E}_T\left[\beta(\tau) X_\tau + \int_0^\tau \beta(s)\, dC_s\right] \leq x \tag{5.7}$$

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On the Pricing of American Options*

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Abstract. The problem of valuation for contingent claims that can be exercised at any time before or at maturity, such as American options, is discussed in the manner of Bensoussan [1]. We offer an approach which both simplifies and extends the results of existing theory on this topic.

1. Introduction

In an important and relatively recent article, Bensoussan [1] presents a rigorous treatment for *American* contingent claims, that can be exercised at any time before or at maturity (in contradistinction to *European* contingent claims which are exercisable only at maturity). He adapts the Black and Scholes [3] methodology of duplicating the cash flow from such a claim to this situation by skillfully managing a self-financing portfolio that contains only the basic instruments of the market, i.e., the stocks and the bond, and that entails no arbitrage opportunities before exercise. Under a condition on the market model called *completeness* (due to Harrison and Pliska [7], [8] in its full generality and rendered more transparent in [1]), Bensoussan shows that the pricing of such claims is

- We consider a financial market without any probabilistic or topological structure but rather with a partial order representing the common beliefs of all agents.
- In this structure, we investigate the proper extensions of the classical notions of arbitrage and viability or the economic equilibrium.
- Our contributions are an extension of classical works of Harrison & Kreps'79 and Kreps'81 to incorporate Knightian uncertainty and also the unification of several arbitrage definitions given recently in model-free finance.
- We prove equivalent conditions for arbitrage and viability.

Frank Knight in his 1921 book, *Risk, Uncertainty, and Profit*, formalized a distinction between risk and uncertainty.

▷ According to Knight, risk applies to situations where we do not know the outcome of a given situation, but can accurately measure the odds.

▷ Uncertainty applies to situations where we cannot know all the information we need in order to set accurate odds.

There is a fundamental distinction between the reward for taking a known risk and that for assuming a risk whose value itself is not known.

Suppose the agents consider not one probability measure but uncountably many of them as possible measures. As an example, suppose that they consider all volatility processes in an interval $[a, b]$ as possible but cannot make a precise estimation of it. Let $\mathcal{P} = \{\mathbb{P}^\sigma\}$ be the set of all such measures indexed by adapted volatility process σ with values in $[a, b]$.

There is no single dominating measure. Relevant questions ;

- ▷ In this context what are the notions of arbitrage or equilibrium ?
- ▷ How is a single measure chosen, or is it chosen ?
- ▷ How do these measures relate to the preferences of agents ?

In the context of $\mathcal{P}$ possible definitions of a tradable contract X with zero initial cost an arbitrage would be

$$\mathbb{P}(X \geq 0) = 1, \quad \text{and} \quad \mathbb{P}(X > 0) > 0.$$

What do we do here? How do we quantify $\mathbb{P}$?

In the context of $\mathcal{P}$ possible definitions of a tradable contract X with zero initial cost an arbitrage would be

$$\inf_{\mathbb{P}\in\mathcal{P}} \mathbb{P}(X \geq 0) = 1, \quad \text{and} \quad \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{P}(X > 0) > 0.$$

or

$$\inf_{\mathbb{P}\in\mathcal{P}} \mathbb{P}(X \geq 0) = 1, \quad \text{and} \quad \inf_{\mathbb{P}\in\mathcal{P}} \mathbb{P}(X > 0) > 0.$$

Which one is appropriate ? or is there such a notion ? Note

$$\inf_{\mathbb{P}\in\mathcal{P}} \mathbb{P}(A) = 1, \quad \Leftrightarrow \quad \mathbb{P}(A) = 1, \ \forall \mathbb{P} \in \mathcal{P}$$

$$\Leftrightarrow \quad A \text{ holds } \mathcal{P} - \text{quasi-surely.}$$

▷ Harrison & Kreps (1979) consider consumption bundles

$$(r, X) \in \mathbb{R} \times \mathcal{L}^2(\Omega, \mathbb{P}),$$

where $r \in \mathbb{R}$ is the units of consumption at time zero and the random variable $X \in \mathcal{L}^2(\Omega, \mathbb{P})$ is the consumption at date T. Important is that $\mathbb{P}$ is fixed right at the beginning.

▷ Kreps (1981) considers a more abstract set-up. Starting point :
- ▷ cone of positive contracts $\mathcal{K}$;
- ▷ linear pricing functional π on a subspace $\mathcal{M}$.

▷ A preference relation $\preceq \in \tilde{\mathcal{A}}$ provided that it is complete, convex, continuous and is strictly increasing in $\mathcal{K}$, i.e.,

$$X \prec X + k, \quad \forall\, X \in \mathcal{X},\ k \in \mathcal{K}.$$

▷ Then a market $(\mathcal{X}, \mathcal{K}, \pi, \mathcal{M})$ is viable if there exists $\preceq \in \tilde{\mathcal{A}}$ and an optimal contract $m^* = 0$ satisfying,

$$m \preceq 0 = m^*, \quad \text{whenever} \quad m \in \mathcal{M} \text{ and } \pi(m) \leq 0 = \pi(m^*).$$

▷ The optimal contract being zero is not a loss of generality. And the condition $\pi(m) \leq 0$ is the budget constraint.

Theorem (Harrison & Kreps'79, Kreps'81)

A market is viable if and only if there exists an linear, continuous, extension φ of π to whole of $\mathcal{X}$ which is strictly increasing, i.e.,

$$\varphi(k) > 0, \quad \forall\, k \in \mathcal{K}.$$

The extension φ is the equivalent risk neutral measure. In this context strict monotonicity implies that φ is equivalent.
But φ may not be countably additive.

Delbaen & Schachermayer (1994).

The definition of No Free Lunch with Vanishing Risk (NFLVR) is that there are no sequences of admissible, predicable processes H^n so that $f_n := (H \cdot S)_T := \int_0^T H^n \cdot dS$ satisfies

$$f_n^- \to 0 \;\text{ uniformly}, \quad f_n \to f \geq 0, \; \mathbb{P} - a.s. \quad \text{and} \quad \mathbb{P}(f > 0) > 0.$$

NFLVR is equivalent to

$$\mathcal{D}(\xi) := \inf \{ r \in \mathbb{R} \; : \; \exists H \;\; \text{so that} \;\; x + (H \cdot S)_T \geq \xi \; a.s. \; \} > 0,$$

for every $\mathbb{P}(\xi \geq 0) = 1$ and $\mathbb{P}(\xi > 0) > 0$.

The deep analysis shows that there is a countably additive martingale measure iff NFLVR.

In our set-up,

▷ agents consider of preferences and not only probabilities and there is a cloud of preferences that are possible ;

▷ they are presented with market data, i.e., liquidly traded contracts and their prices representing a partial equilibrium ;

▷ there is a unanimous partial order that is consistent with the cloud of preferences ;

▷ they also have beliefs.

- $\mathcal{L}$ is the set of all Borel measurable random variables and any $X \in \mathcal{L}$ represents the cumulative future cash flows.
- $\leq$ is a partial order on a subspace $\mathcal{H} \subset \mathcal{L}$.

$\leq$ is not the pointwise order ; although we assume that is monotone with respect to it. Also

$$X \leq Y \quad \Leftrightarrow \quad X + Z \leq Y + Z, \; \forall \; Z \in \mathcal{L}.$$

If a probability measure is given, then $X \leq Y$ iff $X \leq Y$, $\mathbb{P} - a.s.$

Let $\mathcal{A}$ be the set of a all preference relations (i.e., complete and transitive) satisfying

- monotone with respect to $\leq$; convex; weakly continuous.

Up to now, we are now given $(\mathcal{H}, \leq)$ an ordered vector space.

We now assume that there is a cone of contracts that are liquidly traded with zero initial cost, denoted by $\mathcal{I}$.

Examples of elements of $\mathcal{I}$ are stochastic integrals or liquidly traded options.

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We need an object replacing the "positive" cone $\mathcal{K}$. These are contracts that all agents agree to be positive.

We may simply take the set of all positive contracts, i.e., $P \in \mathcal{P}^+$ if and only if $P \in \mathcal{P} \setminus \mathcal{Z}$. Although this is a plausible choice in some examples it might be too large.

In general, we consider an an arbitrary subset $\mathcal{R}$ of $\mathcal{P}^+$ and call it as the set of relevant contracts. We assume all positive constants are in $\mathcal{R}$.

All agents agree that any contract $R \in \mathcal{R}$ is positive and as such it plays the same role as the positive cone.

Definition (Arbitrage)

A traded contract $\ell \in \mathcal{I}$ is an arbitrage if there exists $R \in \mathcal{R}$,

$$\ell \geq R.$$

Definition (Free Lunch with Vanishing Risk)

A sequence of traded contracts $\{\ell^n\}_n \in \mathcal{I}$ is called a free lunch with vanishing risk if there exists $R \in \mathcal{R}$ and a sequence of real numbers $c_n \to 0$ so that

$$\ell^n + c_n \geq R, \quad \forall\ n = 1, 2, \ldots.$$

We define the super-replication functional by,

$$\mathcal{D}(X) := \inf \{ c \in \mathbb{R} \;:\; \exists\, \ell \in \mathcal{I} \;\text{ so that }\; c + \ell \geq X \;\} .$$

Note that this is a convex functional and is Lipschitz in the supremum norm.

Lemma

There are no-free-lunches-with vanishing-risk (NFLVR), if and only if $\mathcal{D}(R) > 0$ for all $R \in \mathcal{R}$.

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Lemma

There are no-free-lunches-with vanishing-risk (NFLVR), if and only if $\mathcal{D}(R) > 0$ *for all* $R \in \mathcal{R}$.

First recall the definition of Harrison & Kreps : The market $(\mathcal{X}, \mathcal{K}, \pi, \mathcal{M})$ is viable if there exists $\preceq \in \tilde{\mathcal{A}}$ so that

$$m \preceq 0, \quad \text{whenever} \quad m \in \mathcal{M} \text{ and } \pi(m) \leq 0.$$

Moreover, a preference relation $\preceq \in \tilde{\mathcal{A}}$ iff it is convex, continuous and $X \prec X + k$ for all $X \in \mathcal{X},\ k \in \mathcal{K}$. And $\mathcal{K}$ is a cone.

- In our structure $\mathcal{R}$ plays the same role as $\mathcal{K}$.
- The set $\mathcal{I}$ is given by $\{m \in \mathcal{M} \ : \ \pi(m) = 0\}$.
- However, we do not insist on $X \prec X + k$.

A market $(\mathcal{H}, \preceq, \mathcal{I}, \mathcal{R})$ is viable if there exists $\preceq' \in \mathcal{A}$ so that

$$\ell - R \preceq' -R \quad \text{and} \quad -R \prec' 0, \quad \forall \ell \in \mathcal{I}.$$

The second condition is strict monotonicity at the optimal portfolio. (Recall that H&K requires $X - R \preceq' X$ for every X.)

As a corollary to the first condition

$$\ell \preceq' 0, \quad \forall \ell \in \mathcal{I}.$$

This is a manifestation of equilibrium.

A market $(\mathcal{H}, \leq, \mathcal{I}, \mathcal{R})$ is viable if there exists $\preceq' \in \mathcal{A}$ and an an optimal portfolio $X^* \in \mathcal{H}$ so that

$$X^* + \ell - R \preceq' X^* - R \quad \text{and} \quad X^* - R \prec' X^*, \quad \forall \ell \in \mathcal{I}.$$

The weak continuity and first condition above imply that

$$X^* + \ell \preceq' X^*, \quad \forall \ell \in \mathcal{I}.$$

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Theorem (Burzoni, Riedel, Soner, 2017)

A financial market is viable if and only if there are no free lunches with vanishing risk.

Proof : Suppose NFLVR holds. Then the super-replication functional $\mathcal{D}$ is convex and proper. We define

$$X \preceq' Y \quad \Leftrightarrow \quad -\mathcal{D}(-X) \leq -\mathcal{D}(-Y), \quad X, Y \in \mathcal{H}, \quad \text{where}$$

$$\mathcal{D}(X) := \inf\{c \in \mathbb{R} \ : \ \exists\, \ell \in \mathcal{I} \ \text{ so that } \ c + \ell \succeq X\}.$$

Then, one checks easily that $\preceq'$ has all the required properties.

- ▷ Suppose the market is viable and towards a contraposition assume that $\ell^n + c_n \geq R^*$ for some $R^* \in \mathcal{R}$ and $c_n \to 0$. Then, $-c_n \leq \ell^n - R^*$.
- ▷ Since $\preceq'$ is monotone, $-c_n \preceq' \ell - R^*$.
- ▷ Moreover, by viability $\ell - R^* \preceq' -R^*$.
- ▷ Combining we conclude that

$$-c_n \preceq' -R^* \quad \Rightarrow \quad 0 \preceq' -R^*.$$

- ▷ This contradicts with $-R \prec' 0$ for every $R \in \mathcal{R}$.

Theorem (Burzoni, Riedel, Soner, 2017)

Assume $\mathcal{L} = \mathcal{H}$ is the set of bounded, measurable functions. Then, $(\mathcal{H}, \leq, \mathcal{I}, \mathcal{R})$ is viable or equivalently NFLVR if and only if there are linear functionals $\mathcal{Q}$ satisfying :

1. (Consistency) $\varphi(\ell) \leq 0, \quad \forall\ \ell \in \mathcal{I},\ \varphi \in \mathcal{Q};$

2. (Absolute Continuity) $\varphi(P) \geq 0, \quad \forall\ P \in \mathcal{P},\ \varphi \in \mathcal{Q};$

3. (Equivalance) $\forall R \in \mathcal{R} \quad \exists\ \varphi_R \in \mathcal{Q}\ \ s.t.\ \ \varphi_R(R) > 0.$

So the appropriate extension of the market is achieved by the following coherent non-linear expectation and not by a linear one,

$$\mathcal{E}(X) := \sup_{\varphi \in \mathcal{Q}} \ \varphi(X).$$

The super-replication functional

$$\mathcal{D}(X) := \inf \{c \in \mathbb{R} \ : \ \exists\, \ell \in \mathcal{I} \ \text{ so that } \ c + \ell \geq X \ \}$$

is convex, proper and Lipschtiz continuous. Also it is homogenous, i.e., $\mathcal{D}(\lambda X) = \lambda \mathcal{D}(X)$ for every $\lambda > 0$. By Fenchel-Moreau

$$\mathcal{D}(X) := \sup_{\varphi \in \mathcal{Q}} \ \varphi(X),$$

where

$$\mathcal{Q} = \{\varphi \in ba(\Omega) \ : \ \varphi(X) \leq \mathcal{D}(X), \ \ \forall\, X \in \mathcal{H}\}.$$

Then, one easily check that $\mathcal{Q}$ has the stated properties.

One may call the elements of $\mathcal{Q}$ as risk neutral measures.

In all of our examples, $\mathcal{I}$ contains all stochastic integrals :

In finite discrete time,

$$\ell = \sum_{k=1}^{T} H_k \cdot (S_{k+1} - S_k),$$

In continuous time,

$$\ell = \int_0^T H_t \cdot dS_t.$$

Appropriate restrictions on H are placed ; predictable, sometimes bounded, etc.

We may also add liquidly traded options as wel,

$$\ell = h(S_T) - \text{price of } h.$$

In these class of problems, one fixes a probability space $(\Omega, \mathbb{F}, \mathbb{P})$ and stock price process S. Then,

- The partial order $\leq$ is given through $\mathbb{P}$ almost sure inequalities.
- $\mathcal{R}$ is the set of $\mathbb{P}$ almost-surely non-negative functions that are not equal to zero.

Then, one obtains equivalent martingale measures. The fact that they are countably additive is a deep result and depends on results from stochastic integration.

In this case we fix a measurable space $(\Omega, \mathbb{F})$ and a family of probability measures $\mathcal{P}$. Then,

▷ $\leq$ is given through $\mathcal{P}$ quasi-sure inequalities.

▷ The choice of $\mathcal{R}$ is important. The following is used in the literature but other choices are possible as well, $R \in \mathcal{R}$ if

$$\inf_{\mathbb{P}\in\mathcal{P}} \mathbb{P}(R \geq 0) = 1, \quad \text{and} \quad \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{P}(R > 0) > 0.$$

Then, the result is the existence of bounded additive measures $\mathcal{Q}$ consistent with $\mathcal{I}$ and with full support property, i..e, for every $R \in \mathcal{R}$ there is $\varphi_R \in \mathcal{Q}$ so that $\varphi_R(R) > 0$.

Bouchard & Nutz (2014) considers above set-up with $\mathcal{I}$ is the set of all stochastic integrals and finitely many static options. They prove that there is no arbitrage iff there exists a set of countably additive martingale measures $\mathcal{Q}$ so that polar sets of $\mathcal{P}$ and $\mathcal{Q}$ agree.

Same is proved in continuous time with continuous paths in Biagini, Bouchard, Kardaras & Nutz.

Burzoni, Fritelli & Maggis (2015) also consider a similar problem in finite discrete time. They extend the notion of arbitrage and the proof technique is different.

One can present these models in two ways :

- $\mathcal{P} = \mathcal{M}_1$ is the set of all probability measures ;
- Equivalently, $\leq$ is the pointwise order.

Then, the financial market is given through dynamic hedging with the stock and also by static hedging through given liquidly traded options.

The set $\mathcal{I}$ is again set of all stochastic integrals and static positions.

The notions of arbitrage depends on the choice of $\mathcal{R}$.

- $R > 0$: Vienna arbitrage.

$$\inf_{\mathbb{P}\in\mathcal{M}_1} \mathbb{P}(R \geq 0) = 1, \quad \text{and} \quad \inf_{\mathbb{P}\in\mathcal{M}_1} \mathbb{P}(R > 0) > 0.$$

- $R \geq 0$ everywhere, positive at one point : one-point arbitrage.

$$\inf_{\mathbb{P}\in\mathcal{M}_1} \mathbb{P}(R \geq 0) = 1, \quad \text{and} \quad \sup_{\mathbb{P}\in\mathcal{M}_1} \mathbb{P}(R > 0) > 0.$$

In this case, constructed preference relation is

$$X \preceq' Y \quad \Leftrightarrow \quad \inf_{\varphi\in\mathcal{Q}} \varphi(X) \leq \inf_{\varphi\in\mathcal{Q}} \varphi(Y).$$

This preference relation is not strictly increasing in the direction of $\mathcal{R}$; it only satisfies,

$$\ell - R \preceq' -R \prec' 0.$$

In summary, from weakest to strongest we have

- one point arbitrage : strictly positive only at one point Riedel ;
- open arbitrage : strictly positive on an open set Burzoni, Fritelli & Maggis, and Dolinsky, S. ;
- Vienna arbitrage : strictly positive everywhere ;
- uniform arbitrage : uniformly positive. This is the strongest possible ; Bartl, Cheredito & Kupper, and Dolinsky, S.

To eliminate uniform arbitrage one finitely additive martingale measure suffices. While for one point arbitrage, for every point there needs to be a martingale measure which charges that point.

Acciaio, Beiglböck, Penker & Schachermayer 2014 consider a finite discrete time model with $\Omega = \mathbb{R}_+^T$. In addition to dynamic trading, a family of European options $\{h_\alpha(\omega)\}$ are traded with zero price.

THEOREM. (Acciaio *et. al*).
Suppose all h_α's are continuous and there is a power option. Then, there is no arbitrage if and only if there exists a martingale measure $\mathbb{Q}$ consistent with prices, i.e., $\mathbb{E}^{\mathbb{Q}}[h_\alpha] = 0$ for every α.

Arbitrage is strong compared to Bouchard & Nutz. So the martingale measures do not have additional properties. In Bouchard & Nutz, martingale measures also have the same polar sets.

This is related to the notion of smooth ambiguity by Klibanoff, Marinacci, Mukerji, 2005.
Also robust arbitrage is developed by Cuchiero, Klein, Teichmann.

- $\mathfrak{P} = \mathfrak{P}(\Omega)$ is the set of all probability measures on $(\Omega, \mathbb{F})$.
- Let μ be probability measure on $\mathfrak{P}$ (i.e., a measure on measures)
- The partial order is then given by, $X \leq Y$ provided that

$$\mu\left(\mathbb{P} \in \mathfrak{P} \ : \ \mathbb{P}(X \leq Y) = 1\right) = 1.$$

We say $R \in \mathcal{R}$ if

$$\mu\left(\mathbb{P} \in \mathfrak{P} \;:\; \mathbb{P}(R \geq 0) = 1\right) = 1, \quad \text{and}$$
$$\mu\left(\mathbb{P} \in \mathfrak{P} \;:\; \mathbb{P}(R > 0) > 0\right) > 0.$$

Moreover, a Borel set $N \subset \Omega$ is μ polar if

$$\mu\left(\mathbb{P} \in \mathfrak{P} \;:\; \mathbb{P}(N) = 0\right) = 1.$$

Let $\mathcal{N}_\mu$ be the set of all μ polar sets.

Then NFLVR and viability is equivalent to existence of a set of risk neutral measures $\mathcal{Q}$ so that $\mathcal{Q}$ polar sets is equal to $\mathcal{N}_\mu$.

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- The notions Arbitrage and viability depends crucially on the partial order and the beliefs (i.e, relevant contracts).
- If the partial equilibrium (i.e., pricing on $\mathcal{I}$) is extended to whole contracts, this imply that the agents agree one of the preference relations available to them.
- There are possibly many linear extensions, i.e., the set $\mathcal{Q}$ could be large. This is also the case in incomplete markets. However, in this case different possibilities may have different null events.

- No-Free-Lunch-with-Vanishing-Risk is can be equivalently stated through the super-replication functional.
- Without much assumption we show the existence of linear pricing rules that are consistent with the market data, i.e., $\mathcal{I}$.
- However, these functionals are only finitely additive.
- To ensure countably additivity we need to use the structure of $\mathcal{I}$ and in particular stochastic integration as done by Delbaen & Schachermayer. We also achieve this in finite discrete time as done is probabilistic models by Bouchard, Burzoni, Fritelli, Maggis, Nutz. In continuous time by Biagini, Bouchard, Kardaras, Nutz.

- We have showed that for partial equilibrium to extend to a larger set, an appropriate no-arbitrage notion is necessary and sufficient.
- We extend the classical work of Harrison & Kreps by simply relaxing a strict monotonicity condition. This relaxation allows us to incorporate Knightian uncertainty.
- We have shown that equilibrium is possible even in market with orthogonal preferences.

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THANK YOU FOR YOUR ATTENTION

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