On a class of stochastic differential equations in a financial network model

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Motivation:

On $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ let us consider $\mathbb{R}^N$-valued diffusion process $(X_1(t), \ldots, X_N(t))$, $0 \le t < \infty$ induced by the following random graph structure.

Suppose that at time 0 we have a random graph of N vertices $\{1, \ldots, N\}$ and define the strength of connections between vertices i and j by $\mathcal{F}_0$-measurable *random variable* $a_{i,j}$ (whose distribution may depend on N) for every $1 \le i \ne j \le N$ and fix $a_{i,i} = 0$ for $1 \le i \le N$. We shall consider

$$\mathrm{d}X_i(t) = -\frac{1}{N} \sum_{j=1}^{N} a_{i,j} \, (X_i(t) - X_j(t)) \, \mathrm{d}t + \mathrm{d}B_i(t) ;$$

for $i = 1, \ldots, N$, $t \ge 0$,

where $(B_1(t), \ldots, B_N(t))$, $t \geq 0$ is the standard N-dimensional BM, independent of $(X_1(0), \ldots, X_N(0))$ and of the random variables $(a_{i,j})_{1 \leq i,j \leq N}$.

The randomness determined at time 0 affects the diffusion process $(X_1(\cdot), \ldots, X_N(\cdot))$.

Deterministic A in the context of financial network : Carmona, Fouque, Sun ('13), Fouque & Ichiba ('13), ...

The system is solvable as a linear stochastic system for $X(\cdot) := (X_1(\cdot), \ldots, X_N(\cdot))'$.

Let $A^{(N)} := (a_{i,j})_{1 \leq i,j \leq N}$ be the $(N \times N)$ random matrix and $B(\cdot)$ be the $(N \times 1)$-vector valued standard Brownian motion.

Then the system can be rewritten as

$$\mathrm{d}X(t) = -\overline{A}^{(N)} X(t)\mathrm{d}t + \mathrm{d}B(t),$$

$$\overline{A}^{(N)} := \frac{1}{N}\mathrm{Diag}(A^{(N)}\mathbf{1}_N) - \frac{1}{N}A^{(N)},$$

where $\mathbf{1}_N$ is the $(N \times 1)$ vector of ones, and $\mathrm{Diag}(c)$ is the diagonal matrix whose diagonal elements are those elements in the vector c. Note that each row sum of elements in the matrix $\overline{A}^{(N)}$ is zero by definition, i.e.,

$$\overline{a}_{i,i}^{(N)} = -\sum_{j \neq i} \overline{a}_{i,j}^{(N)}$$

for each $i = 1, \ldots, N$, where $\overline{a}_{i,j}^{(N)}$ is the (i, j) element of the random matrix $\overline{A}^{(N)}$.

The solution to this linear equation is given by

$$X(t) = e^{-t\overline{A}^{(N)}} \Big(X(0) + \int_0^t e^{s\overline{A}^{(N)}} \, \mathrm{d}B(s) \Big) \, ; \quad t \ge 0 \, .$$

Here we understand $e^{t\overline{A}^{(N)}}$ is the $(N \times N)$ matrix exponential.

Given the initial value $X(0)$ and $\overline{A}^{(N)}$, the law of $X(\cdot)$ is conditionally an N-dimensional Gaussian law with mean $e^{-t\overline{A}^{(N)}} X(0)$ and variance covariance matrix $\mathrm{Var}(X(t) | A^{(N)})$.

Q. How to understand the case $N \to \infty$ of large network, i.e., what happens if $N \to \infty$?
For example, if $\overline{A}^{(N)} \xrightarrow[N \to \infty]{a.s.} \overline{A}^{(\infty)}$ and

if $\overline{a}_{i,j}^{(\infty)} = -1$ for $j = i+1$, $a_{i,i}^{(\infty)} = 1$, $a_{i,j}^{(\infty)} \equiv 0$, o.w., i.e.,

$$-\overline{A}^{(\infty)} := \begin{pmatrix} -1 & 1 & 0 & \cdots \\ 0 & -1 & 1 & 0 \cdots \\ & \ddots & \ddots & \ddots \end{pmatrix},$$

$$\begin{aligned} \mathrm{d}X_1(t) &= (X_2(t) - X_1(t))\mathrm{d}t + \mathrm{d}W_1(t), \\ \mathrm{d}X_2(t) &= (X_3(t) - X_2(t))\mathrm{d}t + \mathrm{d}W_2(t), \\ &\vdots \end{aligned}$$

then how can we solve?

• Finite N case: Fernholz & Karatzas ('08-'09) studied flow, filtering and pseudo-Brownian motion process in equity markets.

Example as $N \to \infty$

For simplicity let us set $X_i(0) = 0$. Given $X_2(\cdot)$, we have

$$X_1(t) = \int_0^t e^{-(t-s)} X_2(s)\mathrm{d}s + \int_0^t e^{-(t-s)}\mathrm{d}B_1(s),$$

and also, given $X_3(\cdot)$, we have

$$X_2(s) = \int_0^s e^{-(s-u)} X_3(u)\mathrm{d}u + \int_0^s e^{-(s-u)}\mathrm{d}B_2(u)$$

for $t \ge 0$, and hence substituting $X_2(\cdot)$ into the first one,

$$X_1(t) = \int_0^t e^{-(t-s)}\mathrm{d}B_1(s) + \int_0^t \int_0^s e^{-(t-u)}\mathrm{d}B_2(u)\mathrm{d}s$$

$$+ \int_0^t e^{-(t-s)} \int_0^s e^{-(s-u)} X_3(u)\mathrm{d}u$$

for $t \ge 0$.

By the product rule for semimartingales, we observe

$$\int_0^t \int_0^s e^u (s-u)^{k-1} \mathrm{d}B(u)\mathrm{d}s = \int_0^t e^u \frac{(t-u)^k}{k} \mathrm{d}B(u),$$

for $k \in \mathbb{N}$, $t \geq 0$, and hence

$$\int_0^t \int_0^s e^u \mathrm{d}B(u)\mathrm{d}s = \int_0^t e^u (t-u) \mathrm{d}B(u),$$

$$\int_0^t \int_0^{s_k} \cdots \int_0^{s_1} e^u \mathrm{d}B(u)\mathrm{d}s_1 \cdots \mathrm{d}s_k = \int_0^t e^u \frac{(t-u)^k}{k!} \mathrm{d}B(u)$$

for $k \in \mathbb{N}$, $t \geq 0$. Thus for the above example we have

$$X_1(t) = \sum_{k=0}^{\infty} \int_0^t e^{-(t-u)} \cdot \frac{(t-u)^k}{k!} \mathrm{d}B_{k+1}(u)$$

for $t \geq 0$.

$$X_1(t) = \sum_{k=0}^{\infty} \int_0^t e^{-(t-u)} \cdot \frac{(t-u)^k}{k!} \mathrm{d}B_{k+1}(u)$$

is a centered, Gaussian process with covariances

$$\mathbb{E}[X_1(s)X_1(t)] = e^{-(s+t)} \sum_{k=0}^{\infty} \int_0^s \frac{e^{2u}}{(k!)^2} (s-u)^k (t-u)^k \mathrm{d}u$$

$$= e^{-(t-s)} \int_0^s e^{-2v} I_0(2\sqrt{(t-s+v)v}) \mathrm{d}v$$

for $0 \leq s \leq t$, where $I_0(\cdot)$ is the modified Bessel function of the first kind with parameter 0, i.e.,

$$I_\nu(x) := \sum_{k=0}^{\infty} \left(\frac{x}{2}\right)^{2k+\nu} \frac{1}{\Gamma(k+1)\Gamma(\nu+k+1)}$$

for $x > 0$, $\nu \geq -1$.

In particular,

$$\mathrm{Var}(X_1(t)) = \int_0^t e^{-2v} I_0(2v)\mathrm{d}v = te^{-2t}(I_0(2t) + I_1(2t)) < \infty$$

(it grows as $O(t^{1/2})$ for large t, also,

$$\mathbb{E}[X_1(s)X_1(s+t)] = O(e^{-(t-2\sqrt{(t+s)s})}t^{-1/4})\,.)$$

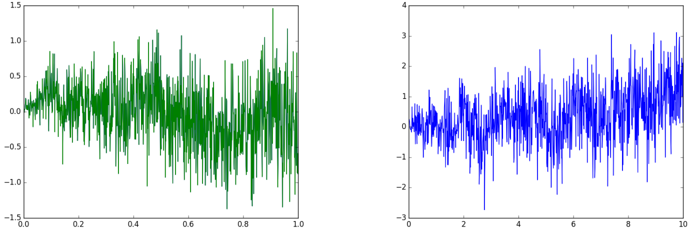
Thus $X_1(\cdot)$ is *not stationary*.
The (marginal) distribution of $X_k(\cdot)$, $k \in \mathbb{N}$ is the same as $X_1(\cdot)$, and hence, we may compute (at least numerically)

$$\mathbb{E}[X_1(t)X_2(u)] = \int_0^t e^{(t-s)}\mathbb{E}[X_2(s)X_2(u)]\mathrm{d}s$$

$$= \int_0^t e^{(t-s)}\mathbb{E}[X_1(s)X_1(u)]\mathrm{d}s$$

and recursively, $\mathbb{E}[X_1(t)X_k(u)]$, $k \in \mathbb{N}$ for $0 \le t, u < \infty$.

Sample path of $(X_1(\cdot), X_2(\cdot))$ generated from the covariance structure.



$X_1(\cdot)$ $\qquad$ $X_2(\cdot)$

$$X_1(t) = \int_0^t e^{-(t-s)} X_2(s)\mathrm{d}s + \int_0^t e^{-(t-s)} \mathrm{d}B_1(s),$$

for $t \geq 0$ and $\mathrm{Law}(X_1(\cdot)) = \mathrm{Law}(X_2(\cdot))$.

Weighted by Poisson probabilities

An interpretation of

$$X_1(t) = \sum_{k=0}^{\infty} \int_0^t e^{-(t-u)} \cdot \frac{(t-u)^k}{k!} \mathrm{d}B_{k+1}(u)$$

$$=: \sum_{k=0}^{\infty} \int_0^t \mathfrak{p}_k(t-u) \mathrm{d}B_{k+1}(u)$$

for $t \geq 0$:

Suppose $N(s), 0 \leq s \leq t$ is a Poisson process with rate 1, independent of $(B_k(\cdot), k \in \mathbb{N})$. Then

$$X_1(t) = \mathbb{E}\Big[\sum_{k=0}^{\infty} \int_0^t \mathbf{1}_{\{N(t-u)=k\}} \mathrm{d}B_{k+1}(u) \Big| \mathcal{F}(t) \Big],$$

where $\mathcal{F}(t) := \sigma(B_k(s), 0 \leq s \leq t, k \in \mathbb{N})$, $t \geq 0$.

If we replace the Poisson probability by compound Poisson probability, i.e.,

$$\widetilde{N}(t) := \sum_{k=1}^{N(t)} \xi_k ,$$

where $(\xi_k, k \in \mathbb{N})$ are I.I.D. integer-valued R.V.'s with $\mathbb{P}(\xi_1 = i) = p_i$, $1 \le i \le q$, $\sum_{i=1}^{q} p_i = 1$ for some $q \in \mathbb{N}$, independent of $N(\cdot)$ and $(B_k(\cdot), k \in \mathbb{N})$, then

$$\widetilde{X}_1(t) := \mathbb{E}\Big[\sum_{k=0}^{\infty} \int_0^t \mathbf{1}_{\{\widetilde{N}(t-u)=k\}} \mathrm{d}B_{k+1}(u) \Big| \mathcal{F}(t) \Big]$$

$$= \sum_{k=0}^{\infty} \int_0^t \widetilde{\mathfrak{p}}_k(t-u) \mathrm{d}B_{k+1}(u) ,$$

where

$$\widetilde{\mathfrak{p}}_k(t) := \frac{\partial^k}{\partial z^k} \Big[\exp\Big(\sum_{i=1}^{q} p_i t (z^i - 1) \Big) \Big] \Big|_{z=0}$$

for $k \in \mathbb{N}$, $t \ge 0$

corresponds to the modified matrix

$$-\widetilde{\overline{A}^{(\infty)}} := \begin{pmatrix} -1 & p_1 & p_2 & \cdots & p_q & 0 & \cdots \\ 0 & -1 & p_1 & p_2 & \cdots & p_q & \ddots \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix},$$

and

$$\mathrm{d}X_k(t) = \Big(- X_k(t) + \sum_{i=1}^{q} p_i X_{i+k}(t) \Big)\, \mathrm{d}t + \mathrm{d}B_k(t)$$

with $X_k(0) = 0$ for $k \in \mathbb{N}$, $t \geq 0$.

- In particular, if $q = 2$, $p_1 = p_2 = 1/2$, then

$$\widetilde{\mathfrak{p}}_k(t) = \sum_{j=0}^{\lfloor k/2 \rfloor} \frac{e^{-t} t^{k-j}}{2^{k-j}(k-2j)!\, j!}; \quad t \geq 0, k \in \mathbb{N}.$$

Another modification:

$$\mathrm{d}X_1(t) = (X_1(t) - X_2(t))\mathrm{d}t + \mathrm{d}B_1(t)\,,$$

$$\mathrm{d}X_2(t) = (X_2(t) - X_3(t))\mathrm{d}t + \mathrm{d}B_2(t)\,,$$

$$\ldots$$

We may use the same reasoning in this case to obtain

$$X_1(t) = \int_0^t \sum_{k=0}^{\infty} e^{t-s} \cdot \frac{(-1)^k (t-s)^k}{k!} \mathrm{d}B_{k+1}(s)$$

with exponentially growing variance

$$\mathrm{Var}(X_1(t)) = te^{2t}(I_0(2t) - I_1(2t))\,; \quad t \geq 0\,.$$

A formulation of equation with identical distribution

Let us consider $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}(t), t \geq 0\})$ on which $X_0(\cdot)$ is an adapted stochastic process which is a weak solution to

$$\mathrm{d}X_0(t) = b(t, X_0(t), X_1(t))\mathrm{d}t + \sigma(t, X_0(t), X_1(t))\mathrm{d}B_0(t)\,; \quad t \geq 0\,,$$

where r-dimensional standard Brownian motion $B_0(\cdot)$ is independent of d-dimensional process $X_1(\cdot)$ which has the same distribution as $X_0(\cdot)$ on $[0, T]$ i.e.,

$$\mathrm{Law}(X_0(s), 0 \leq s \leq T) = \mathrm{Law}(X_1(s), 0 \leq s \leq T)\,,$$

and also $\mathbb{P}$ a.s. $X_0(0) = x_0 \in \mathbb{R}^d$ and

$$\int_0^T (\|b(t, X_0(t), X_1(t))\| + \|\sigma_{i,j}(t, X_0(t), X_1(t))\|^2)\mathrm{d}t < +\infty$$

for $1 \leq i \leq d$, $1 \leq j \leq r$ and $T \geq 0$. Here we assume

$b : \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^{d \times r}$ are Lipschitz continuous with at most linear growth, i.e., there exist a constant $K > 0$ such that

$$\|b(t,x,y) - b(t,\tilde{x},\tilde{y})\| + \|\sigma(t,x,y) - \sigma(t,\tilde{x},\tilde{y})\| \leq K(\|x - \tilde{x}\| + \|y -$$

and

$$\|b(t,x,y)\|^2 + \|\sigma(t,x,y)\|^2 \leq K^2(1 + \|x\|^2 + \|y\|^2)$$

for every $(t,x,y) \in \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d$.

We also assume that $X_1(\cdot)$ is adapted to the filtration $\{\mathcal{F}_1(t), t \geq 0\}$ generated by the Brownian motions $(B_k(t), k \geq 1, t \geq 0)$ augmented by the $\mathbb{P}$-null sets.

We shall solve this system with distributional identity.

- When $b(t,x,y) = x - y$ and $\sigma(t,x,y) = 1$, it reduces to the first example

$$X_1(t) = \int_0^t e^{-(t-s)} X_2(s)\mathrm{d}s + \int_0^t e^{-(t-s)} \mathrm{d}B_1(s)\,; \quad t \ge 0\,.$$

- In the linear case we may consider the corresponding $\overline{A}^{(\infty)}$ to the example of the block matrix form

$$\overline{A}^{(\infty)} = \begin{pmatrix} A_{1,1} & A_{1,2} & 0 & \cdots & \\ 0 & A_{1,1} & A_{1,2} & 0 & \ddots \\ & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$

- It looks similar to the nonlinear diffusion

$$\mathrm{d}X(t) = b(X(t), \mathbb{E}[X(t)])\mathrm{d}t + \sigma(X(t), \mathbb{E}[X(t)])\mathrm{d}B(t)\,, \quad t \ge 0$$

of mean-field which appears as Mckean-Vlasov limit of interacting particles (Mckean ('67), Kac ('73), Sznitman ('89), Tanaka ('84), Shiga & Tanaka ('85), ...), but is different.

Proposition.

On some probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}(t), t \geq 0\})$ there is a unique weak solution to

$$\mathrm{d}X_0(t) = b(t, X_0(t), X_1(t))\mathrm{d}t + \sigma(t, X_0(t), X_1(t))\mathrm{d}B_0(t)\,; \quad t \geq 0\,,$$

with $\mathrm{Law}(X_1(t), 0 \leq t \leq T) = \mathrm{Law}(X_0(t), 0 \leq t \leq T)$ and

$$\alpha(t) = \mathbb{E}[\sup_{0 \leq s \leq t} \|X_0(s) - X_1(s)\|^2]\,; \quad t \geq 0$$

satisfies

$$\int_0^t \alpha(s)\mathrm{d}s + \int_0^t \beta_0 e^{\beta_0(t-s)}\Big(\int_0^s \alpha(u)\mathrm{d}u\Big)\mathrm{d}s \leq c_1\alpha(t)\,,$$

for $0 \leq t \leq T$, $T > 0$, where $\beta_0 := 4K^2(\Lambda_1 + T)$,

$$c_0 := 9\max(1, K^2(\Lambda_1 + T)(1 \vee T))\,, \quad c_1 := \frac{1 - e^{(c_0 - \beta_0)T}}{c_0 - \beta_0}$$

and Λ_1 is a global constant form the BURKHOLDER-DAVIS-GUNDY inequality.

Idea of proof:
Given $B_0(\cdot)$ and $X_1(\cdot)$, we may construct $X_0(\cdot)$ by the method of Picard iteration, i.e., there exists a map

$$\Phi : C([0,\infty),\mathbb{R}^d) \times C([0,\infty),\mathbb{R}^r) \to C([0,\infty),\mathbb{R}^d)$$

with $X_0(t) = \Phi_t(X_1, B_0)$ for $t \geq 0$.
We shall find a fixed point of $\Phi_\cdot(\cdot, B_0)$, i.e.,

$$\mathrm{Law}(X_1(\cdot)) = \mathrm{Law}(\Phi_\cdot(X_1, B_0)) = \mathrm{Law}(X_0(\cdot))$$

by evaluating the Wassestein distance

$$W_{2,T}(\mu, \tilde{\mu}) := \inf_{\nu} \mathbb{E}_\nu[\sup_{0\leq t\leq T} \|\xi(t) - \tilde{\xi}(t)\|^2]$$

where $\mu = \mathrm{Law}(\xi(\cdot))$, $\tilde{\mu} = \mathrm{Law}(\tilde{\xi}(\cdot))$ and the infimum is taken over the joint law ν of $(\xi(\cdot), \tilde{\xi}(\cdot))$, and using Banach fixed point theorem.

$$\mathfrak{S} := \Big\{ \alpha(\cdot) \, : \, \int_0^t \alpha(s) \mathrm{d}s + \int_0^t \beta_0 e^{\beta_0 (t-s)} \Big(\int_0^s \alpha(u) \mathrm{d}u \Big) \mathrm{d}s \leq c_1 \alpha(t) \, , \, 0 \leq t \leq T \, \Big\}$$

• Note that $c_0 > \beta_0$, $(1 - e^{-(c_0 - \beta_0) T}) / (c_0 - \beta_0) < 1$ and, $\alpha_1(t) := e^{c_0 t}$ satisfies

$$\int_0^t \alpha_1(s) \mathrm{d}s + \int_0^t \beta_0 e^{\beta_0 (t-s)} \Big(\int_0^s \alpha_1(u) \mathrm{d}u \Big) \mathrm{d}s = c_1 \alpha_1(t) \, , \quad 0 \leq t \leq T \, ,$$

and so, $\alpha_1(\cdot) \in \mathfrak{S}$ and $\mathfrak{S}$ is non-empty.

If $f \in \mathfrak{S}$, then $c \cdot f \in \mathfrak{S}$ for every positive constant $c > 0$.

Also, if $f, g \in \mathfrak{S}$, then $a \cdot f + (1-a) \cdot g \in \mathfrak{S}$ for every $a \in [0,1]$, and hence, $\mathfrak{S}$ is convex.

Moreover, since $(1 - e^{-(x-\beta_0)T})/(x - \beta_0)$ is a non-decreasing function of x for $x > \beta_0$,
$\alpha_2(t) := e^{c_2 t}$ with $0 < c_2 \le c_0$ satisfies

$$\int_0^t \alpha_2(s)\mathrm{d}s + \int_0^t \beta_0 e^{\beta_0(t-s)} \Big(\int_0^s \alpha_2(u)\mathrm{d}u \Big)\mathrm{d}s = \frac{1 - e^{-(c_2-\beta_0)T}}{c_2 - \beta_0} \cdot \alpha_2(t)$$

$$\le c_1 \alpha_2(t); \quad 0 \le t \le T,$$

and hence $\alpha_2(\cdot) \in \mathfrak{S}$ for every $0 < c_2 \le c_0$.

□

- We may extend to the case of the form

$$\mathrm{d}X_0(t) = b(t, X_0(t), \ldots, X_q(t))\mathrm{d}t + \sigma(t, X_0(t), \ldots, X_q(t))\mathrm{d}B_0(t)$$

with $\mathrm{Law}(X_0(\cdot)) = \mathrm{Law}(X_1(\cdot)) = \cdots = \mathrm{Law}(X_q(\cdot))$ for some $q \in \mathbb{N}$ and with Lipschitz coefficients, where $X_i(\cdot)$ is adapted to the filtration generated by $(B_k(\cdot), k \geq i)$ for $i \in \mathbb{N}$.

Coming back to the diffusions on the graph

- We say the infinite dimensional matrix $x = (x_{i,j})_{(i,j)\in\mathbb{N}^2}$ is row-finite if for each $i \in \mathbb{N}$ there is $k(i) \in \mathbb{N}$ such that $x_{i,j} = 0$ for every $j \geq k(i)$.

- We say the infinite dimensional matrix $x = (x_{i,j})_{(i,j)\in\mathbb{N}^2}$ is uniformly row-finite, if there is $n_0 \in \mathbb{N}$ such that $x_{i,j} = 0$ for every $i \in \mathbb{N}$ and every j with $|i-j| \geq n_0$.

- We also say the infinite dimensional matrix $(x_{i,j})_{(i,j)\in\mathbb{N}^2}$ is (uniformly) column-finite, if its transpose $(x_{i,j})'_{(i,j)\in\mathbb{N}^2}$ is (uniformly) row finite.

- Let us denote by $\mathcal{A}$ the class of uniformly positive definite, bounded, infinite dimensional matrices which are both uniformly row and uniformly column finite.

- Suppose that there exist $\mathrm{u} > \mathrm{d} > 0$ such that all the eigenvalues of $\overline{A}^{(N)}$ are bounded above by u and below by d for every N, and as $N \to \infty$, each (i,j) element $\overline{a}_{i,j}^{(N)}$ of $\overline{A}^{(N)}$ converges to an (i,j) element $\overline{a}_{i,j}^{(\infty)}$ of fixed matrix $\overline{A}^{(\infty)} \in \mathcal{A}$ almost surely for every $(i,j) \in \mathbb{N}^2$, i.e.,

$$\lim_{N \to \infty} \overline{a}_{i,j}^{(N)} = \overline{a}_{i,j}^{(\infty)} .$$

- Assume the first k elements $X^{(k,N)}(0) = (X_1(0), \ldots, X_k(0))$ of initial random variables $X^{(N)}(0)$ converges weakly to an $\mathbb{R}^k$-valued random vector $\eta^{(k)}$ for every $k \in \mathbb{N}$.

- We also assume that $\sup_N \mathbb{E}[\|X^{(k,N)}(0)\|^4] < \infty$ for every $k \in \mathbb{N}$.

Then for every $k \in \mathbb{N}$ and $T > 0$, as $N \to \infty$, the law of the first k elements $X^{(k,N)}(\cdot) = (X_1(\cdot), \dots, X_k(\cdot))'$ of $X^{(N)}(\cdot) = (X_1(\cdot), \dots, X_N(\cdot))'$ converges weakly in $C([0, T])$ to the law of the first k-dimensional stochastic process $Y^{(k)}(\cdot) := (Y_1(\cdot), \dots, Y_k(\cdot))'$ of $Y(\cdot) := (Y_i(\cdot))'_{i \in \mathbb{N}}$ defined by

$$Y(t) = e^{-t\overline{A}^{(\infty)}} \Big(Y(0) + \int_0^t e^{s\overline{A}^{(\infty)}} \, \mathrm{d}\boldsymbol{W}(s) \Big) ; \quad t \geq 0 ,$$

where $\mathrm{Law}(Y^{(k)}(0)) = \mathrm{Law}(\eta^{(k)})$ for every $k \in \mathbb{N}$ and $\boldsymbol{W}(\cdot)$ is the $\mathbb{R}^\infty$-valued standard Brownian motion.

Summary

Thank you all for your attentions, and Happy Birthday!

- Examples of linear systems on infinite graph
- A class of stochastic differential equations with restrictions in their distribution

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