Interesting one-dimensional diffusions that arise in stochastic games

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Aaron Kolb (2016), Strategic real options, working paper, Kelly School of Business, Indiana Univ. (submitted for publication)

Brendan Daley and Brett Green (2012), Waiting for news in the market for lemons, *Econometrica*, 80, 1433-1504.

George Akerlof (1970), The market for “lemons”: Quality uncertainty and the market mechanism, *Quart. J. Econ*, 84, 488-500

Five Model Components Added Sequentially

1. Adoption decision
2. Learning
3. Strategic seller (exit)
4. Private information
5. Endogenous quality (upgrades)

1. Adoption Decision

Asset has type $\theta \in \{H, L\}$, buyer has prior $p_0 = \mathbb{P}(\theta = H)$.

Buyer chooses whether to adopt or not.
If she adopts, she gets $1_{\{\theta = H\}} - k$, where $k \in (0, 1)$; otherwise, 0.

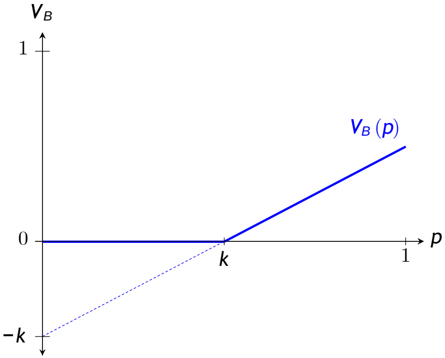


Figure: Buyer Value Function

2. Learning

Continuous time, $t \in [0, \infty)$, discount rate $r > 0$.

Players observe news process: $dX_t = \mu_\theta dt + \sigma dW_t$, where $\mu_H > \mu_L$.

Assume for simplicity that $\varphi \equiv \frac{\mu_H - \mu_L}{\sigma} = 1$ (signal-to-noise ratio).

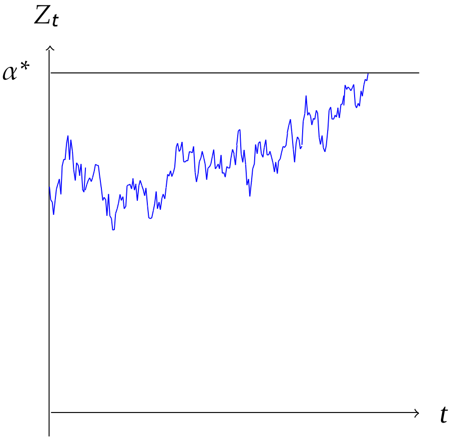
- Posterior belief process $\pi_t \equiv P\{\theta = H \mid X_s, 0 \le s \le t\}$
  satisfies $d\pi_t = \pi_t (1 - \pi_t)\, dB_t$ where B is another standard BM

- State process $Z_t \equiv \log\left(\frac{\pi_t}{1-\pi_t}\right), t \ge 0$

Buyer chooses stopping time ρ to adopt

- $\rho = \inf\{t \ge 0 : Z_t \ge \alpha^*\}$

Optimal Adoption Policy in Model with Learning



Sample Path of State Process Z

3. Strategic Seller

Seller has type $\theta \in \{H, L\}$, common prior $p_0 = \mathrm{P}(\theta = H)$.

Seller has flow cost $c > 0$.
Seller chooses stopping time τ to exit.
Payoffs (excluding flow cost and discounting):

- $(0,0)$ if $\tau \leq \rho$
- $(1_{\{\theta = H\}} - k, k)$ if $\rho < \tau$

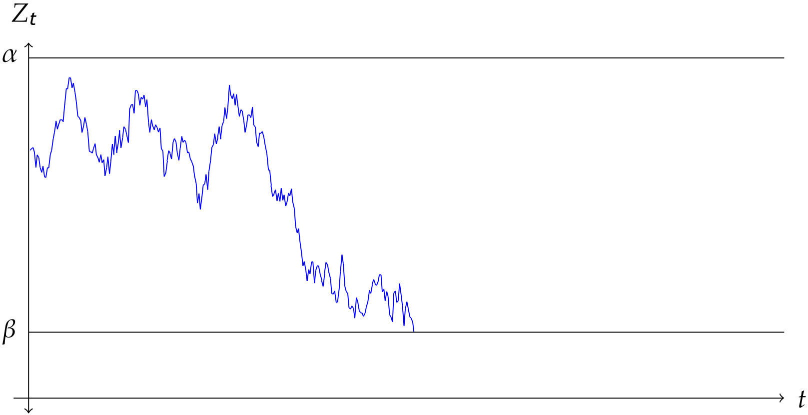
Seller exits when $Z_t \geq \beta$

Buyer adopts when $Z_t \geq \alpha$

$\beta < \alpha < \alpha^*$

Buyer is made worse off

Equilibrium pair (α,β) with learning, no private info



Sample Path of State Process Z

4. Asymmetric Information

Seller knows his type, buyer has known prior $p_0 = \mathrm{P}(\theta = H)$.

Seller types choose (or randomize over) stopping times τ_H, τ_L.

It suffices to consider selling strategies of the following form:

$\tau_H = \infty$,

$\tau_L = \inf\{t \geq 0: L_t \geq \xi\}$,

where $\{L_t, t \geq 0\}$ is ↑ and adapted to X,

$\xi \sim \exp(1)$, and ξ is independent of X.

Using the log-likelihood transformation, $Z_t \equiv \log\left(\frac{\pi_t}{1-\pi_t}\right), t \geq 0$,

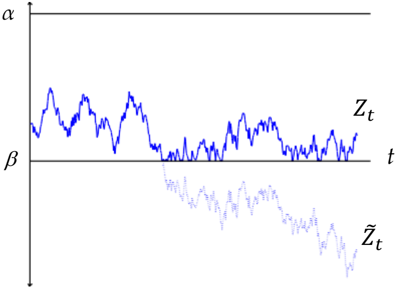
$$Z_t = \tilde{Z}_t + L_t$$

State process = State based on news alone + Conditioning on no exit

Equilibrium strategy pair with asymmetric information

Buyer: $\rho = \inf\{t \geq 0 \colon Z_t \geq \alpha\}$

Seller: $L_t = L_t^Z(\beta) =$ local time of Z at level β



Reflecting Equilibrium

There is killing in local time at the reflecting boundary (killing rate 1)

Equilibrium necessarily involves randomization

- Consider a putative equilibrium of the following form ($\beta < \alpha$): buyer adopts when $Z \geq \alpha$ and low-type seller exits when $Z \leq \beta$.

- Then buyer will adopt whenever $Z \leq \beta$, because seller's non-exit in that region guarantees $\theta = H$.

- Thus an equilibrium in pure (non-randomized) strategies cannot have the hypothesized form, and continued reasoning shows that it cannot have any other form either.

5. Endogenous Quality

Suppose that L can privately upgrade to H for lump-sum cost $K \in (0, 1)$.

The seller now chooses an exit time τ_L for use if low type, *and* an upgrade time υ_L for use if low type ($\tau_H = \upsilon_H = \infty$).

$\upsilon_L = \inf\{t \geq 0 : Q_t \geq \zeta\}$, where $\{Q_t, t \geq 0\}$ is $\uparrow$ and adapted to X, $\zeta \sim \exp(1)$, independent of X and ξ.

Seller type is now a process $\{\theta_t, t \geq 0\}$, and news arrives as

$$dX_t = \mu_{\theta_t} dt + \sigma dW_t.$$

Buyer's beliefs incorporate hidden upgrade possibility:

$$Z_t = \tilde{Z}_t + L_t + Q_t .$$

Three possible forms of equilibrium in the model with endogenous quality

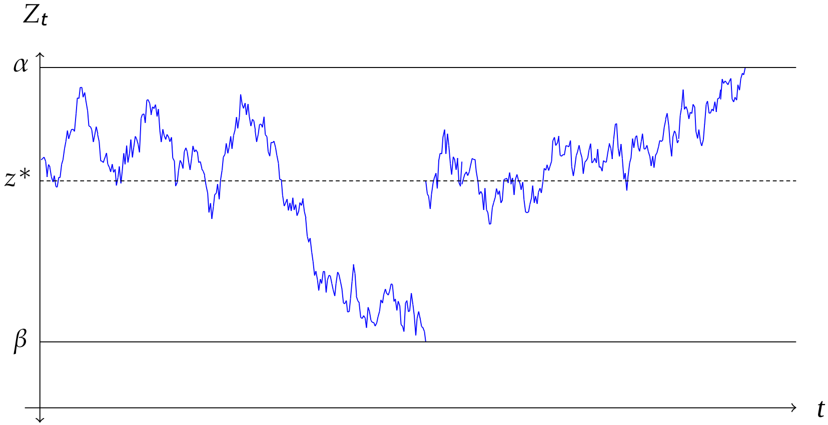
Critical values K* and K** satisfy $0 < K^{**} < K^* < \infty$.

- $0 \le K \le K^{**}$ $\Rightarrow$ *resetting equilibrium* with parameters α, β and z^* $(0 < \beta < z^* < \alpha < \infty)$

- $K \ge K^*$ $\Rightarrow$ reflecting equilibrium with parameters α and β $(0 < \beta < \alpha < \infty)$

- $K^{**} < K < K^*$ $\Rightarrow$ *skew-resetting equilibrium* with parameters α, β, $\hat{z}$, z^* and δ $(0 < \beta < \hat{z} < z^* < \alpha < \infty$ and $\delta > 0)$

Resetting Equilibrium



Q_t = sum of jumps, each of size ($z*$-β), initiated at successive times when Z = β.

Local Time and Skew-Brownian Motion

- Define *local time* of process Z at level z as follows:

$$L_t^Z(z) = \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \, meas\{s \in [0,t] : |Z(s) - z| \le \varepsilon\}$$

- An SDE involving own local time at z :

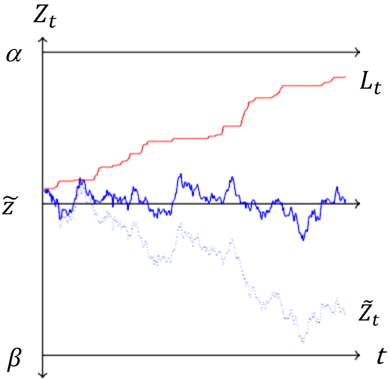
(1) $$Z_t = W_t + \delta L_t^Z(z)$$

- Harrison and Shepp (1981, *Annals of Probability*): (1) has a solution iff $|\delta| \le 1$, in which case solution is unique

- Limit of a rescaled binary random walk that is symmetric except for one distinguished point:

P{up} = 1- P{down} $= \frac{1+\delta}{2}$ at the distinguished point

Skew-Resetting Equilibrium



Sample Path of State Process Z

$$Z_t = \tilde{Z}_t + L_t + Q_t$$

$$L_t = \delta\, L_t^Z,\ |\delta| < 1$$

There is killing in local time at level $\tilde{z}$ (killing rate δ)

Q_t = sum of jumps, each of size (z*-β), initiated at successive times when Z = β.

Highlights

- Unique equilibrium involves randomization
- Surprising appearance of a “punched” or “partially reflected” diffusion process
- Novel phenomenon: (partial) reflection with killing in local time
- Novel interpretation of (partial) reflection: informational displacement