# Equilibrium Liquidity Premia

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# Introduction

## Equilibrium Models and Trading Costs

- Frictionless analysis of Karatzas/Lehozky/Shreve '90:
  - *The goal of equilibrium analysis is to establish the existence and uniqueness of equilibrium prices, and to characterize these prices as well as the decisions made by the individual agents. [..] The result is a major increase in knowledge about not only the existence, but also about the uniqueness and the structure of equilibrium.*
- Much less tractability with frictions. Cvitanić/Karatzas '96:
  - *Our approach gives different insights and can be applied to the case of time-dependent and random market coefficients, but it provides no explicit description of optimal strategies, except for the cases in which it is optimal to not trade at all.*

# Introduction

## Liquidity Premia

- Equilibrium models with trading costs – why?
- Less liquid stocks have higher returns.
- "Liquidity premia". Consistent empirical observation.
  - E.g., Amihud/Mendelson '86; Brennan/Subrahmanyan '98; Pástor/Stambaugh '03.
  - One possible explanation for the "size effect" that stocks of smaller companies have higher returns even after controlling for risk.
  - A different model based on the stability of the capital distribution curve; Fernholz/Karatzas '06.
- Theoretical underpinning?
- Dependence of equilibrium asset returns on trading costs?

# Introduction

## This Paper

- Bouchard/Fukasawa/Herdegen/M-K:
  - Simple, tractable equilibrium model with trading costs.
  - *Existence* and *uniqueness*. *Characterization* in terms of matrix functions and conditional expectations.
  - Explicit formulas for concrete specifications.
- To make this possible, model is taylor-made for tractability:
  - Agents have *local* mean-variance preferences as in Kallsen '98; Garleanu/Pedersen '13, '16; Martin '14.
  - Trading costs are quadratic. Tractable without asymptotics as in Garleanu/Pedersen '13, '16; Bank/Soner/Voss '17.
  - Interest rate and volatility are exogenous. Only returns determined in equilibrium as in Kardaras/Xing/Zitković '15; Zitković/Xing '17.

# Introduction

## Related Literature

- Partial equilibrium models for liquidity premia.
  - Constantinides '86; Lynch/Tan '11; Jang/Koo/Liu/Loewenstein '07; Dai/Li/Liu/Wang '16.
  - Returns chosen to match frictionless to frictional performance rather than to clear markets.
- Numerical solution of discrete-time models.
  - Heaton/Lucas '96. Buss/Dumas '15; Buss/Vilkov/Uppal '15.
- No risky assets or constant asset prices.
  - Vayanos/Vila '99; Weston '16; Lo/Mamaysky/Wang '04.
- Other linear-quadratic models:
  - Garleanu/Pedersen '16. Only one strategic agent.
  - Sannikov/Skrzypacz '17: endoegneous trading costs as in Kyle '85. Existence? Uniqueness?

# Model

## Frictionless Benchmark

- Exogenous savings account. Normalized to one.
- Zero net supply of $d$ risky assets with Itô dynamics:

$$dS_t = \mu_t dt + \sigma dW_t$$

  - Constant covariance matix $\Sigma = \sigma^\top \sigma$ given exogenously.
  - Risky returns $\mu_t$ to be determined in equilibrium.
  - Similar to models of Zitković '12, Choi/Larsen'15, Kardaras/Xing/Zitković '15, Garleanu/Pedersen '16.

- $N$ agents with partially spanned endowments:

$$dY_t = \nu_t dt + \zeta_t \sigma dW_t + dM_t^\perp$$

- Frictionless wealth dynamics of a trading strategy $\varphi$:

$$\varphi_t dS_t + dY_t$$

# Model

## Frictionless Benchmark ct'd

- Equilibria are generally intractable even for CARA preferences.
  - Abstract existence results if market is complete, or almost complete (Kardaras/Xing/Zitković '15).
  - Some partial very recent existence results for the general incomplete case (Xing/Zitković '17).
  - Only few examples that can be solved explicitly (e.g., Christensen/Larsen/Munk '12, Christensen/Larsen '14).
- Tractability issues exacerbated by trading frictions.
- Need simpler frictionless starting point.
- Use *local* mean-variance preferences over changes in wealth:

$$E\left[\int_0^T (\varphi_t dS_t + dY_t) - \frac{\gamma}{2}\int_0^T \langle \varphi_t dS_t + dY_t \rangle\right] \to \max!$$

# Model

## Frictionless Benchmark ct'd

- Optimizers readily determined by pointwise optimzation of

$$E\left[\int_0^T \left(\varphi_t^\top \mu_t + \nu_t - \frac{\gamma}{2}(\varphi_t + \zeta_t)^\top \Sigma(\varphi_t + \zeta_t)\right)dt + \frac{\gamma}{2}\langle M^\perp\rangle_T\right]$$

- Merton portfolio plus mean-variance hedge:

$$\varphi_t = \frac{\Sigma^{-1}\mu_t}{\gamma} - \zeta_t$$

- Myopic. Available in closed form for *any* risky return.
- Leads to CAPM-equilibrium by summing across agents:

$$0 = \sum_{i=1}^{N} \varphi_N^i \quad \Rightarrow \quad \mu_t = \frac{\Sigma(\zeta_t^1 + \ldots + \zeta_t^N)}{1/\gamma_1 + \ldots + 1/\gamma_N}$$

# Model

## Transaction Costs

- This model has been studied with *small* proportional transaction costs by Martin/Schöneborn '11, Martin '14.
  - Simplification compared to CARA utility is closed-form solution for frictionless problem.
  - But frictional problem is no longer myopic. Transaction costs of similar complexity in both models (Kallsen/M-K '15).
- But asymptotics can be avoided for *quadratic* costs:
  - Garleanu/Pedersen '13, '16: explicit solutions for infinite-horizon model with linear-quadratic dynamics.
  - Trade towards (discounted) *average of expected future frictionless target*. "Aim in front of the moving target".
  - Bank/Soner/Voss '17: same structure remains true in general, not even necessarily Markovian, tracking problems.
  - This will be heavily exploited in our analysis here.

# Model

## Transaction Costs ct'd

- Optimization criterion with quadratic costs:

$$E\left[\int_0^T (\varphi_t dS_t + dY_t) - \frac{\gamma}{2}\int_0^T \langle \varphi_t dS_t + dY_t\rangle_t - \frac{\lambda}{2}\int_0^T \dot{\varphi}_t^2 dt\right] \to \text{max!}$$

  - Linear price impact proportional to trade size *and* speed.
  - Standard model in optimal execution (Almgren/Chriss '01).
  - Recently used for portfolio choice (Garleanu/Pedersen '13, '16; Guasoni/Weber '15; Almgren/Li '16; Moreau/M-K/Soner '16).
  - No longer myopic with trading costs. Current position becomes state variable.
- Equilibrium returns with transaction costs?
- Liquidity premia compared to frictionless benchmark?

# Individual Optimality

## First-Order Condition

- Need to choose risky returns $\mu_t$ so that purchases equal sales:

$$0 = \dot{\varphi}_t^1 + \ldots + \dot{\varphi}_t^N$$

- First step: determine individually optimal trading strategies.
- Adapt convex analysis argument of Bank/Soner/Voss '17.
  - Compute Gateaux deriviative $\lim_{\rho \to 0} \frac{1}{\rho}(J(\varphi + \rho\psi) - J(\varphi))$ of goal functional $J$.
  - Necessary and sufficient condition for optimality: needs to vanish for *any* direction $\psi$:

$$0 = E_t \left[ \int_0^T \left( \mu_t^\top \int_0^t \dot{\psi}_u du - \gamma(\varphi_t + \zeta_t)^\top \Sigma \int_0^t \dot{\psi}_u du - \lambda \dot{\varphi}_t \dot{\psi}_t \right) dt \right]$$

- Rewrite using Fubini's theorem.

# Individual Optimality

## First-Order Condition ct'd

- Necessary and sufficient condition for optimality:

$$0 = E_t \left[ \int_0^T \left( \int_t^T \left( \mu_u^\top - \gamma(\varphi_u + \zeta_u)^\top \Sigma \right) du - \lambda \dot{\varphi}_t^\top \right) \dot{\psi}_t dt \right]$$

- Has to hold for any perturbation $\psi_t$.
- Whence, tower property of conditional expectation yields:

$$\begin{aligned} \dot{\varphi}_t &= \frac{1}{\lambda} E_t \left[ \int_t^T \left( \mu_u - \gamma \Sigma (\varphi_u + \zeta_u) \right) du \right] \\ &= M_t - \frac{1}{\lambda} \int_0^t \left( \mu_u - \gamma \Sigma (\varphi_u + \zeta_u) \right) du \end{aligned}$$

for a martingale $M_t$.

# Individual Optimality

## Linear FBSDEs and Riccati ODEs

- Thus, individually optimal strategy solves *linear* FBSDE:

$$d\varphi_t = \dot{\varphi}_t dt, \quad \varphi_0 = \text{initial condition}$$
$$d\dot{\varphi}_t = dM_t - \frac{1}{\lambda}\Big(\mu_t - \gamma\Sigma(\varphi_t + \zeta_t)\Big)dt, \quad \dot{\varphi}_T = 0$$

- Backward component is special case of

$$d\dot{\varphi}_t = dM_t + B(\varphi_t - \xi_t)dt, \quad \dot{\varphi}_T = 0$$

for mean-reversion matrix $B$ and vector target process $\xi_t$.

- Bank/Soner/Voss '17: one-dimensional case can be reduced to Riccati equation using the ansatz

$$\dot{\varphi}_t = F(t)(\hat{\xi}_t - \varphi_t), \quad \hat{\xi}_t = K_1(t)E_t\left[\int_t^T K_2(s)\xi_s ds\right]$$

# Individual Optimality

## Linear FBSDEs and Riccati ODEs

- Higher dimensions lead to coupled but still linear FBSDEs.
  - Many risky assets here. Many agents later.
- Ansatz still allows to reduce to matrix-valued Riccati ODEs.
- Can be solved by matrix power series, e.g.:

$$F(t) = -G'(t)G^{-1}(t) \quad \text{where } G(t) = \sum_{n=0}^{\infty} \frac{1}{2n!} B^n (T-t)^{2n}$$

- Matrix versions of univariate hyperbolic functions in Bank/Soner/Voss '17.
- To prove that the solutions are well-defined in general:
  - Need that $B$ is invertible and has only positive eigenvalues.
  - For individual optimality, $B = \frac{\gamma}{\lambda}\Sigma$. Follows from assumptions on covariance matrix.

# Equilibrium

## Market Clearing

- Recall: need to choose returns $(\mu_t)_{t\in[0,T]}$ such that

$$
\begin{aligned}
0 &= \dot{\varphi}^1_t + \ldots + \dot{\varphi}^N_t \\
&= \frac{N}{\lambda} E_t\left[\int_t^T \left(\mu_u - \frac{1}{N}\sum_{i=1}^N \Sigma(\gamma^i \zeta^i_u + \gamma^i \varphi^i_u)\right) du\right]
\end{aligned}
$$

- In equilibrium, $\varphi^N_s = -\varphi^1_s - \ldots - \varphi^{N-1}_s$, so that

$$
0 = E_t\left[\int_t^T \left(\Sigma^{-1}\mu_u - \sum_{i=1}^N \frac{\gamma^i}{N}\zeta^i_u + \sum_{i=1}^{N-1} \frac{\gamma^N - \gamma^i}{N}\varphi^i_u\right) du\right]
$$

- Whence, equilibrium if (and only if)

$$
\Sigma^{-1}\mu_t = \sum_{i=1}^N \frac{\gamma^i}{N}\zeta^i_t + \sum_{i=1}^{N-1} \frac{\gamma^i - \gamma^N}{N}\varphi^i_t
$$

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# Equilibrium

## Linear FBSDEs

- For homogenous agents with the same risk aversion:
  - Same equilibrium return $\mu_t = \frac{\gamma}{N}\Sigma \sum_{i=1}^{N} \zeta_t^i$ as without costs. No liquidity premium.
  - Same result in general if costs are split appropriately.
  - Asymptotic result of Herdegen/M-K '16 holds exactly here.
  - Agents are not indifferent to costs, but same asset prices still clear the market.
- With heterogenous agents:
  - Plug back formula for $\mu_t$ into clearing condition.
  - Again leads to a system of coupled but *linear* FBSDEs:

$$\dot{\varphi}_t^i = \frac{\Sigma}{\lambda} E_t \left[ \int_t^T \left( \sum_{j=1}^{N} \frac{\gamma^j}{N} \zeta_u^j + \sum_{j=1}^{N-1} \frac{\gamma^j - \gamma^N}{N} \varphi_u^j - \gamma^i \zeta_u^i - \gamma^i \varphi_u^i \right) du \right]$$

- Solution like for individual optimality?

# Equilibrium

## Linear FBSDEs ct'd

- Difficulty: need to verify that

$$B = \begin{pmatrix} \left(\frac{\gamma^N - \gamma^1}{N} + \gamma^1\right)\frac{\Sigma}{\lambda} & \cdots & \frac{\gamma^N - \gamma^{N-1}}{N}\frac{\Sigma}{\lambda} \\ \vdots & \ddots & \vdots \\ \frac{\gamma^N - \gamma^1}{N}\frac{\Sigma}{\lambda} & \cdots & \left(\frac{\gamma^N - \gamma^{N-1}}{N} + \gamma^{N-1}\right)\frac{\Sigma}{\lambda} \end{pmatrix} \in \mathbb{R}^{d(N-1)\times d(N-1)}$$

  is invertible and has only positive eigenvalues.
- To check this:
  - First reduce to the case of diagonal $\Sigma$ by multiplying with appropriate orthogonal block matrices.
  - Then use a result of Silvester '00 for the computation of determinants of matrices with elements from the commutative subring of diagonal matrices in $\mathbb{C}^{d\times d}$.
- Existence then follows as for individual optimality. Solution of Riccati ODEs in terms of power series.

# Equilibrium

## Summary

- In summary:
  - Define $\varphi^1_t, \dots, \varphi^{N-1}_t$ as the solution of the FBSDE.
  - Then, the unique equilibrium return process is given by

$$\Sigma^{-1}\mu_t = \sum_{i=1}^{N} \frac{\gamma^i}{N}\zeta^i_t + \sum_{i=1}^{N-1} \frac{\gamma^i - \gamma^N}{N}\varphi^i_t$$

  - $\varphi^i_t$ and in turn $\mu_t$ can be expressed explicitly in terms of solutions of matrix-valued Riccati ODEs.
- To obtain fully explicit examples:
  - Only need to compute conditional expectations of the endowment exposures!
  - Simplest case: exposures follow arithmetic Brownian motions as Lo/Mamaysky/Wang '04.

# Example

## Concrete Endowments

- Simplest nontrivial example:
  - No aggregate endowments. Individual exposures follow

    $$\zeta_t^1 = -\zeta_t^2 = at + N_t,$$

    for a constant $a$ and a Brownian motion $N$.
  - To obtain simpler stationary solutions: $T = \infty$.
  - Problem remains well posed after introducing discount rate $\delta > 0$. Only adds one extra term to FBSDE, allows to replace terminal with limiting transversality condition.
  - Trading rates become constant, discounting becomes exponential.
- (Discounted) conditional expectations of endowment exposures can be readily computed in closed form.
- Lead to explicit dynamics of the equilibrium return.

# Example

## Equilibrium Return

- Equilibrium return has Ornstein-Uhlenbeck dynamics:

$$d\mu_t = \left(\sqrt{\frac{\gamma_1+\gamma_2}{2}\frac{\Sigma}{2\Lambda} + \frac{\delta^2}{4}} - \frac{\delta}{2}\right)\left(2\frac{\gamma_1-\gamma_2}{\gamma_1+\gamma_2}\delta\Lambda a - \mu_t\right)dt \\ + \frac{(\gamma_1-\gamma_2)\Sigma}{2}dN_t$$

- Average liquidity premium vanishes for equal risk aversions. Generally proportional to relative difference times impatience.
- Positive premium if more risk averse agent is a net seller.
  - Has stronger motive to trade, therefore provides extra compensation.
- Average premium is $O(\Lambda)$. Standard deviation is $O(\sqrt{\Lambda})$.
- Mean reversion even for martingale endowments. Induced by sluggishness of frictional portfolios.

# Summary

## Equilibrium Liquidity Premia

- Tractable model with local mean-variance preferences and quadratic trading costs.
- Equilibrium liquidity premia characterized as unique solution of coupled system of linear FBSDEs.
- Can be solved in terms of matrix power series.
- Explicit examples show:
  - Returns becomes mean-reverting with illiquidity.
  - Sign of liquidity premium determined by trading needs of more risk averse agents.
- Extensions:
  - Noise traders can be included. Recaptures model of Garleanu/Pedersen '16 as a special case.
  - Asymptotically equivalent to exponential equilibrium?
  - Endogenous volatility?

# Last but not Least..

Happy Birthday Ioannis!