Supply and Shorting in Speculative Markets

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Static Model

Consider

- Agents $i \in \{1, 2, \ldots, n\}$
- Using distributions $Q_i$ for the state $X(t)$
- Trading an asset with a single payoff $f(X(T))$ at time $T$
- The asset cannot be shorted and is in supply $s > 0$

Static case: Suppose the agents trade only once, at time $t = 0$

Equilibrium:

- Determine an equilibrium price $p$ for $f$ and portfolios $q_i \in \mathbb{R}_+$
- such that $q_i$ maximizes $q(E_i[f(X(T))] - p)$ over $q \geq 0$, for all $i$
- and the market clears: $\sum_i q_i = s$
Solution: The most optimistic agent determines the price (Miller ’77),

\[ p = \max_i E_i[f(X(T))]. \]

- Let \( i_* \in \{1, 2, \ldots, n\} \) be the maximizer.
- With portfolios \( q_{i_*} = s \) and \( q_i = 0 \) for \( i \neq i_* \), this in an equilibrium.
- It is unique (modulo having several maximizers).

Note:
- At price \( p \), the optimist is invariant and will accept any portfolio.
- All other agents want to have \( q_i = 0 \).
- Price not affected by supply.
Preview: The Resale Option (Harrison and Kreps ’78)

- When there are several trading dates, the relatively most optimistic agent depends on date and state.

- Option to resell the asset to another agent at a later time.

- Adds to the static price: speculative bubble.

Scheinkman and Xiong ’03, ’04
A Continuous-Time Model

Asset can be traded on \([0, T]\).

Agents:
- Risk-neutral agents \(i \in \{1, \ldots, n\}\) using models \(Q_i\)
- Here: agent \(i\) uses a local vol model \(Q_i\) for \(X\),
  \[
  dX(t) = \sigma_i(t, X(t)) \, dW_i(t), \quad X(0) = x
  \]

Equilibrium:
- Find a price process \(P(t)\) with \(P(T) = f(X(T))\) \(Q_i\)-a.s.
- Agents choose portfolio processes \(\Phi\)
- such as to optimize expected P&L: \(E_i[\int_0^T \Phi(t) \, dP(t)]\)
- Market clearing \(\sum_i \Phi_i(t) = s\)
Existence

Theorem: There exists a unique equilibrium price $P(t) = v(t, X(t))$, and $v$ is the solution of

$$v_t(t, x) + \sup_{i \in \{1, \ldots, n\}} \frac{1}{2} \sigma_i^2(t, x) v_{xx}(t, x) = 0, \quad v(T, \cdot) = f.$$ 

The optimal portfolios $\Phi_i(t) = \phi_i(t, X(t))$ are given by

$$\phi_i(t, x) = \begin{cases} 
s, & \text{if } i \text{ is the maximizer at } (t, x) \\
0, & \text{else} \end{cases}$$

- Derivative held by the locally most optimistic agent at any time
- Agents trade as this role changes
Control Problem and Speculative Bubble

- \( v \) is also characterized as the value function

\[
v(t, x) = \sup_{\theta \in \Theta} E[f(X^{t,x}_\theta(T))]
\]

\( \Theta \) is the set of \( \{1, \ldots, n\} \)-valued, progressive processes

- \( X^{t,x}_\theta(r), r \in [t, T] \) is the solution of

\[
dX(r) = \sigma_{\theta(r)}(r, X(r)) \, dW(t), \quad X(t) = x.
\]

Bubble:

- The control problem (or comparison) shows that

\[
P(0) \geq \max_i E_i[f(X(T))]
\]

- Thus, the price exceeds the static equilibrium

- This “speculative bubble” can be attributed to the resale option
Remarks

Note:
- Price is again independent of supply
- No-shorting was essential

Comparison with UVM:
- \( \nu \) is the uncertain volatility (UV) or \( G \)-expectation price corresponding to the interval

\[
[\sigma, \overline{\sigma}] = \left[ \inf_i \sigma_i(t, x), \sup_i \sigma_i(t, x) \right]
\]

\( \rightarrow \) In our model, the UV price arises as an equilibrium price of risk-neutral agents, instead of a superhedging price
Outline

1. Part I: Resale Option
2. Part II: Supply
3. Part III: Short-Selling
Model

Supply:
- Supply should diminish price, not reflected in the model of Part I
- Need (risk) aversion against large positions

Add Cost-of-Carry: For holding a position $y = \Phi(t)$ at time $t$, agents must pay an instantaneous cost

$$c(y) = \begin{cases} \frac{1}{2\alpha_+} y^2, & y \geq 0 \\ \infty, & y < 0 \end{cases}$$

Equilibrium: Agents optimize expected P&L − cost:

$$E_i \left[ \int_0^T \Phi(t) \, dP(t) - \int_0^T c(\Phi(t)) \, dt \right]$$
Existence

Theorem: • There exists a unique equilibrium price $P(t) = v(t, X(t))$, and $v$ is the solution of

$$v_t + \sup_{\emptyset \neq J \subseteq \{1, \ldots, n\}} \left\{ \frac{1}{|J|} \sum_{i \in J} \frac{1}{2} \sigma_i^2 v_{xx} - \frac{s}{|J|^{\alpha_+}} \right\} = 0$$

• The optimal portfolios $\Phi_i(t) = \phi_i(t, X(t))$ are unique and given by

$$\phi_i(t, x) = \left\{ \alpha_+ \mathcal{L}^i v(t, x) \right\}^+$$

where $\mathcal{L}^i v(t, x) = \partial_t v(t, x) + \frac{1}{2} \sigma_i^2 \partial_{xx} v(t, x)$

Supply: enters as a running cost, $\kappa = \frac{s}{|J|^{\alpha_+}}$
Delay Effect

- Again, one can consider a static version of the equilibrium: price is

\[ p = \max_{\emptyset \neq J \subseteq \{1, \ldots, n\}} \left( \frac{1}{|J|} \sum_{i \in J} E_i[f(X(T))] - \frac{sT}{|J| \alpha_+} \right). \]

- The resale option is still present and increases the dynamic price.

- Novel: Delay Effect

- If many agents expect to increase positions over time, they may anticipate the increase in the static case.

- The resulting demand pressure raises the static price.

- This effect may dominate, causing a “negative bubble.”
Delay Effect

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Outline

1. Part I: Resale Option
2. Part II: Supply
3. Part III: Short-Selling
In securities markets, shorting is often possible, though at a cost. Not modeled in the existing literature.

Asymmetric Cost-of-Carry:
For holding a position $y = \Phi(t)$ at time $t$, instantaneous cost

$$c(y) = \begin{cases} 
\frac{1}{2\alpha_+} y^2, & y \geq 0 \\
\frac{1}{2\alpha_-} y^2, & y < 0 
\end{cases}$$

Short is more costly than long: $\alpha_- \leq \alpha_+$
Existence

**Theorem:** • There exists a unique equilibrium price \( P(t) = v(t, X(t)) \), and \( v \) is the solution of

\[
\begin{align*}
\frac{\partial}{\partial t} v(t, x) + \sup_{I \subseteq \{1, \ldots, n\}} \left\{ \frac{1}{2} \Sigma^2_I(t, x) \partial_{xx} v(t, x) - \kappa_I(t, x) \right\} = 0, \\
v(T, \cdot) = f,
\end{align*}
\]

where the coefficients are defined as

\[
\kappa_I(t, x) = \frac{s(t, x)}{|I| \alpha_- + |I^c| \alpha_+},
\]

\[
\Sigma^2_I(t, x) = \frac{\alpha_-}{|I| \alpha_- + |I^c| \alpha_+} \sum_{i \in I} \sigma^2_i(t, x) + \frac{\alpha_+}{|I| \alpha_- + |I^c| \alpha_+} \sum_{i \in I^c} \sigma^2_i(t, x)
\]

• The optimal portfolios \( \Phi_i(t) = \phi_i(t, X(t)) \) are unique and given by

\[
\phi_i(t, x) = \alpha_{\text{sign}(L^i v(t, x))} L^i v(t, x), \quad L^i v(t, x) = \frac{\partial}{\partial t} v(t, x) + \frac{1}{2} \sigma^2_i \partial_{xx} v(t, x).
\]
Control Representation

- HJB equation of the control problem

\[ \nu(t, x) = \sup_{\mathcal{I} \in \Theta} E \left[ f(X^{t,x}_\mathcal{I}(T)) - \int_0^T \kappa_{\mathcal{I}(r)}(r, X^{t,x}_\mathcal{I}(r)) \, dr \right] \]

- \( \Theta \) is the set of \( 2^{\{1, \ldots, n\}} \)-valued, progressive processes
- \( X^{t,x}_\mathcal{I}(r), r \in [t, T] \) is the solution of

\[ dX(r) = \Sigma_{\mathcal{I}(r)}(r, X(r)) \, dW(t), \quad X(t) = x. \]

- Interpretation?
A Principal Agent Problem

- At each state \((t, x)\), principal assigns a cost coefficient \(\alpha_i \in \{\alpha_-, \alpha_+\}\) to every agent \(i \in \{1, \ldots, n\}\).
- This assignment will play the role of a contract (Second Best).
- With these coefficients given, agents maximize

\[
E_i \left[ \int_0^T \Phi(t) \, dP(t) - \int_0^T c_i(t, X(t), \Phi(t)) \, dt \right]
\]

where \(c_i(t, x, y) = \alpha_i(t, x)y^2\) irrespectively of \(y\) being long or short.

- An assignment can be summarized as a set

\[
l(t, x) = \{i \in \{1, \ldots, n\} : \alpha_i(t, x) = \alpha_-\}.
\]

I.e., \(l = \{\text{agents with } \alpha_-\}\), \(l^c = \{\text{agents with } \alpha_+\}\).
Principal Agent Problem: Solution

Theorem:

(i) For any assignment \( I(t) = l(t, X(t)) \) of the principal, there exists a unique equilibrium price \( P_{\mathcal{I}}(t) = v_{\mathcal{I}}(t, X(t)) \), and

\[
v_{\mathcal{I}}(t, x) = E \left[ f(X_{\mathcal{I}}^{t,x}(T)) - \int_0^T \kappa_{\mathcal{I}}(r)(r, X_{\mathcal{I}}^{t,x}(r)) \, dr \right]
\]

(ii) If the principal’s aim is to maximize the price,

- the optimal value is our previous equilibrium price \( v(t, x) \)
- the optimal contract assigns, in equilibrium, \( \alpha^- \) to short positions and \( \alpha^+ \) to long positions

→ Interpretation for \( \Sigma_I, \kappa_I \) in our PDE for \( v(t, x) \)
Comparative Statics and Limiting Cases

- The price is **decreasing wrt. supply**
- The price is **increasing wrt.** $\alpha_+$ (when $\alpha_-$ is fixed)
- The price is **decreasing wrt.** $\alpha_-$ (when $\alpha_+$ is fixed)

**Infinite Cost for Short:** As $\alpha_- \to 0$, the price $v^{\alpha_-,\alpha_+}$ converges to the price from **Part II**:

$$v_t + \sup_{\emptyset \neq J \subseteq \{1,\ldots,n\}} \left\{ \frac{1}{2} \frac{1}{|J|} \sum_{i \in J} \sigma_i^2 v_{xx} - \frac{s}{|J|\alpha_+} \right\} = 0$$

**Zero Cost for Long:** As $\alpha_+ \to \infty$, the price $v^{\alpha_-,\alpha_+}$ converges to the price from **Part I**:

$$v_t + \sup_{i \in \{1,\ldots,n\}} \frac{1}{2} \sigma_i^2 v_{xx} = 0$$

In particular, the limit is independent of $\alpha_-$ and $s$. 
Comparison of Dynamic and Static Models

- Again, we can compare with the static version

- Resale and delay options now apply to long and short positions
- The resale option for short positions depresses the dynamic price
- “Bubble” may have either sign

- In the limits
  \[ \alpha_+ \to \infty \quad \text{and/or} \quad \alpha_- \to 0 \quad \text{and} \quad s \to 0, \]
  the bubble is always nonnegative, as in Part I
- Main difference to previous models: increasing marginal cost of carry
Conclusion

Part I:
- Resale option leads to UVM price and speculative bubble

Parts II–III: A tractable model where
- Supply affects the price as a running cost
- Delay effect can depress the dynamic equilibrium price
- Short-selling is possible and may further depress the price

Happy Birthday, Ioannis!
Conclusion

Part I:
- Resale option leads to UVM price and speculative bubble

Parts II–III: A tractable model where
- Supply affects the price as a running cost
- Delay effect can depress the dynamic equilibrium price
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Happy Birthday, Ioannis!