Recent results in game theoretic mathematical finance

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Based on joint work with R. Łochowski (Warsaw), D. Prömel (Zurich)

Motivation

Game theoretic approach formulates probability / math finance without measure theory.

- Kolmogorov's approach powerful but sometimes not well justified (frequentist vs subjective probability).
- Martingales usually introduced as "fair games":
  - not obvious from definition;
  - which parts of martingale theory come from "fair game" description, which from measure theoretic modelling?
- Model free math finance also eliminates reference probability $\Rightarrow$ connections to game-theoretic approach.

Scope of Vovk's approach

Vovk's Vovk '08 approach

- convenient book-keeping for model free math finance.
- qualitative properties of "typical price paths":
  variation regularity Vovk '11, quadratic variation Vovk '12, Vovk '15, Łochowski-P.-Prömel '16, local times P.-Prömel '15, rough paths P.-Prömel '16.
- measure free stochastic calculus:
  P.-Prömel '16, Łochowski '15, Vovk '16, Łochowski-P.-Prömel '16,
- quantitative results:
  pathwise Dambis Dubins-Schwarz theorem Vovk '12;
  model free pricing-hedging duality Beiglböck-Cox-Huesmann-P.-Prömel '15, Bartl-Kupper-Prömel-Tangpi '17.

Outline

1 Definition and basic properties

2 Overview of some nice results

3 Measure free stochastic calculus

4 Pathwise stochastic calculus

Vovk's approach

- $\Omega := C([0,\infty),\mathbb{R})$ (or $C([0,T],\mathbb{R})$, $D_+([0,T],\mathbb{R}^d)$, ...);
- $S_t(\omega) = \omega(t)$; $\mathcal{F}_t = \sigma(S_s : s \leq t)$;
- simple strategy H:
  - stopping times $0 = \tau_0 < \tau_1 < \dots$
  - $\mathcal{F}_{\tau_n}$-measurable $F_n \colon \Omega \to \mathbb{R}$.
- Well-defined integral:

$$(H \cdot S)_t(\omega) = \sum_{n=0}^{\infty} F_n(\omega)[S_{\tau_{n+1}\wedge t}(\omega) - S_{\tau_n \wedge t}(\omega)]$$

H is λ-admissible ($\in \mathcal{H}_\lambda$) if $(H \cdot S)_t(\omega) \geq -\lambda \,\forall\, \omega, t$.

Definition (Vovk '09 / P-Prömel '15)

Outer measure $\overline{P}$ of $A \subseteq \Omega$ is

$$\overline{P}(A) := \inf\left\{\lambda : \exists (H^n)_n \subseteq \mathcal{H}_\lambda \text{ s.t. } \liminf_{n\to\infty}(\lambda + (H^n \cdot S)_\infty(\omega)) \geq \mathbb{1}_A(\omega) \forall \omega\right\}.$$

Game-theoretic martingales are the capital processes $\lambda + (H \cdot S)$, $H \in \mathcal{H}_\lambda$.

Link with measure-theoretic martingales

Lemma (Vovk '12)

$$\sup_{\mathbb{P}\ MM} \mathbb{P}(A) \leq \overline{P}(A), \qquad A \in \mathcal{F}_\infty.$$

- For $\lambda > \overline{P}(A)$ we find $(H^n) \subseteq \mathcal{H}_\lambda$ with

$$\mathbb{1}_A(\omega) \leq \liminf_{n\to\infty} (\lambda + (H^n \cdot S)_\infty(\omega)).$$

- Throw martingale measure $\mathbb{P}$ at both sides:

$$\begin{aligned}\mathbb{P}(A) &\leq \mathbb{E}_\mathbb{P}\left[\liminf_{n\to\infty} (\lambda + (H^n \cdot S)_\infty)\right] \\ &\leq \liminf_{n\to\infty} \mathbb{E}_\mathbb{P}\left[(\lambda + (H^n \cdot S)_\infty)\right] \leq \lambda.\end{aligned}$$

Link with (NA1)

- By scaling: $\overline{P}(A) = 0$ iff $\exists (H^n) \subset \mathcal{H}_1$ with

$$\liminf_{n\to\infty}(1 + (H^n \cdot S)_\infty) \geq \infty \cdot \mathbb{1}_A.$$

- Recall: $\mathbb{P}$ satisfies (NA1) (= (NUPBR)) if

$$\{1 + (H \cdot S)_\infty : H \in \mathcal{H}_1\}$$

  bounded in $\mathbb{P}$-probability.
- $\sup_{\mathbb{P}\ (\mathrm{NA1})} \mathbb{P}(A) \not\leq \overline{P}(A)$, but:

Lemma (P-Prömel '15)

Let $A \in \mathcal{F}_\infty$. If $\overline{P}(A) = 0$, then $\mathbb{P}(A) = 0$ for all $\mathbb{P}$ with (NA1).

(NA1) is minimal assumption any market model should fulfill.

(Ankirchner '05, Karatzas-Kardaras '07, Ruf '13, Fontana-Runggaldier '13, Imkeller-P. '15...)

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4. Pathwise stochastic calculus

Typical price paths

Property (P) holds for typical price paths if it is violated on a null set.

Observations due to Vovk:

- Typical price paths have no points of increase.
- Typical price paths have finite p-variation for $p > 2$.
- Typical price paths have a quadratic variation $[S]$.

Observations due to P.-Prömel:

- Typical price paths are rough paths in the sense of Lyons.
- Typical price paths have nice local times.

Typical price paths have quadratic variation

$$
\begin{aligned}
[S]^n_t := \sum_{k=0}^{\infty} (S_{\tau^n_{k+1}\wedge t} - S_{\tau^n_k \wedge t})^2 \\
= S_t^2 - S_0^2 - 2\sum_{k=0}^{\infty} S_{\tau^n_k\wedge t}(S_{\tau^n_{k+1}\wedge t} - S_{\tau^n_k\wedge t}) \\
= S_t^2 - S_0^2 - 2(S^n \cdot S)_t
\end{aligned}
$$

- Deterministic τ^n_k: no chance for convergence.
- Set $\tau^n_0 = 0$, $\tau^n_{k+1} = \inf\{t \geq \tau^n_k : |S_t - S_{\tau^n_k}| \geq 2^{-n}\}$;
- $[S]^{n+1}_t - [S]^n_t = 2((S^n - S^{n+1}) \cdot S)_t$.
- Bounds on $(S^n - S^{n+1})$ and $S_{\tau^n_{k+1}} - S_{\tau^n_k}$
  + a priori control on $\#\{\tau^n_k : k\}$
  + pathwise Hoeffding inequality:
  convergence of $[S]^n(\omega)$ for typical price paths ω Vovk '12
  (continuous paths or bounded jumps).

Pathwise Dambis Dubins-Schwarz Theorem

$\Omega = C([0,\infty),\mathbb{R})$, define time-change operator $\mathfrak{t}\colon \Omega \to \Omega$:

$$[\mathfrak{t}(\omega)]_t = t, \quad t \in [0,\infty).$$

Theorem (Vovk '12)

$\mathbb{W}$ *Wiener measure, $F \geq 0$ measurable, $c \in \mathbb{R}$:*

$$\overline{E}[(F \circ \mathfrak{t})\mathbb{1}_{\{S_0=c,[S]_\infty=\infty\}}] = \int_\Omega F(c+\omega)\mathbb{W}(\mathrm{d}\omega),$$

where

$$\overline{E}(F) := \inf\left\{\lambda : \exists (H^n)_n \subseteq \mathcal{H}_\lambda \text{ s.t. } \liminf_{n\to\infty}(\lambda + (H^n \cdot S)_\infty(\omega)) \geq F(\omega)\forall\omega\right\}.$$

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Model free concentration of measure

$\Omega = C([0, T], \mathbb{R}^d)$.

- Want "stochastic integral".
- For step functions F ok. Extension?

Lemma (Łochowski-P.-Prömel '16)

F adapted step function, then

$$\overline{P}\left(\|F \cdot S\|_\infty \geq a\sqrt{b}, \int_0^T F_t^{\otimes 2} \mathrm{d}[S]_t \leq b\right) \leq 2e^{-a^2/2}.$$

Pathwise Hoeffding: $a_1, \dots, a_n \in \mathbb{R}$ with $|a_n| \leq c$, then $\forall \lambda$ there exist $b_\ell = b_\ell(a_1, \dots, a_{\ell-1}, c, \lambda)$ with

$$1 + \sum_{k=1}^{\ell} b_k a_k \geq \exp\left(\lambda \sum_{k=1}^{\ell} a_k - \frac{\lambda^2}{2} \ell c^2\right) \forall \ell.$$

Now discretize S and apply Hoeffding.

Topologies on path space

- $d_{\mathrm{QV}}(F,G) := \overline{E}\Big(\int_0^T (F_t - G_t)^{\otimes 2} \mathrm{d}[S]_t \wedge 1\Big)$:
  complete metric space of integrands.
- $d_\infty(X,Y) := \overline{E}(\|X-Y\|_\infty \wedge 1)$:
  complete metric space of (possible) integrals.
- $F \mapsto F \cdot S$ continuous on (step functions, d_{QV}), extends to closure.
- No idea how closure looks like. Need to localize:

$$d_{\mathrm{QV,loc}}(F,G) := \sum_{n=1}^\infty 2^{-n} \overline{E}\Big(\Big(\int_0^T (F_t - G_t)^{\otimes 2} \mathrm{d}[S]_t \wedge 1\Big) \mathbb{1}_{[S]_T \le n}\Big).$$

  Now closure contains càglàd paths, open problem if also bounded predictable processes.
- Convergence of integrals for typical price paths, Itô's formula, integral is independent of approximating sequence, ...

What about jumps?

Strategy H is λ-admissible if

$$(H \cdot S)_t(\omega) = \sum_{n=0}^{\infty} F_n(\omega)[S_{\tau_{n+1}\wedge t}(\omega) - S_{\tau_n \wedge t}(\omega)] \geq -\lambda \qquad \forall t, \omega.$$

- $\Omega = D([0,T], \mathbb{R}^d)$: no admissible H!
- Ω paths with bounded jumps: Vovk '12.

Canonical: $D_+([0,T], \mathbb{R}^d)$ (positive càdlàg paths).

- means no short-selling;
- want $[S]$, but all constructions of $[S]$ use short-selling.
- Way out: relax problem to allow "little bit of short-selling". Take relaxation away $\Rightarrow$ $[S]$ ex for typical positive càdlàg price paths Łochowski-P.-Prömel '16.

Integration with jumps

$\Omega = D_{S_0,+}([0,T],\mathbb{R}^d)$.

- Again canonical definition of $F \cdot S$ for step functions F. Extension?
- Pathwise Hoeffding no longer works: $F_{\tau_k}(S_{\tau_{k+1}} - S_{\tau_k})$ unbounded.

Instead: pathwise B-D-G inequality of Beiglböck-Siorpaes '15

$a_1, \dots, a_n \in \mathbb{R}$, then there exist $b_\ell = b_\ell(a_1, \dots, a_{\ell-1})$ with

$$\sum_{k=1}^{\ell} b_k a_k \geq \max_{m \leq \ell} \Big| \sum_{k=1}^{m} a_k \Big| - 6 \sqrt{\sum_{k=1}^{\ell} a_k^2} \qquad \forall \ell.$$

From here:

$$\overline{P}\Big(\|F \cdot S\|_\infty \geq a, \int_0^T F_t^{\otimes 2} \mathrm{d}[S]_t \leq b, \|F\|_\infty \leq c \Big) \leq (1 + |S_0|) \frac{6\sqrt{b} + 2c}{a}.$$

Extension to càglàd F as before.

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Pathwise stochastic calculus

- Measure free calculus excludes “nontypical price paths” at every step $\Rightarrow$ not pathwise.
- Föllmer '81: pathwise Itô calculus.
- Lyons '98 and Gubinelli '04: generalization to rough paths.
- Can we implement / extend this here?

Pathwise Itô formula (no probability)

Consider $f \in C^2(\mathbb{R}, \mathbb{R})$, partition π. Taylor expansion:

$$
\begin{aligned}
f(S(t)) - f(S(0)) &= \sum_{t_j \in \pi} f(S(t_{j+1})) - f(S(t_j)) \\
&= \sum_{t_j \in \pi} f'(S(t_j))(S(t_{j+1}) - S(t_j)) + \frac{1}{2} \sum_{t_j \in \pi} f''(S(t_j))(S(t_{j+1}) - S(t_j))^2 \\
&\quad + \sum_{t_j \in \pi} \varphi(|S(t_{j+1}) - S(t_j)|)(S(t_{j+1}) - S(t_j))^2.
\end{aligned}
$$

Pathwise Itô formula (Föllmer (1981))

S has quadratic variation along sequence of partitions (π^n) if

$$\sum_{t_j \in \pi^n} (S(t_{j+1}) - S(t_j))^2 \delta_{S_{t_j}}$$

converges vaguely to (non-atomic) μ. Write $[S](t) := \mu([0,t])$.

Theorem (Föllmer '81)

If S has quadratic variation along (π^n) and $f \in C^2$, then

$$f(S(t)) = f(S(0)) + \int_0^t f'(S(s))\mathrm{d}S(s) + \frac{1}{2}\int_0^t f''(S(s))\mathrm{d}[S](s).$$

Pathwise Itô formula

Without probability Föllmer constructed

$$\int_0^t f'(S(s))\mathrm{d}S(s) := \lim_{n\to\infty} \sum_{t_j\in\pi^n} f'(S(t_j))(S(t_{j+1}\wedge t) - S(t_j\wedge t)).$$

Natural (pathwise) extensions:

1. Higher dimensions:

   Lyons '98, Gubinelli '04, P.-Prömel '16

2. Path-dependent functionals f:

   Cont-Fournié '10, Imkeller-Prömel '15

3. Less regular functions f:

   Wuermli '80, P.-Prömel '15, Davis-Obłój-Siorpaes '15

$\Rightarrow$ Applications to robust and model-free finance:

Bick-Willinger '94, Lyons '95, ..., Davis-Obłój-Raval '14, Schied-Voloshchenko '15,...

Pathwise Tanaka formula (Wuermli (1980))

Let $f(x) = \int_0^x f'(y)\mathrm{d}y$ and $b \geq a$:

$$f(b) - f(a) = f'(a)(b-a) + \int_{(a,b]} (f'(x) - f'(a))\mathrm{d}x$$
$$= f'(a)(b-a) + \int_{(a,b]} (b-u)\mathrm{d}f'(u).$$

So for $S \in C([0,\infty), \mathbb{R})$ and any partition π:

$$f(S(t)) - f(S(0)) = \sum_{t_j \in \pi} f'(S(t_j \wedge t))(S(t_{j+1} \wedge t) - S(t_j \wedge t))$$
$$+ \int_{-\infty}^{\infty} \sum_{t_j \in \pi} \left(\mathbb{1}_{(\!|S(t_j \wedge t), S(t_{j+1} \wedge t)]\!]}(u) |S(t_{j+1} \wedge t) - u| \right) \mathrm{d}f'(u).$$

Pathwise local time

Define discrete pathwise local time

$$L_t^\pi(S,u) := \sum_{t_j\in\pi} \mathbb{1}_{(\!(S(t_j\wedge t),S(t_{j+1}\wedge t)]\!]}(u)|S(t_{j+1}\wedge t)-u|.$$

Then:

$$f(S(t))-f(S(0)) = \sum_{t_j\in\pi} f'(S(t_j\wedge t))(S(t_{j+1}\wedge t)-S(t_j\wedge t)) + \int_{-\infty}^{\infty} L_t^\pi(S,u)\mathrm{d}f'(u).$$

L^p-local time

Let (π^n) be sequence of partitions with mesh size $\Rightarrow 0$.

$L(S)\colon [0,\infty) \times \mathbb{R} \to \mathbb{R}$ is a L^p-local time of S along (π^n)
if $L_t^{\pi^n}(S,\cdot)$ converge weakly in $L^p(\mathrm{d}u)$ to $L_t(S,\cdot)$ for all $t \in [0,\infty)$.

Theorem (Wuermli '80, Davis-Obłój-Siorpaes '15)

For $f \in W^{2,q}$ (Sobolev space) with $1/q + 1/p = 1$ we have

$$f(S(t)) = f(S(0)) + \int_0^t f'(S(s))\mathrm{d}S(s) + \int_{-\infty}^{\infty} f''(u)L_t(S,u)\mathrm{d}u.$$

Remark: Existence of L^p-local time implies quadratic variation along (π^n).
Converse is wrong!

Continuous local time

Let (π^n) be sequence of partitions with mesh size $\Rightarrow 0$.

S has a continuous local time along (π^n) if

- $L_t^{\pi^n}(S, \cdot)$ converges uniformly to continuous limit $L_t(S, \cdot)$ $\forall t$,
- $(t, u) \mapsto L_t(S, u)$ is continuous.

Theorem (P.-Prömel '15)

Let f be absolutely continuous with f' of bounded variation. Then

$$f(S(t)) = f(S(0)) + \int_0^t f'(S(u))\mathrm{d}S(u) + \int_{-\infty}^{\infty} L_t(u)\mathrm{d}f'(u).$$

Local time of finite p-variation

Recall

$$\|f\|_{p\text{-var}} = \sup\left\{\left(\sum_{k=1}^{n}|f(u_k)-f(u_{k-1})|^p\right)^{1/p} : -\infty < u_0 < \ldots < u_n < \infty\right\}.$$

For $p \geq 1$ the set $\mathcal{L}_{c,p}(\pi^n)$ consists of all $S \in C([0,T],\mathbb{R})$

- having a continuous local time $L_t(S,u)$ with
- discrete local times $(L_t^{\pi^n})$ of uniformly bounded p-variation, uniformly in $t \in [0,T]$ for all $T > 0$, i.e.

$$\sup_{n\in\mathbb{N}}\sup_{t\in[0,T]}\|L_t^{\pi^n}(\cdot)\|_{p\text{-var}} < \infty.$$

Pathwise generalized Itô formula

Theorem (P.-Prömel '15)

Let $p, q \geq 1$ be such that $\frac{1}{p} + \frac{1}{q} > 1$ and let $S \in \mathcal{L}_{c,p}(\pi^n)$.

Let $f : \mathbb{R} \to \mathbb{R}$ be absolutely continuous with f' of locally finite q-variation. Then

$$f(S(t)) = f(S(0)) + \int_0^t f'(S(s))\mathrm{d}S(s) + \int_{-\infty}^{\infty} L_t(u)\mathrm{d}f'(u),$$

where $\mathrm{d}f'(u)$ denotes Young integration and where

$$\int_0^t f'(S(s))\mathrm{d}S(s) := \lim_{n\to\infty} \sum_{t_j \in \pi^n} f'(S(t_j))(S(t_{j+1} \wedge t) - S(t_j \wedge t)).$$

But do such nice local times exist?

Consider $\Omega = C([0,T],\mathbb{R})$ and define random partition π^n via

$$\tau_0^n := 0, \quad \tau_{k+1}^n := \inf\{t \geq \tau_k^n : |S_t - S_{\tau_k^n}| \geq 2^{-n}\}.$$

Then

$$L_t^{\pi^n}(S,u) = (S_t - u)^- - (S_0 - u)^- + \sum_{j=0}^{\infty} \mathbb{1}_{(-\infty,u)}(S_{\tau_j^n})[S_{\tau_{j+1}^n \wedge t} - S_{\tau_j^n \wedge t}],$$

where we recall that

$$L_t^{\pi^n}(S,u) = \sum_{j=0}^{\infty} \mathbb{1}_{(\!(S_{\tau_j^n \wedge t}, S_{\tau_{j+1}^n \wedge t}]\!]}(u)|S_{\tau_{j+1}^n \wedge t} - u|.$$

Local times for typical price paths

Theorem (P.-Prömel '15)

Let $T > 0$, $\alpha \in (0, 1/2)$*. For typical price paths* $\omega \in \Omega$*,*

- *the discrete local time* L^{π^n} *converges uniformly in* $(t, u) \in [0, T] \times \mathbb{R}$ *to a limit* $L \in C([0, T], C^\alpha(\mathbb{R}))$*,*
- *there exists* $C = C(\omega) > 0$ *with*

$$\|L^{\pi^n} - L\|_{L^\infty([0,T]\times\mathbb{R})} \leq C 2^{-n\alpha},$$

- L^{π^n} *has uniformly bounded p-variation for* $p > 2$*:*

$$\sup_{n\in\mathbb{N}} \sup_{t\in[0,T]} \|L_t^{\pi^n}(\cdot)\|_{p\text{-var}} < \infty.$$

Conclusion

- Vovk formulates continuous time math finance without probability.
- Get interesting properties of "typical price paths"...
- ...but also quantitative results (pathwise Dambis Dubins-Schwarz, model free pricing-hedging duality).
- Probability free stochastic calculus based on model free analogues of Itô's isometry.
- Pathwise calculus of Föllmer extended via pathwise local times, those exist for typical price paths.

Thank you