Equilibrium large deviations for mean-field systems with translation invariance

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Motivation and outline

Study of the **fluctuations** of **large systems** with **mean-field** interactions,

from **Statistical Physics**...

- Large deviation theory **Freidlin, Wentzell – '79**
- McKean-Vlasov models and propagation of chaos, **Dawson, Gärtner – Mem. AMS '89**

...to **Stochastic Portfolio Theory**.

- Atlas and first-order models **Fernholz – '02**, **Banner, Fernholz, Karatzas – '05**
- with mean-field interactions **Shkolnikov – SPA '12**, **Jourdain, R. – AF '15**, **Bruggeman – PhD Thesis**

Outline

The Dawson-Gärtner Theory for confined systems

Translation invariant systems

Application to capital distribution

Definition of the particle system

Consider the system of n SDEs

$$\mathrm{d}X_i(t) = -\nabla V(X_i(t))\mathrm{d}t - \frac{1}{n}\sum_{j=1}^{n} \nabla W(X_i(t) - X_j(t))\mathrm{d}t + \sigma \mathrm{d}\beta_i(t) \qquad \text{in } \mathbb{R}^d,$$

with:

- $V : \mathbb{R}^d \to [0, +\infty)$ **external potential**;
- $W : \mathbb{R}^d \to [0, +\infty)$ even **interaction potential**;
- $\sigma^2 > 0$ a **temperature** parameter.

The interactions between particles are of **mean-field** type, and the configuration is encoded by the **empirical measure**

$$\mu_n(t) = \frac{1}{n}\sum_{i=1}^{n} \delta_{X_i(t)} \in \mathcal{P}(\mathbb{R}^d).$$

Natural questions:

- **large-scale** ($n \to +\infty$) and **long time** ($t \to +\infty$) behaviour;
- both at the level of **typical** behaviour and **fluctuations**.

Dawson, Gärtner – Mem. AMS '89 as a continuous version of Curie-Weiss model,
Garnier, Papanicolaou, Yang – SIFIN '13 for an application to systemic risk.

The Dawson-Gärtner Theory

Dawson, Gärtner – Mem. AMS '89: write the evolution of

$$\mu_n(t) = \frac{1}{n}\sum_{i=1}^{n} \delta_{X_i(t)} \in \mathcal{P}(\mathbb{R}^d).$$

as a formal **infinite-dimensional SDE**

$$\mathrm{d}\mu_n(t) = -\mathrm{Grad}\,\mathcal{F}[\mu_n(t)]\mathrm{d}t + \frac{\sigma}{\sqrt{n}}\mathrm{d}\beta(t) \qquad \text{in } \mathcal{P}(\mathbb{R}^d),$$

where:

- $\mathcal{F}$ is the **free energy** defined on $\mathcal{P}(\mathbb{R}^d)$ by

$$\begin{aligned}\mathcal{F}[\mu] &= \frac{\sigma^2}{2}\int \mu\log\mu + \int V\mu + \frac{1}{2}\int (W*\mu)\mu \\ &= \frac{\sigma^2}{2}\underbrace{\mathcal{S}[\mu]}_{\text{Entropy}} + \underbrace{\mathcal{V}[\mu] + \mathcal{W}[\mu]}_{\text{Energy}}.\end{aligned}$$

- Grad is the gradient with respect to some 'Riemannian metric' on $\mathcal{P}(\mathbb{R}^d)$ adapted to the covariance of the noise $\beta(t)$. (related with **quadratic Wasserstein distance** by **Jordan-Kinderlehrer-Otto, Carrillo-McCann-Villani, Ambrosio-Gigli-Savaré**...)

The Dawson-Gärtner Theory

Formal **infinite-dimensional SDE** $\mathrm{d}\mu_n(t) = -\operatorname{Grad}\mathcal{F}[\mu_n(t)]\mathrm{d}t + \frac{\sigma}{\sqrt{n}}\mathrm{d}\beta(t)$.

When $n \to +\infty$:

- **LLN**: μ_n converges to the solution of the **McKean-Vlasov PDE**

$$\partial_t \mu = -\operatorname{Grad}\mathcal{F}[\mu] = \frac{\sigma^2}{2}\Delta\mu + \operatorname{div}\left(\mu\left(\nabla V + \nabla W * \mu\right)\right),$$

  which is also a **propagation of chaos** result.

- The **invariant measure** $\begin{cases} \text{writes } \exp\left(-\frac{2n}{\sigma^2}\mathcal{F}\right), & \text{(formal)} \\ \text{satisfies a LDP with rate function } \frac{2}{\sigma^2}\mathcal{F} + \mathrm{Cte}. \end{cases}$

- Extension of the **Freidlin-Wentzell theory**: definition of an **action functional**, identification of the free energy as a **quasipotential**.

Main message

- The **dynamical** behaviour of the large-scale system, both **typical** (LLN) and **atypical** (LDP), is described the **free energy**.
- The latter quantity is only derived from the **stationary** distribution.

Stationary measure for the particle system

The particle system $\mathbf{X}(t) = (X_1(t), \dots, X_n(t)) \in (\mathbb{R}^d)^n$ defined by

$$\mathrm{d}X_i(t) = -\nabla V(X_i(t))\mathrm{d}t - \frac{1}{n}\sum_{j=1}^{n} \nabla W(X_i(t) - X_j(t))\mathrm{d}t + \sigma \mathrm{d}\beta_i(t)$$

can be rewritten

$$\mathrm{d}\mathbf{X}(t) = -n\boldsymbol{\nabla} U_n(\mathbf{X}(t))\mathrm{d}t + \sigma\mathrm{d}\boldsymbol{\beta}(t)$$

where, for $\mathbf{x} = (x_1, \dots, x_n) \in (\mathbb{R}^d)^n$ and $\mu_n(\mathbf{x}) = \frac{1}{n}\sum_{i=1}^{n} \delta_{x_i}$,

$$U_n(\mathbf{x}) = \frac{1}{n}\sum_{i=1}^{n} V(x_i) + \frac{1}{2n^2}\sum_{i,j=1}^{n} W(x_i - x_j) = \mathcal{V}[\mu_n(\mathbf{x})] + \mathcal{W}[\mu_n(\mathbf{x})].$$

Assume and define

$$z = \int_{x\in\mathbb{R}^d} \exp\left(-\frac{2V(x)}{\sigma^2}\right)\mathrm{d}x < +\infty, \qquad \mathrm{d}\nu(x) = \frac{1}{z}\exp\left(-\frac{2V(x)}{\sigma^2}\right)\mathrm{d}x.$$

- The process $\mathbf{X}$ has a unique stationary distribution P_n on $(\mathbb{R}^d)^n$.
- Letting $Q_n = \nu^{\otimes n}$, we have $\frac{\mathrm{d}P_n}{\mathrm{d}Q_n}[\mathbf{x}] \propto \exp\left(-\frac{2n}{\sigma^2}\mathcal{W}[\mu_n(\mathbf{x})]\right)$ on $(\mathbb{R}^d)^n$.

Equilibrium large deviations for the empirical measure

Let $\mathbb{P}_n = P_n \circ \mu_n^{-1}$ and $\mathbb{Q}_n = Q_n \circ \mu_n^{-1}$ be probability measures on $\mathcal{P}(\mathbb{R}^d)$.
Then

$$\frac{\mathrm{d}P_n}{\mathrm{d}Q_n}[\mathbf{x}] \propto \exp\left(-\frac{2n}{\sigma^2}\mathcal{W}[\mu_n(\mathbf{x})]\right) \qquad \Rightarrow \qquad \frac{\mathrm{d}\mathbb{P}_n}{\mathrm{d}\mathbb{Q}_n}[\mu] \propto \exp\left(-\frac{2n}{\sigma^2}\mathcal{W}[\mu]\right),$$

so that

$$\mathrm{d}\mathbb{P}_n[\mu] \propto \exp\left(-\frac{2n}{\sigma^2}\mathcal{W}[\mu]\right) \mathrm{d}\mathbb{Q}_n[\mu] \asymp \exp\left(-\frac{2n}{\sigma^2}\mathcal{W}[\mu] - n\mathcal{R}[\mu|\nu]\right)$$

where, by **Sanov's Theorem**, $\mathcal{R}[\mu|\nu]$ is the **relative entropy**

$$\mathcal{R}[\mu|\nu] = \int_{\mathbb{R}^d} \mathrm{d}\mu \log\left(\frac{\mathrm{d}\mu}{\mathrm{d}\nu}\right) = \mathcal{S}[\mu] + \frac{2}{\sigma^2}\mathcal{V}[\mu] + \mathrm{Cte}.$$

As a consequence, $\mathbb{P}_n$ **satisfies a LDP on** $\mathcal{P}(\mathbb{R}^d)$ with rate function

$$\mathcal{I}[\mu] = \mathcal{R}[\mu|\nu] + \frac{2}{\sigma^2}\mathcal{W}[\mu] + \mathrm{Cte} = \frac{2}{\sigma^2}\mathcal{F}[\mu] + \mathrm{Cte}.$$

- Rigorous formulation based on the **Laplace-Varadhan Lemma**, see **Léonard – SPA '87**, **Dawson-Gärtner – Mem. AMS '89**;
- variations on topology and assumptions on the regularity and integrability of V and W, culminating in **Dupuis, Laschos, Ramanan – arXiv:1511.06928**.

Partial conclusion

For **mean-field** particle systems with an **equilibrium Gibbs measure**:

- both the **dynamical** and **static** behaviour at large scales are described by the **free energy**,
- which can be derived from the **equilibrium** distribution by an elementary **'Sanov+Laplace-Varadhan'** procedure.

Preview of the sequel of the talk:

- **Robert Fernholz' talk**: systems of **rank-based interacting diffusions** (equivalently: **first-order models**, **competing particles**) allow to recover empirical capital distribution curves;
- for **large markets**, it can be argued that **mean-field interactions** provide a correct approximation of such models through **propagation of chaos**;
- it is therefore natural to look for a **free energy** for such models!

Main technical issue: **lack of equilibrium** due to **translation invariance**.

Outline

Systems without external potential

We want to address the situation where $V \equiv 0$, *i.e.*

$$\mathrm{d}X_i(t) = -\frac{1}{n}\sum_{j=1}^{n} \nabla W(X_i(t) - X_j(t))\mathrm{d}t + \sigma \mathrm{d}\beta_i(t) \qquad \text{in } \mathbb{R}^d.$$

Malrieu – AAP '03, **Cattiaux, Guillin, Malrieu – PTRF '08**: link with granular media equation.
Fouque, Sun – '13: model of inter-bank borrowing and lending.

- Trajectorial LLN and LDP on $[0, T]$ remain valid, the associated free energy writes

$$\mathcal{F}[\mu] = \frac{\sigma^2}{2}\mathcal{S}[\mu] + \mathcal{W}[\mu].$$

- The drift is **invariant by translation**, and the centre of mass

$$\Xi(t) = \frac{1}{n}\sum_{i=1}^{n} X_i(t)$$

  is a Brownian motion: **no equilibrium!**

Malrieu – AAP '03: the system **seen from its centre of mass** is ergodic under suitable assumptions on W.

System seen from its centre of mass

Define

$$\widetilde{X}_i(t) = X_i(t) - \Xi(t),$$

then $\widetilde{\mathbf{X}} = (\widetilde{X}_1, \ldots, \widetilde{X}_n)$ is a diffusion process in the linear subspace

$$M_{d,n} = \{\widetilde{\mathbf{x}} = (\widetilde{x}_1, \ldots, \widetilde{x}_n) \in (\mathbb{R}^d)^n : \widetilde{x}_1 + \cdots + \widetilde{x}_n = 0\},$$

the Lebesgue measure on which is denoted by $\mathrm{d}\widetilde{\mathbf{x}}$.

Invariant measure for the centered system

If $\exp(-2W/\sigma^2)$ is integrable, then $\widetilde{\mathbf{X}}$ is reversible with respect to the probability measure

$$\mathrm{d}\widetilde{P}_n(\widetilde{\mathbf{x}}) = \frac{1}{\widetilde{Z}_n} \exp\left(-\frac{2n}{\sigma^2} W_n(\widetilde{\mathbf{x}})\right) \mathrm{d}\widetilde{\mathbf{x}}, \qquad W_n(\widetilde{\mathbf{x}}) = \frac{1}{2n^2} \sum_{i,j=1}^n W(\widetilde{x}_i - \widetilde{x}_j) = \mathcal{W}[\widetilde{\mu}_n].$$

- Define
$$\widetilde{\mathbb{P}}_n := \widetilde{P}_n \circ \widetilde{\mu}_n^{-1},$$
which gives full measure to the set of **centered** probability measures $\widetilde{\mathcal{P}}(\mathbb{R}^d)$.
- What is the link between the **free energy** and the **large deviations** of $\widetilde{\mathbb{P}}_n$?

Large deviations for $\widetilde{\mathbb{P}}_n$

Essential remark:

- because of the constraint that

$$\widetilde{x}_1 + \cdots + \widetilde{x}_n = 0,$$

  $\widetilde{\mathbb{P}}_n$ **cannot be compared to a product measure**.

- The 'Sanov + Laplace-Varadhan' procedure fails.

Alternative idea: comparison with a system with **small external potential**, recentered.

Comparison with weakly confined system

Principle of a proof: consider a McKean-Vlasov particle system with interaction potential W and **external potential ηV, $\eta > 0$**.

1. By the 'Sanov + Laplace-Varadhan' procedure (**Dupuis, Laschos, Ramanan – arXiv:1511.06928**), the associate sequence $\mathbb{P}_n^\eta$ satisfies a LDP with rate function
$$\mathcal{I}^\eta[\mu] = \frac{2}{\sigma^2}\mathcal{F}^\eta + \mathrm{Cte}, \qquad \mathcal{F}^\eta = \frac{\sigma^2}{2}\mathcal{S} + \eta\mathcal{V} + \mathcal{W}.$$
2. If the LDP holds on a topology making the **centering map**
$$\mathrm{T} : \mathcal{P}(\mathbb{R}^d) \to \widetilde{\mathcal{P}}(\mathbb{R}^d)$$
continuous, then the **Contraction Principle** implies a LDP for
$$\widetilde{\mathbb{P}}_n^\eta := \mathbb{P}_n^\eta \circ \mathrm{T}^{-1},$$
with rate function
$$\widetilde{\mathcal{I}}^\eta[\widetilde{\mu}] = \inf_{\mu\in\mathcal{P}(\mathbb{R}^d):\mathrm{T}\mu=\widetilde{\mu}} \mathcal{I}^\eta[\mu]$$
$$= \mathcal{S}[\widetilde{\mu}] + \frac{2}{\sigma^2}\left(\mathcal{W}[\widetilde{\mu}] + \eta\inf_\tau \mathcal{V}[\tau\widetilde{\mu}]\right) + \mathrm{Cte}.$$
3. If $\widetilde{\mathbb{P}}_n^\eta$ is a good approximation of $\widetilde{\mathbb{P}}_n$ at the exponential scale when $\eta \downarrow 0$, then $\widetilde{\mathbb{P}}_n$ is expected to satisfy a LDP on $\widetilde{\mathcal{P}}(\mathbb{R}^d)$ with rate function
$$\widetilde{\mathcal{I}}[\widetilde{\mu}] = \mathcal{S}[\widetilde{\mu}] + \frac{2}{\sigma^2}\mathcal{W}[\widetilde{\mu}] + \mathrm{Cte} = \frac{2}{\sigma^2}\mathcal{F}[\widetilde{\mu}] + \mathrm{Cte}.$$

Large deviations for $\widetilde{\mathbb{P}}_n$: influence of ℓ

Take $W(x) = \kappa|x|^\ell$ + perturbation, $\ell \geq 1$: the **larger** ℓ, the **stronger** the interaction.

- In order for $\widetilde{\mathbb{P}}_n^\eta$ to be close to $\widetilde{\mathbb{P}}_n$, V **must not grow faster than** W: $V(x) = |x|^\ell$.
- The centering map T is continuous on any **Wasserstein space** $\mathcal{P}_p(\mathbb{R}^d)$, $p \geq 1$.
- **Wang, Wang, Wu – SPL '10**: Sanov's Theorem on $\mathcal{P}_p(\mathbb{R}^d)$ for $\mathbb{Q}_n$ **if and only if** $p < \ell$.

Theorem: case $\ell > 1$

If $\ell > 1$, then for all $p \in [1, \ell)$, the sequence $\widetilde{\mathbb{P}}_n$ satisfies a LDP on $\widetilde{\mathcal{P}}_p(\mathbb{R}^d)$ with rate function

$$\widetilde{\mathfrak{I}}[\widetilde{\mu}] = \frac{2}{\sigma^2}\mathcal{F}[\widetilde{\mu}] + \mathrm{Cte}.$$

By contraction, the LDP also holds on $\mathcal{P}(\mathbb{R}^d)$ with rate function

$$\mathfrak{I}[\mu] = \begin{cases} \widetilde{\mathfrak{I}}[\mu] & \text{if } \mu \in \widetilde{\mathcal{P}}_1(\mathbb{R}^d), \\ +\infty & \text{otherwise.} \end{cases}$$

When $\ell = 1$, does this LDP holds in the weak topology?
No: the rate function may fail to have compact level sets!

Why is $\ell = 1$ interesting?

We now let $d = 1$ and consider **system of rank-based interacting diffusions**

$$\mathrm{d}X_i(t) = \sum_{k=1}^{n} b_n(k) \mathbb{1}_{\{X_i(t) = X_{(k)}(t)\}} \mathrm{d}t + \sigma \mathrm{d}\beta_i(t),$$

with order statistics $X_{(1)}(t) \leq \cdots \leq X_{(n)}(t)$ and **mean-field coefficients**

$$b_n(k) = \frac{1}{1/n} \int_{u=(k-1)/n}^{k/n} b(u)\mathrm{d}u \simeq b\left(\frac{k}{n}\right), \qquad b : [0,1] \to \mathbb{R}.$$

- **Fernholz – '02**, **Banner, Fernholz, Karatzas – AAP '05**: **first-order approximation** of log-capitalisations in **asymptotically stable markets**.
- Many other applications (statistical physics, queuing systems, etc.): see **R. – arXiv:1705.08140** for a partial review.

Define and assume

$$B(u) := \int_{v=0}^{u} b(v)\mathrm{d}v, \qquad B(1) = 0.$$

Then the drift is translation invariant and the center of mass is a Brownian motion.

Why is $\ell = 1$ interesting?

Pal, Pitman – AAP '08, **Jourdain, Malrieu – AAP '08**: if

$$B(u) = \int_{v=0}^{u} b(v)\mathrm{d}v > 0, \qquad u \in (0,1),$$

then for all $n \geq 2$, for all $\ell \in \{1, \dots, n-1\}$,

$$\frac{1}{\ell}\sum_{k=1}^{\ell} b_n(k) > \frac{1}{n-\ell}\sum_{k=\ell+1}^{n} b_n(k),$$

so that the centered particle system $\widetilde{\mathbf{X}}$ is reversible with respect to the probability measure

$$\mathrm{d}\widetilde{P}_n(\widetilde{\mathbf{x}}) = \frac{1}{\widetilde{Z}_n}\exp\left(\frac{2}{\sigma^2}\sum_{k=1}^{n} b_n(k)\widetilde{x}_{(k)}\right)\mathrm{d}\widetilde{\mathbf{x}} \qquad \text{on } M_{1,n}.$$

- **Exponential tails**, similarly to McKean-Vlasov model with $W(x) = \kappa|x|$.
- Denoting by $F_{\widetilde{\mu}_n}$ the Cumulative Distribution Function of $\widetilde{\mu}_n$:

$$\sum_{k=1}^{n} b_n(k)\widetilde{x}_{(k)} = n\sum_{k=1}^{n}\left(B\left(\frac{k}{n}\right) - B\left(\frac{k-1}{n}\right)\right)\widetilde{x}_{(k)}$$
$$= -n\int_{x\in\mathbb{R}} B(F_{\widetilde{\mu}_n}(x))\mathrm{d}x =: -n\mathcal{W}[\widetilde{\mu}_n].$$

Why is $\ell = 1$ interesting?

Systems of rank-based interacting diffusions can be addressed in the same framework as McKean-Vlasov models, with **free energy**

$$\mathcal{F}[\mu] = \frac{\sigma^2}{2}\mathcal{S}[\mu] + \mathcal{W}[\mu], \qquad \mathcal{W}[\mu] = \int_{x\in\mathbb{R}} B(F_\mu(x))\mathrm{d}x.$$

- Propagation of chaos: **Bossy, Talay – AAP '96, MC '97**, **Jourdain – '97–'02**, **Shkolnikov – SPA '12**, **Jourdain, R. – SPDE '13**, **Bruggeman – PhD Thesis**;
- Central Limit Theorem: **Jourdain – MCAP '00**, **Kolli, Shkolnikov – arXiv:1608.00814**;
- Large Deviation Principle: **Dembo, Shkolnikov, Varadhan, Zeitouni – CPAM '16**.

Study of **equilibrium large deviations**:

- **translation invariance** and **exponential tails** make it similar to McKean-Vlasov model with $V \equiv 0$ and $W(x) = \kappa|x|$;
- in fact with $d = 1$ and $W(x) = \kappa|x|$,

$$\mathcal{W}^{\mathrm{MV}}[\mu] = \frac{\kappa}{2}\iint_{x,y\in\mathbb{R}} |x-y|\mathrm{d}\mu(x)\mathrm{d}\mu(y) = \kappa\int_{x\in\mathbb{R}} F_\mu(x)(1-F_\mu(x))\mathrm{d}x = \mathcal{W}^{\mathrm{RB}}[\mu],$$

with $B(u) = \kappa u(1-u)$.

Case $\ell = 1$

Consider either McKean-Vlasov particle system, or (any) rank-based model. The corresponding interaction functional is

$$\mathcal{W}[\mu] = \frac{1}{2} \iint_{x,y \in \mathbb{R}^d} W(x-y)\mathrm{d}\mu(x)\mathrm{d}\mu(y) \qquad \text{or} \qquad \mathcal{W}[\mu] = \int_{x \in \mathbb{R}} B(F_\mu(x))\mathrm{d}x.$$

- Let $\overline{\mathcal{P}}(\mathbb{R}^d)$ the **orbit space** of $\mathcal{P}(\mathbb{R}^d)$ under action of **translations** $\{\tau_y, y \in \mathbb{R}^d\}$.
- Interaction functional is **translation invariant**: for any $y \in \mathbb{R}^d$, $\mathcal{W}[\tau_y \mu] = \mathcal{W}[\mu]$.
- The free energy $\mathcal{F} = \frac{\sigma^2}{2}\mathcal{S} + \mathcal{W}$ also translation invariant.
- We denote by $\overline{\mathcal{F}}$ the induced functional on $\overline{\mathcal{P}}(\mathbb{R}^d)$.

Case $\ell = 1$

Alternative description of the particle system

Instead of considering the particle system **seen from its centre of mass**, we look at the **orbit** $\overline{\mu}_n$ of its empirical measure in $\overline{\mathcal{P}}(\mathbb{R}^d)$.

A similar idea in **Mukherjee, Varadhan – AP '16** for slightly different framework.

- We replace the use of the **not continuous** centering operator

$$\mathrm{T} : \mathcal{P}(\mathbb{R}^d) \to \widetilde{\mathcal{P}}(\mathbb{R}^d)$$

  with the use of the **continuous** orbit map

$$\rho : \mathcal{P}(\mathbb{R}^d) \to \overline{\mathcal{P}}(\mathbb{R}^d).$$

- The Contraction Principle can now be employed to transfer the LDP from $\mathbb{P}_n^\eta$ to

$$\overline{\mathbb{P}}_n^\eta := \mathbb{P}_n^\eta \circ \rho^{-1},$$

- and the remainder of the argument holds without any assumption on the strength of the interaction.

Final theorem

Let $\mathcal{W} : \mathcal{P}(\mathbb{R}^d) \to [0, +\infty)$ be the interaction functional of:

- either the McKean-Vlasov model with $W(x) = \kappa|x|^\ell +$ perturbation, $\ell \geq 1$;
- or the rank-based model with $B(0) = B(1) = 0$ and $B(u) > 0$.

Define the sequences $\widetilde{\mathbb{P}}_n$ on $\widetilde{\mathcal{P}}_p(\mathbb{R}^d)$ for any $p \geq 1$, and $\overline{\mathbb{P}}_n := \widetilde{\mathbb{P}}_n \circ \rho^{-1}$ on $\overline{\mathcal{P}}(\mathbb{R}^d)$.

Main result

- The sequence $\overline{\mathbb{P}}_n$ satisfies a LDP on $\overline{\mathcal{P}}(\mathbb{R}^d)$ with rate function $\frac{2}{\sigma^2}\overline{\mathcal{F}} + \mathrm{Cte}$.
- In the McKean-Vlasov case, if $\ell > 1$, then for any $p \in [1, \ell)$, the sequence $\widetilde{\mathbb{P}}_n$ satisfies a LDP on $\widetilde{\mathcal{P}}_p(\mathbb{R}^d)$ with rate function $\frac{2}{\sigma^2}\mathcal{F} + \mathrm{Cte}$.

- Under the assumptions of the latter statement, the former is obtained by contraction, which makes it **weaker**;
- but it holds for a **larger** class of models, including **rank-based interacting diffusions**.

Bottom-line

- For systems of **rank-based interacting diffusions**, the appropriate space in which the equilibrium large deviations can be expressed in terms of the free energy is **the orbit space** under the action of translations.
- We now consider **capital distribution curves** at equilibrium, and apply our result to the computation of the probability of an **atypical concentration of capital**.

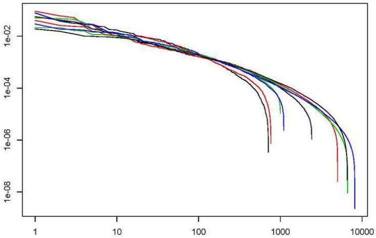
Outline

Capital distribution curve

Define the **market weights**

$$\mu_i(t) = \frac{S_i(t)}{S_1(t) + \cdots + S_n(t)} = \frac{\exp(X_i(t))}{\exp(X_1(t)) + \cdots + \exp(X_n(t))}$$

and plot the **capital distribution curve** $\ell \mapsto \mu_{(n-\ell+1)}(t)$.

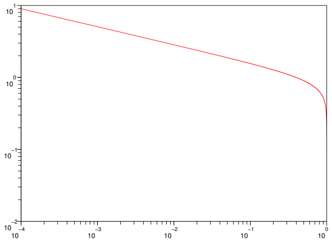


(Well-known!) picture by Robert Fernholz.

- The shape of the **rescaled** curve $\ell/n \mapsto \mu_{(n-\ell+1)}(t)$ seems **stationary**.
- Suggests to take $(\widetilde{X}_1, \dots, \widetilde{X}_n) \sim \widetilde{P}_n$ for some underlying rank-based model.
- Notice that **the curve is a function of $\rho(\widetilde{\mu}_n) = \overline{\mu}_n$ only**!

Hydrodynamic limit and typical capital distribution

- **R. – ECP '15**: when $n \to +\infty$, $\overline{\mu}_n \to \overline{\mu}$ which is **deterministic** and **explicit**.
- Equivalently: $\overline{\mu}$ is the **unique minimiser** on $\overline{\mathcal{P}}(\mathbb{R})$ of the free energy $\overline{\mathcal{F}}$.
- Up to a phase transition already described in **Chatterjee, Pal – PTRF '10**, the associated capital distribution curve looks like



see **Jourdain, R. – AF '15**.

We take $\overline{\mu}$ as the definition of a **typical** concentration of capital.

IID model

Remark: the **original model** with distribution $\widetilde{P}_n$ or a sample of centered **iid random variables** with law $\widetilde{\mu}$ such that $\rho(\widetilde{\mu}) = \overline{\mu}$ have the same law of large numbers.

In some situations, the 'iid model' is more amenable:

- **Jourdain, R. – AF '15** for functionally generated portfolios on large markets,
- **Bruggeman – PhD Thesis** for hitting times, etc.

Valid for the study of **typical** behaviour.

Question: can we compare the **large deviations** of both models?

- With the original model, $\frac{1}{n} \log \mathbb{P}(\overline{\mu}_n \simeq \overline{\nu}) \simeq -\overline{\mathfrak{I}}[\overline{\nu}]$;
- with the iid model, $\frac{1}{n} \log \mathbb{P}(\overline{\mu}_n \simeq \overline{\nu}) \simeq -\mathcal{R}[\overline{\nu}|\overline{\mu}]$.

Quick computation

If B is **concave**, then $\overline{\mathfrak{I}}[\overline{\nu}] \leq \mathcal{R}[\overline{\nu}|\overline{\mu}]$.

The probability of atypical concentration is **underestimated** by the iid model.

Conclusion

- Connection between the **equilibrium large deviation principles** of the **empirical measure of mean-field systems** with **translation invariance** and the **free energy** of such systems.
- Statement in the **orbit space** of the action or translations, or in the **space of centered measures** with Wasserstein topology **depending on the strength of the interaction**.
- Application to **capital distribution**: for systems having the same law of large numbers, **the interactions between stocks tend to increase the probability of atypical concentration of capital** when compared to independent stocks.

Thank you for your attention, et Joyeux Anniversaire Ioannis !