Boxcounting

Combinatorial questions and fun facts about D-dimensional partitions form a nice projection from what puzzles/excites modern mathematical physicist to the plane of what can be easily grasped and, hopefully, appreciated by all those who love mathematics

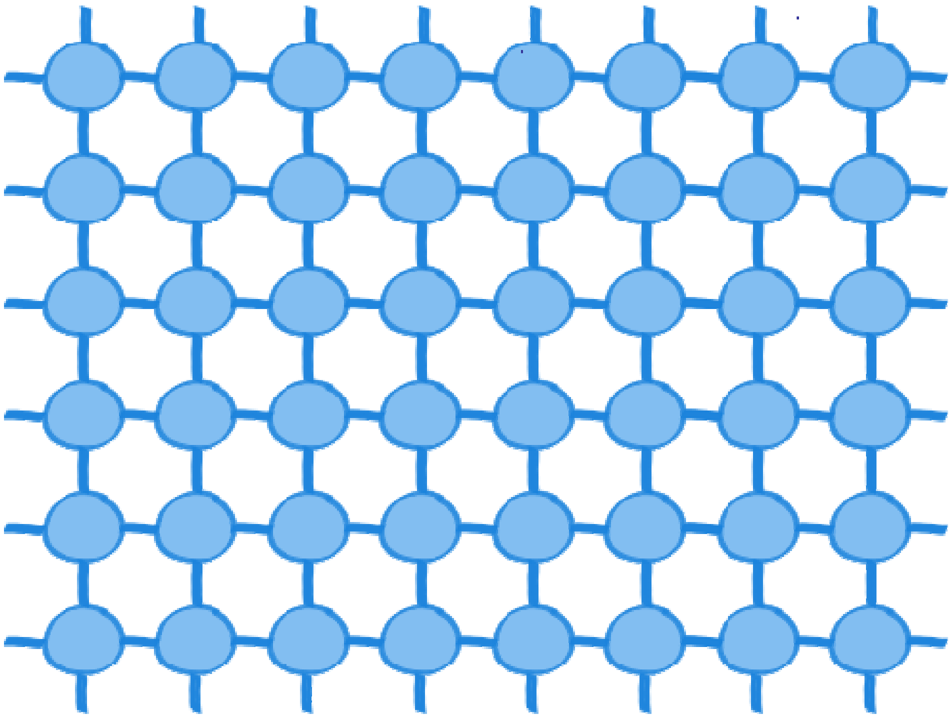
I. Kinematics

d-dimensional partitions may be explained e.g. by relating them to the corner of a d-dimensional cubic crystal

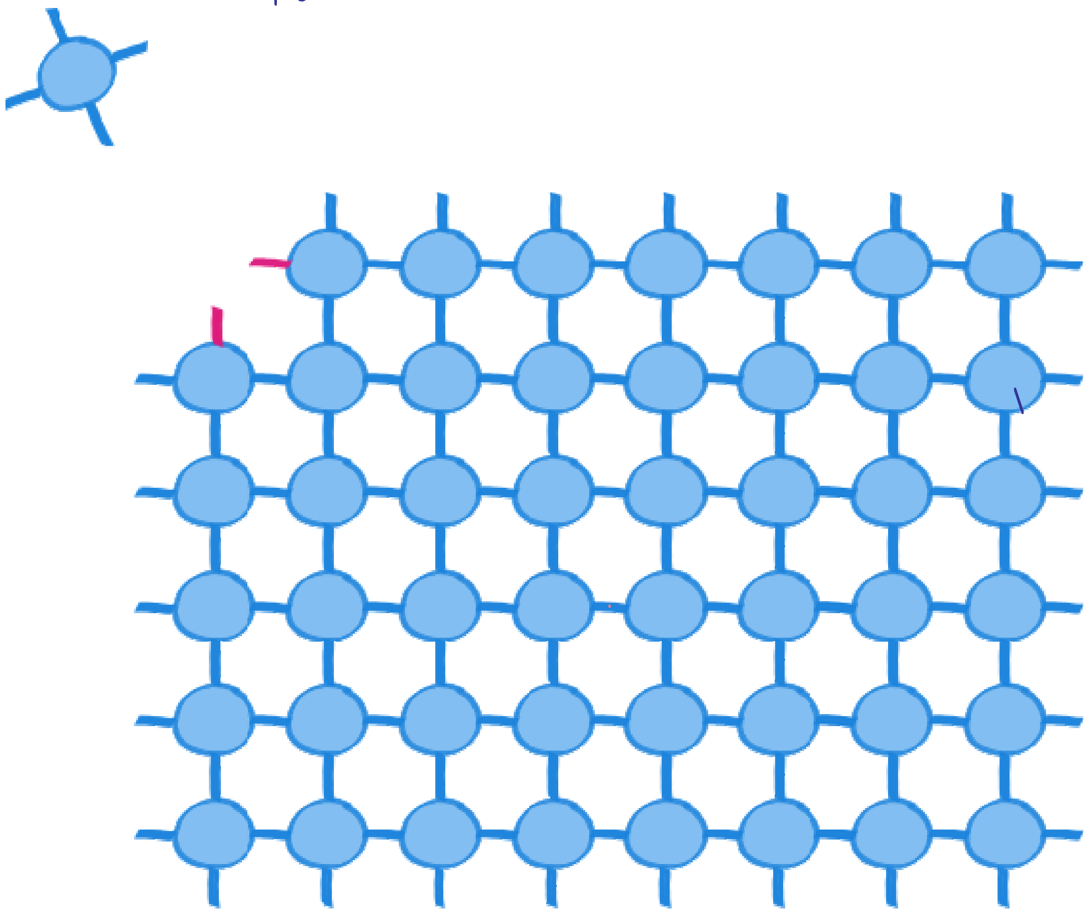
Here $d=2$

Every dot in the interior has $2d$ neighbors

Bonds along the boundary broken

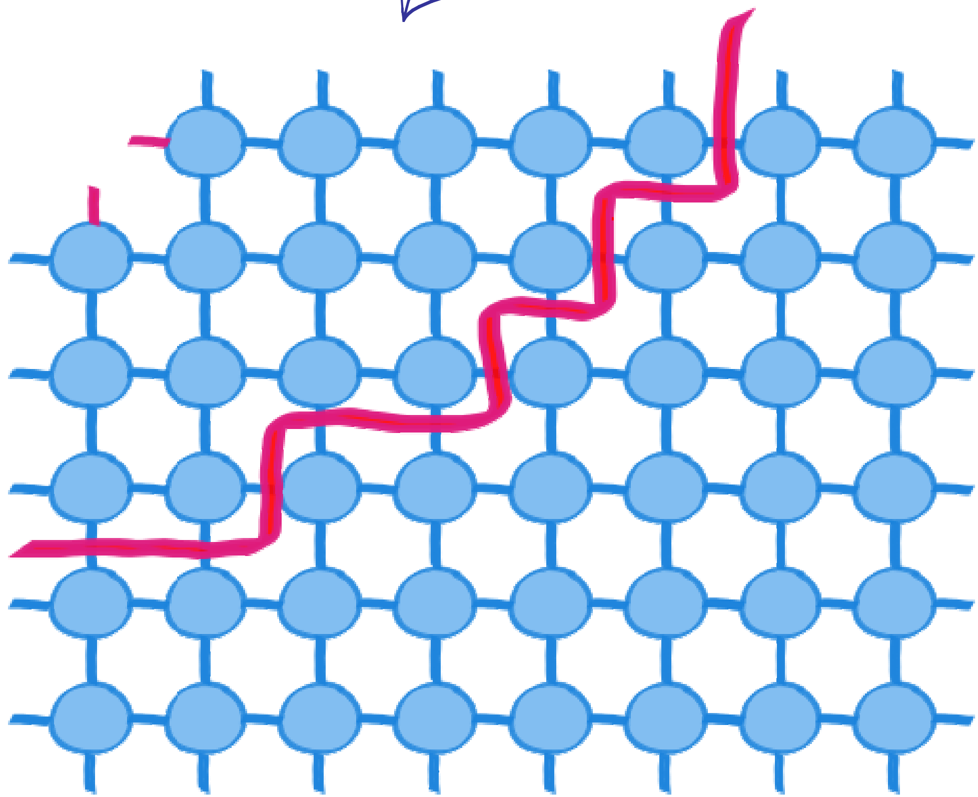


if the corner atom is removed



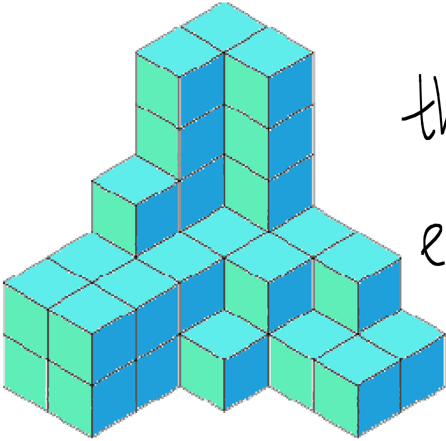
the number of broken bonds (= energy) remains the same

same with removing all these atoms



nonzero entropy at zero temperature

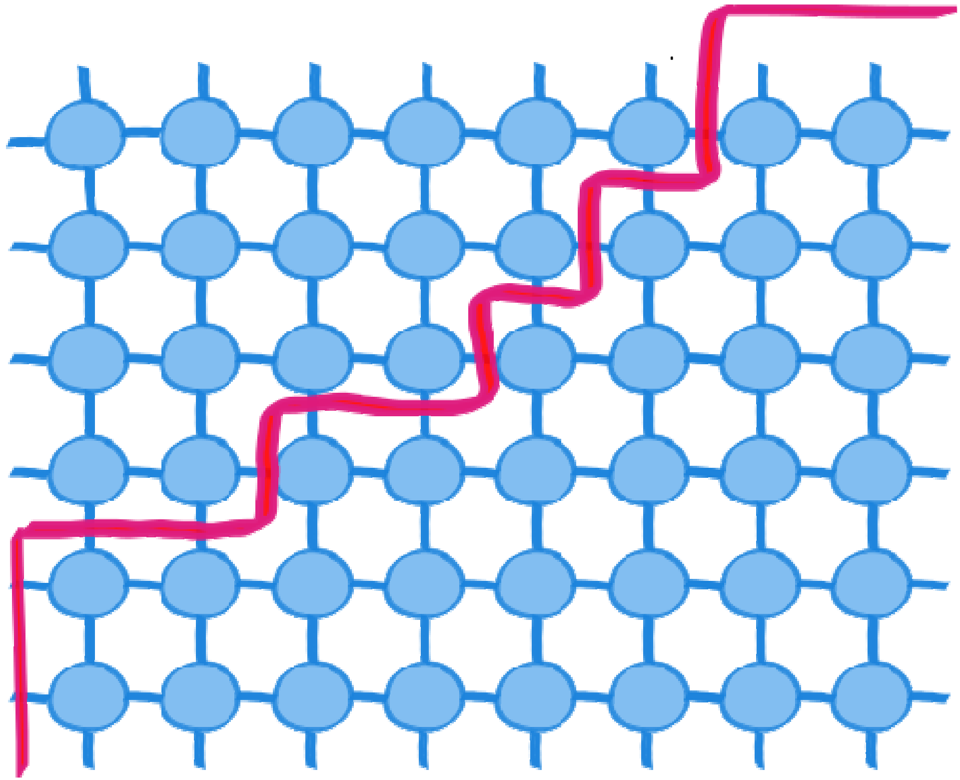
to be revisited later



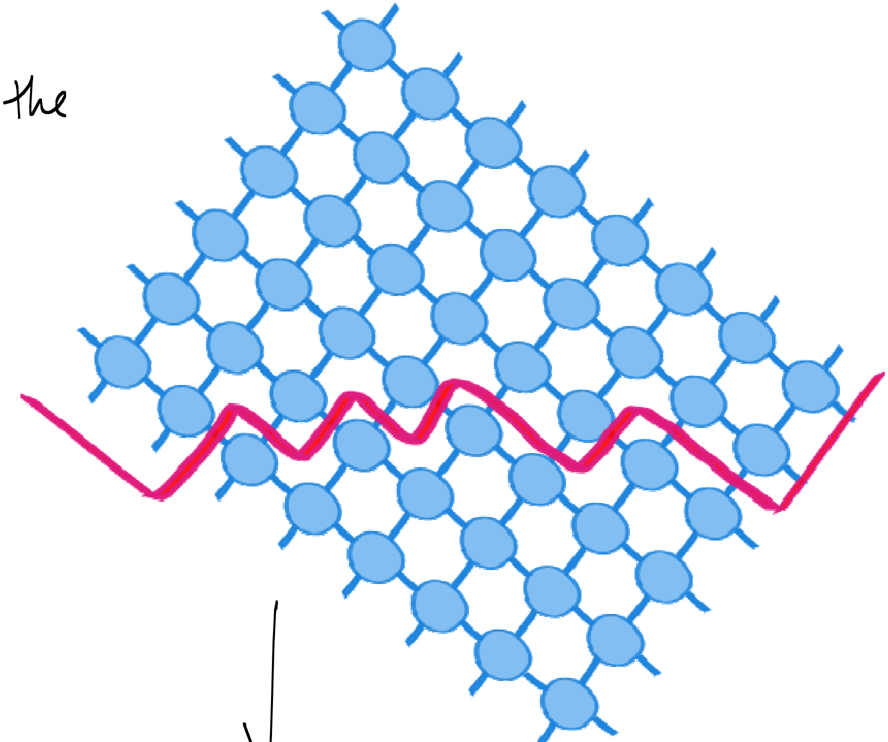
these configurations may be called partitions by extension of the d=2 concept

$$6 + 5 + 4 + 2 = 17$$

# of missing atoms

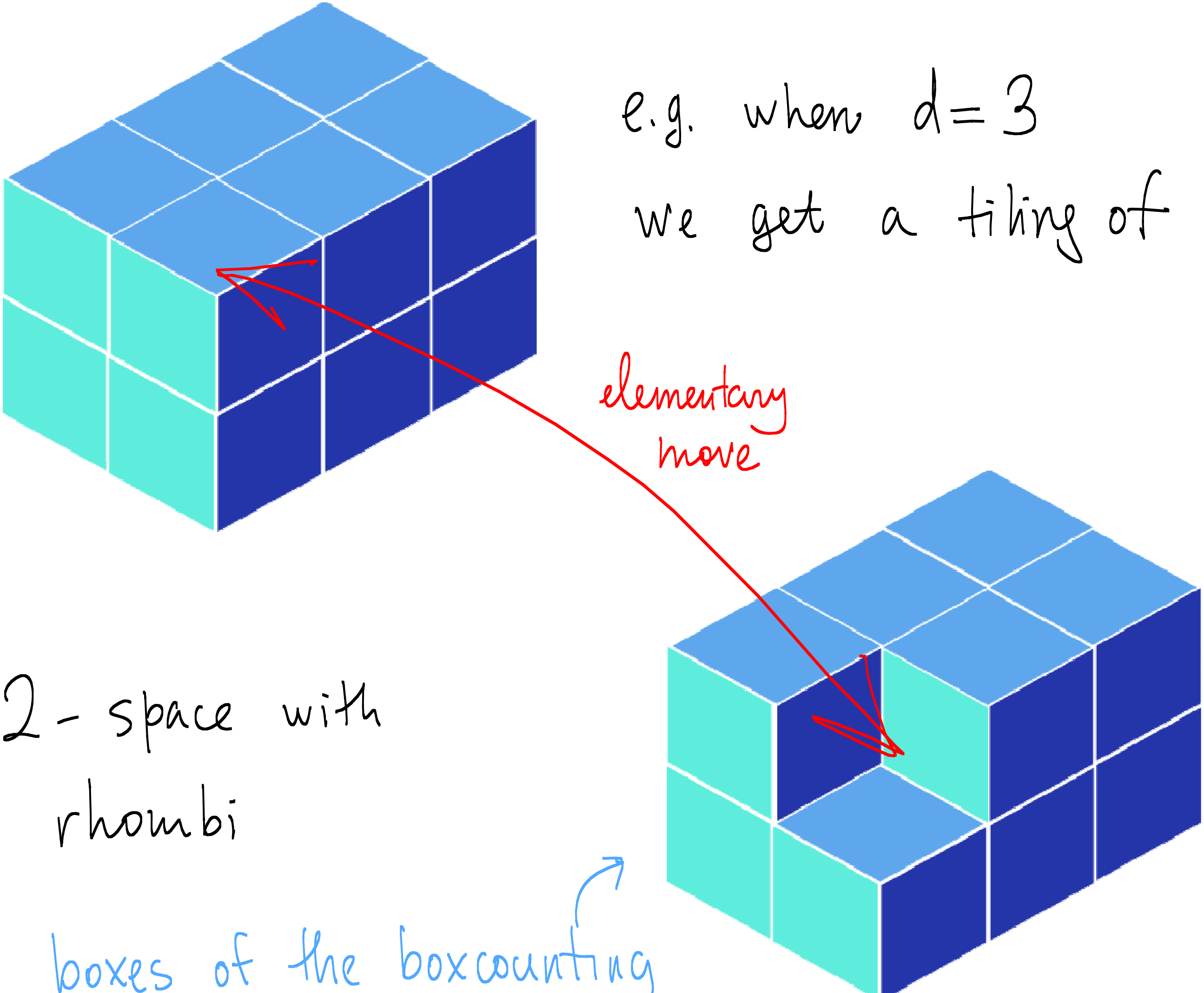


via projection in the
$(1,1,\ldots,1)$
direction is
related to



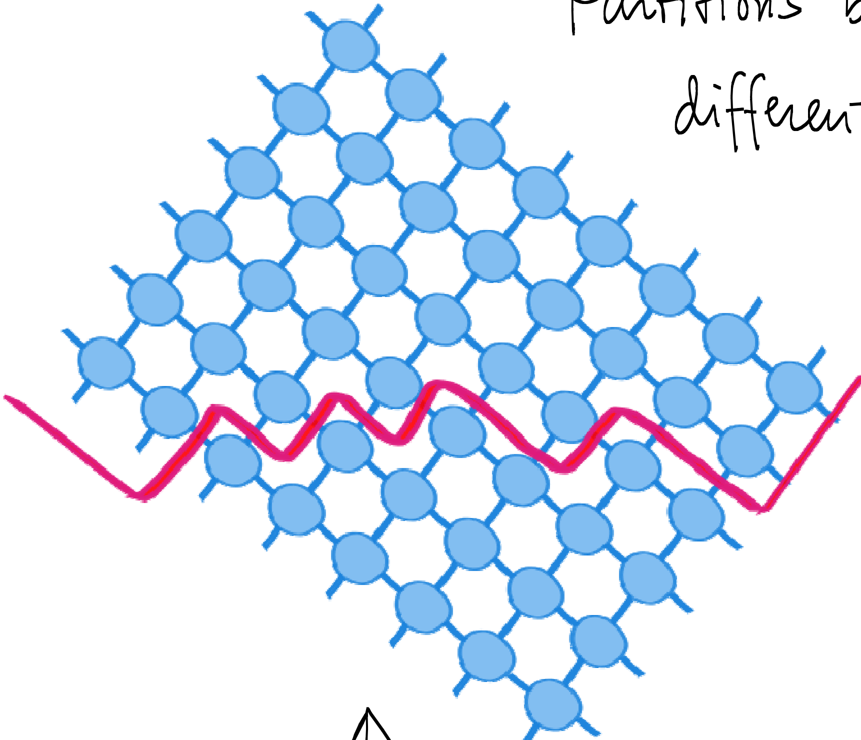
tiling in $(d-1)$-space

e.g. when $d=3$ we get a tiling of 2-space with rhombi



boxes of the boxcounting

Partitions behave very differently in different dimensions



(d-1)-dimensional Gaussian field

scales

a random walk with (d-1)-dimensional time fluctuations of a $(d-1)$-dimensional Gaussian are given by

Brownian motion

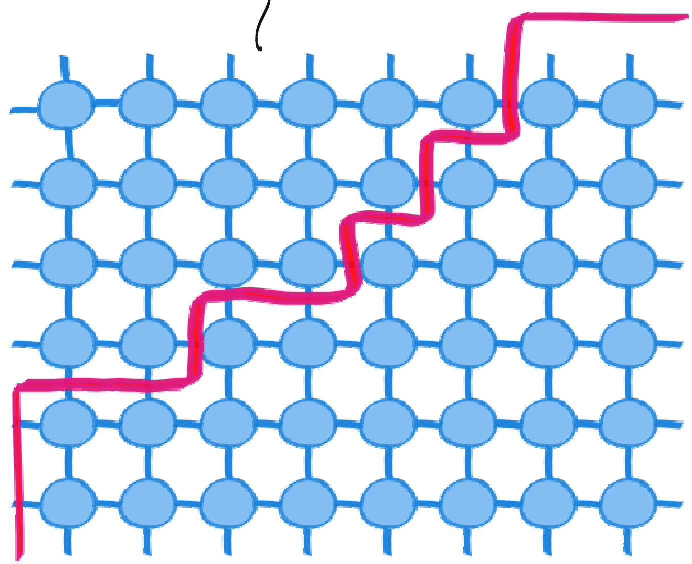
Green function of Δ $= \begin{cases} \sim r, & d=2 \\ \ln r, & d=3 \quad \text{CFT} \\ \dfrac{1}{r^{d-3}} & d>3 \end{cases}$

height fluctuations decay

rigidity of tilings ?

This is reflected in combinatorial and other properties of the basic partition function

$$Z = \sum_{\substack{d\text{-dimensional} \\ \text{partitions } \pi}} q^{|\pi|}$$



where $(-\ln q)$ may be interpreted as the energy price of removing an atom

$d=2$ Euler $\quad Z_2 = \prod_{n=1}^{\infty} \frac{1}{1-q^n}$

$d=3$ McMahon $\quad Z_3 = \prod_{n\geq 1} \frac{1}{(1-q^n)^n}$

$d \geq 4$ no similar formula, even $q \to 1$ behavior unclear

we will be interested in similar formulas with weights

and also

why they factor?

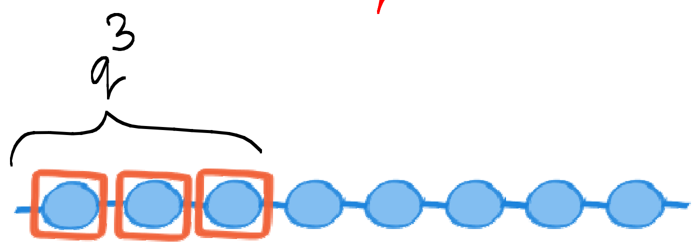
factorizations means an identification

partitions $\longleftrightarrow$ some independent degrees of freedom

This is obvious for $d = 1$

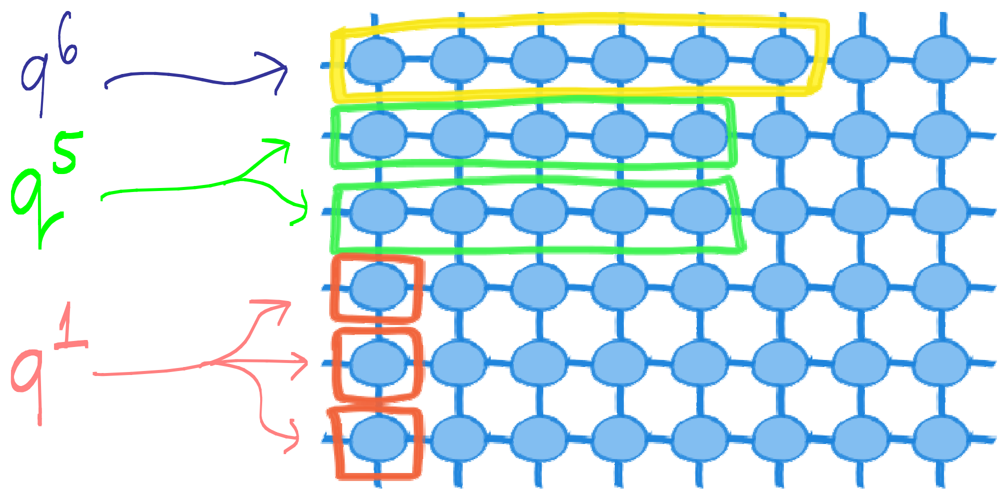
$$Z_1 = 1 + q + q^2 + q^3 + \dots = \frac{1}{1-q}$$

one boson of weight q



Similarly obvious for $Z_{d=2} = \prod_{n\geq 1} \frac{1}{1-q^n}$

which suggests bosons of weights $q, q^2, q^3, \dots$



but this is not so easy for $d=3$. Try!

note that the independent degrees of freedom for $d = 1, 2, 3$ are naturally identified with polynomials in $n = 0, 1, 2$ variables

$d = 2$

$$\begin{array}{cccccc} 1 & x & x^2 & x^3 & x^4 & \cdots \\ q & q^2 & q^3 & q^4 & q^5 & \cdots \end{array} \qquad \frac{q}{1-q}$$

$$\frac{1}{(1-q)(1-q^2)(1-q^3)\cdots}$$

$d = 3$

$$\begin{array}{cccc} 1 & x & x^2 & x^3 \\ y & xy & x^2y & x^3y \\ y^2 & xy^2 & x^2y^2 & \\ y^3 & xy^3 & & \end{array}$$

$q \quad q^2 \quad q^3 \quad q^4$

$$\frac{q}{(1-q)^2}$$

$$\frac{1}{(1-q)(1-q^2)^2(1-q^3)^3\cdots}$$

what really fluctuates is a bosonic field φ

and its derivatives $\dfrac{\partial^5 \varphi}{\partial^3 x \, \partial^2 y} \rightsquigarrow q^6$

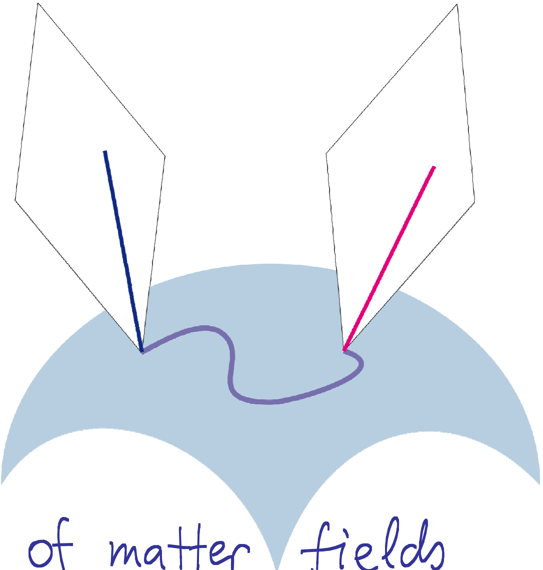
in some number of dimensions

In more complicated cases there will be many bosons and fermions in place of φ, so Z will be a more complicated, but similarly infinite product

Such interpretations come from the study of

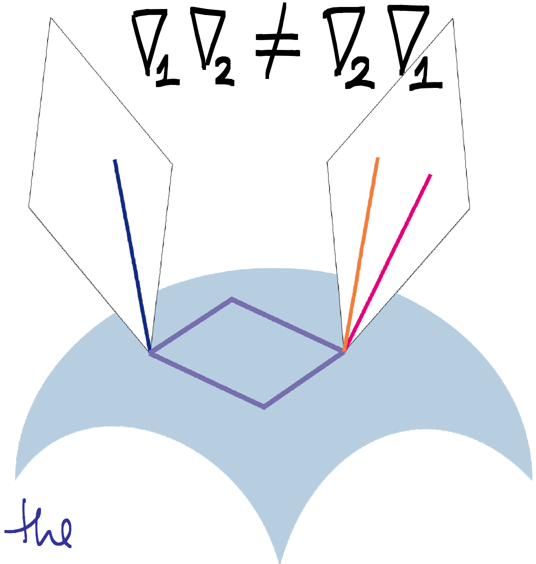
II. Partitions in gauge theories

gauge theories are absolutely fundamental to all modern high-energy physics



gauge fields are connections on bundles of matter fields

Poetically, they hold them together while prosaically they let us differentiate and so quantify kinetic energy connections can have curvature and the L^2-norm $\|F\|^2$ of the curvature F is the natural (Yang-Mills) energy of the gauge field



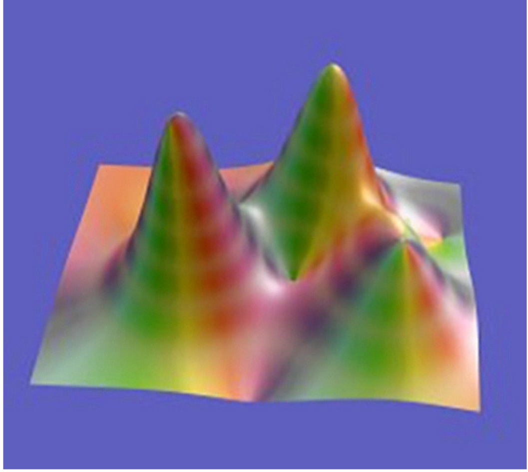
E.g. the familiar $\|dA\|^2$ energy of the electro-magnetic field, i.e. U(1)-gauge theory

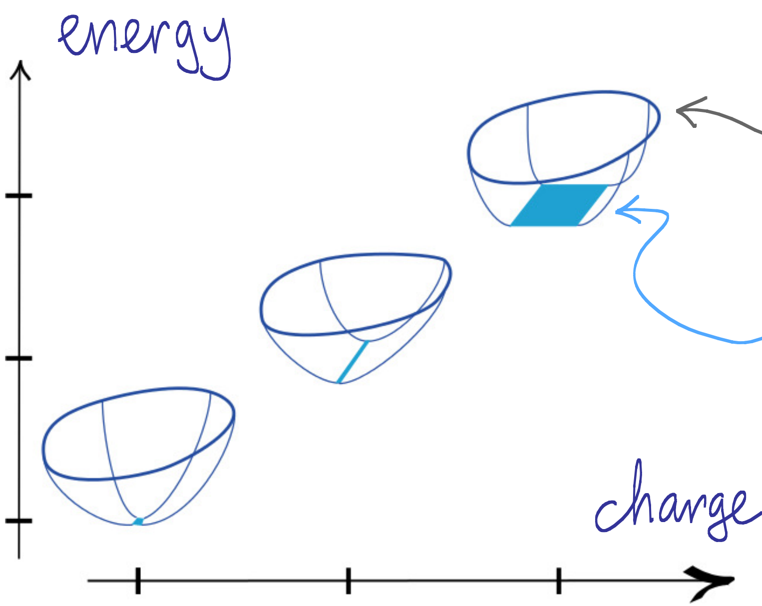
For nonabelian gauge fields, $F = dA + [A, A]$ reflecting the nonlinear self-interacting nature of the theory

degree 4 in $\|F\|^2$

Instantons in 4d Euclidean gauge theories exactly minimize the energy for given topological charge

(= 3 here)



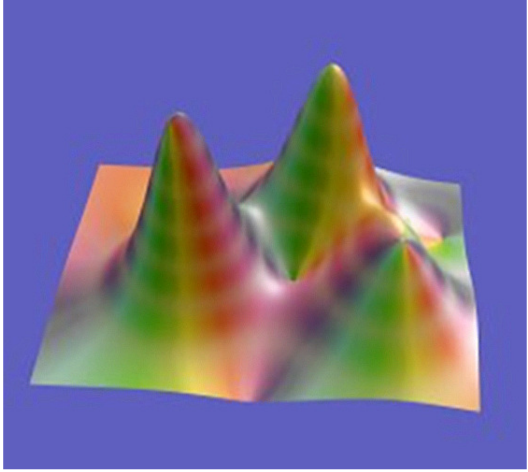


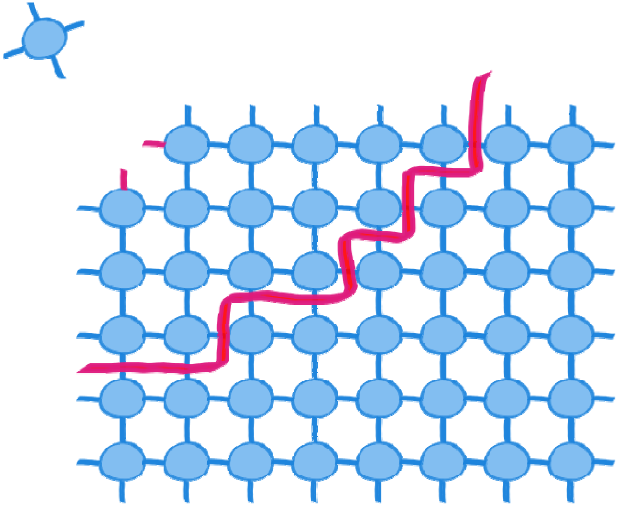
all connections of given charge / equivalence

instantons, a finite dimensional manifold

dim $\propto$ charge

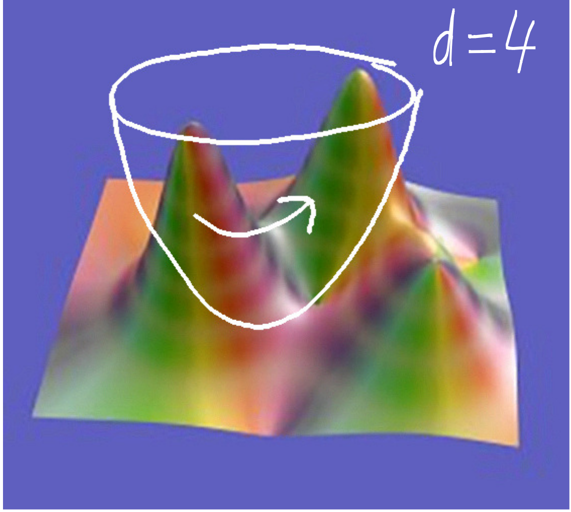
again
nonzero entropy
at zero temperature !





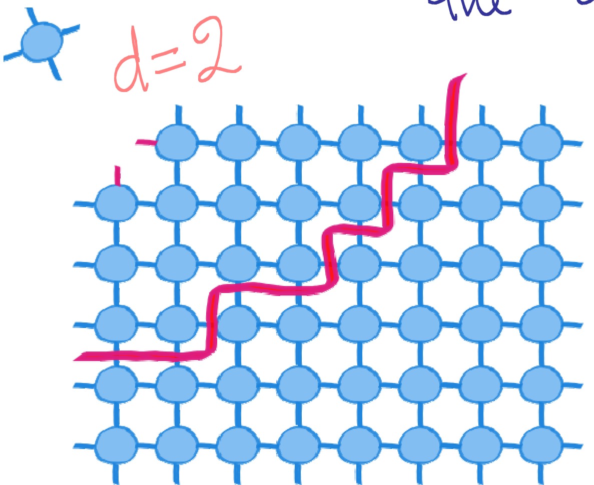
instantons may be seen
as a very non linear gas
of bumps, each having a
certain variable size and
gauge degrees of freedom

it is unnatural to put the instanton gas in a box. Instead (following Nekrasov) one puts in in a well by adding the kinetic energy of rotations



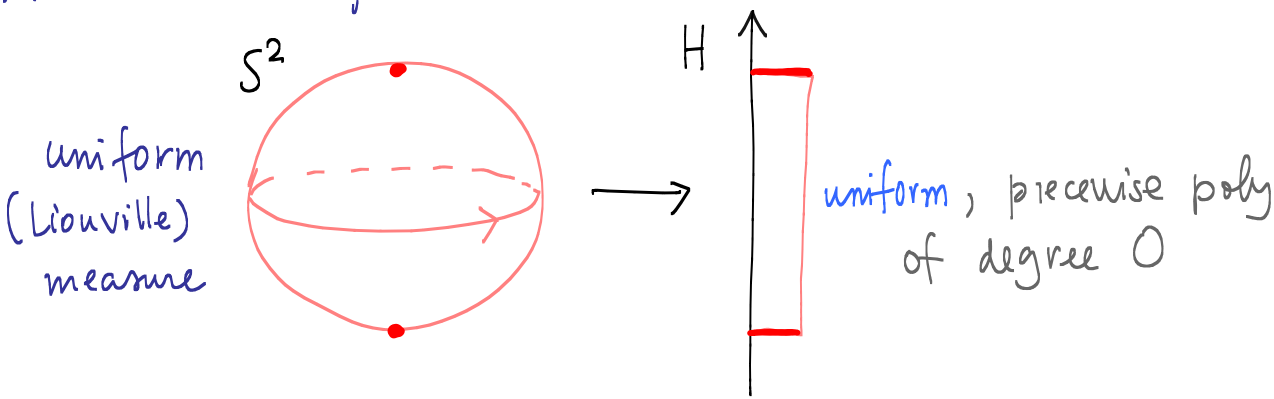
$$\omega = \begin{pmatrix} 0 & -\omega_1 & & & \\ \omega_1 & 0 & & & \\ & & 0 & -\omega_2 & \\ & & \omega_2 & 0 & \\ & & & & \ddots \end{pmatrix}$$

Then: (*) partition function may be computed exactly in the $\omega \to \infty$ limit (localization)



(**) in the $\omega \to \infty$ limit all instantons shrink to size $= 0$ and get on top of each other at the origin in a pattern described by $d/2$-dimensional partitions

This exact computation is based on a general theorem of Archimedes - Atiyah - Bott - Duistermaat - Heckman - Berline - Vergne



which says that the energy (Hamiltonian H) of a periodic motion (rotation) is distributed with a piecewise polynomial density

hence $\int e^{\beta H}$ may be computed exactly from its $\beta \to \infty$ limit from the contributions of jumps = fixed points so

here $d = 4, 6, \underline{\underline{8}}$

partition function Z of
an instanton gas in
d dimensions

partition function
of $d/2$ - dimensional
partition with weight
that depends on $\omega_1, \dots, \omega_{d/2}$
and parameters
in the gauge theory

product

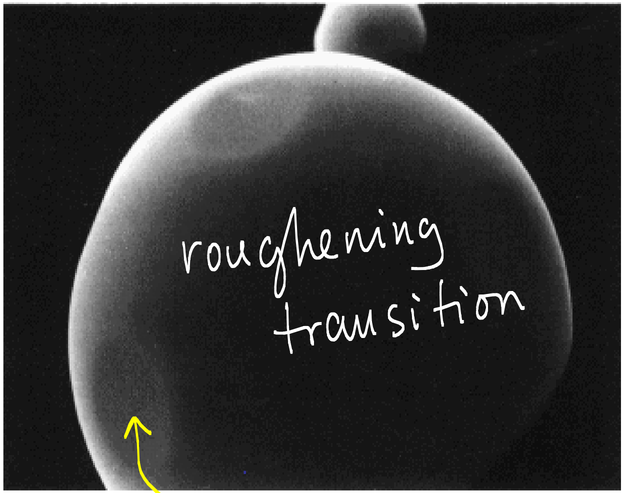
in some dual description,
fluctuating fields

$$\frac{\partial^7 \varphi}{\partial x_1^2 \, \partial x_2^3 \dots}$$

in some other number of
dimensions

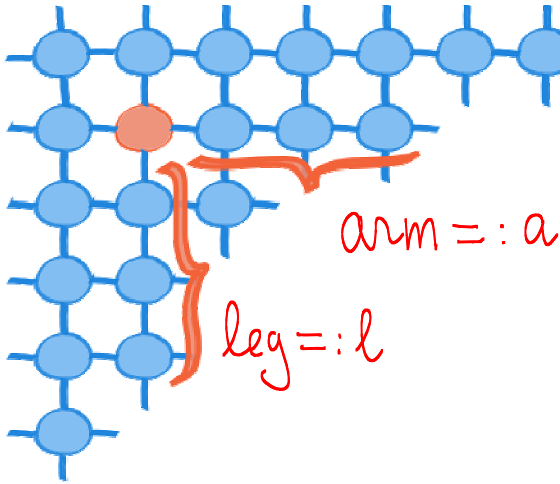
in this discussion, we've been ignoring the gauge variables which sometimes translate into periodic structures in the crystal, allowing for interesting parallels like

singularities of the partition function $\longleftrightarrow$



mass of some particle $\longleftrightarrow$ area of a facet $\longrightarrow 0$

III Example



Weight a 2-dimensional partition by the following

$$\text{weight} = \prod_{\text{all dots } \bullet} q \, \frac{1 - m\, \omega_1^{a(\bullet)+1}\, \omega_2^{-\ell(\bullet)}}{1 - \omega_1^{a(\bullet)+1}\, \omega_2^{-\ell(\bullet)}} \times$$

same with $a \leftrightarrow \ell$

a 3-parameter deformation of random matrices

Theorem (Nekrasov-O-Carlsson)

$$Z := \sum_{\text{2d partition}} \text{weight} \stackrel{!}{=} \prod_{n} \frac{1}{(1-E_n)}$$

where

$$\sum_{n} E_n = q \frac{(1-m\omega_1)(1-m\omega_2)}{(1-mq)(1-\omega_1)(1-\omega_2)}$$

2 bosons $(q, qm^2\omega_1\omega_2)$ and 2 fermions $(qm\omega_1, qm\omega_2)$ in 3 dimensions

analogous formulas for 3d partitions conjectured by Nekrasov and proven by A.O.

3d partition $\longrightarrow$ fields of M-theory in 10 dimensions

very recently the same story for 4d partitions

4d partition $\longrightarrow$ fields of ?-theory in 12 dimensions

does not specialize to just q^{size}