Mean-Field optimization problems and non-anticipative optimal transport

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The story in a nutshell

Given

- a (finite or infinite) set of agents
- who need to choose their actions/strategies
- and face a cost depending on their own type, action, and on the symmetric interaction with each other:

\[ \text{cost}(i) = \text{fct}\left(\text{type}(i), \text{action}(i), \text{(empirical) distrib. actions}\right) \]

Aim to

- find/characterize equilibria
- through connections with non-anticipative optimal transport
Outline

1. First setting: hidden/no dynamics
   - Problem formulation
   - Connection with non-anticipative optimal transport
   - Existence and uniqueness results

2. Second setting: state dynamics
   - Problem formulation
   - Connection with non-anticipative optimal transport
   - First results

3. Conclusions
Setting

- **Time set**: \( T = \{0, ..., T\} \), or \( T = [0, T] \)
- **\( X \)**: agents *types*
  - \( X \subseteq X^{\|T\|} \): agents types evolutions
- **\( Y \)**: agents’ *actions*
  - \( Y \subseteq Y^{\|T\|} \): agents’ actions evolutions
- *e.g.* \( X = Y = \mathbb{R} \), and \( X = Y = \mathbb{R}^{T+1} \) or \( X = Y = C([0, T]; \mathbb{R}) \)
- \( \eta \in \mathcal{P}(X) \): known a priori distribution over types

→ **Cost function**: \( \tilde{c}(x, y, \nu) \) (for each agent)

\[
\begin{align*}
& \uparrow \quad \uparrow \quad \downarrow \\
& \text{type} \quad \text{action} \quad \text{actions’ distribution} \\
& x \in X \quad y \in Y \quad \nu \in \mathcal{P}(Y)
\end{align*}
\]
Separable structure: \[ \tilde{c}(x, y, \nu) = c(x, y) + V[\nu](y) \]

- idiosyncratic part
- mean-field interaction

with \( c : X \times Y \to \mathbb{R}_+ \) l.s.c., \( V : \mathcal{P}(Y) \to \mathcal{B}(Y; \mathbb{R}_+) \)

- **congestion effect:** \( V_c[\nu](y) = f\left(y, \frac{d\nu}{dm}(y)\right) \), with \( m \in \mathcal{P}(Y) \) reference meas. w.r.t. which congestion measured, \( f(y, .) \)

- **attractive effect:** \( V_a[\nu](y) = \int_Y \phi(y, z)\nu(dz) \), with \( \phi \) symmetric, convex, minimal on the diagonal

Static case: Blanchet-Carlier 2015
Pure adapted strategies

pure strategy: all players of type $x \in \mathcal{X}$ choose the same strategy
$$y = A(x) = (A_t(x))_{t \in \mathbb{T}}$$

adapted strategy: $A_t(x) = T^t(x_{0:t})$ for some measurable $T^t$

Denote by $\mathcal{A}$ the set of pure adapted strategies $A : \mathcal{X} \rightarrow \mathcal{Y}$

- type distribution: $\eta \in \mathcal{P}(\mathcal{X})$ (known)
- strategy distribution: $\nu = A\#\eta = T\#\eta \in \mathcal{P}(\mathcal{Y})$, $T = (T^t)_{t \in \mathbb{T}}$ (will be determined in equilibrium)
Pure equilibrium

Social planner perspective: minimize average cost

For every $\nu \in \mathcal{P}(\mathcal{Y})$, denote

$$P(\nu) := \inf_{A \in \mathcal{A}} \int \left\{ c(x, A(x)) + V[\nu](A(x)) \right\} \eta(dx)$$

Definition

An element $A \in \mathcal{A}$ is called a pure equilibrium if

- $A$ attains $P(\nu)$,
- where $\nu = A \# \eta$. 

**Remark.** Let $\mathbb{T} = \{0, 1, \ldots, T\}$ (analogous in continuous time). Let $c(x, y) = \sum_{t=0}^{T} c_t(x_{0:t}, y_t)$ and $V[\nu](y) = \sum_{t=0}^{T} V_t[\nu_t](y_t)$, then

pure equilibrium for social planner = Cournot-Nash equilibrium ($\eta$-a.s. each agent acts as best response to other agents’ actions)

The equilibria are described by the set $\{A^\nu : A^\nu_#\eta = \nu\}$, where

$$A^\nu_t(x) = T^\nu_t(x_{0:t}) := \arg\min_z \{c_t(x_{0:t}, z) + V_t[\nu_t](z)\}.$$
From pure to mixed-strategy equilibrium

adapted pure strategy = adapted Monge transport
From pure to mixed-strategy equilibrium

non-anticipative mixed strategy = causal Kantorovich transport
Mixed non-anticipative strategy

mixed-strategy: players of same type can choose different actions

non-anticipative: $A_t(x) = \text{fct}(x_{0:t}) + \text{sth indep. of } x$

\[\downarrow\]

Non-anticipative (causal) transport: $\pi \in \mathcal{P}(X \times Y)$ s.t. $\rho_1 \# \pi = \eta$, and for all $t$ and $D \in \mathcal{F}_t^Y$, the map $X \ni x \mapsto \pi^X(D)$ is $\mathcal{F}_t^X$-measurable (where $(\mathcal{F}_t^X), (\mathcal{F}_t^Y)$ canonical filtr. in $X,Y$, and $\pi^X$ reg. cond. kernel)

Denote by $\Pi_c(\eta, \nu)$ the set of causal transports between $\eta$ and $\nu$, and let $\Pi_c(\eta, .) := \bigcup_{\nu \in \mathcal{P}(Y)} \Pi_c(\eta, \nu)$

Note that $\pi = (id, T) \# \eta \in \Pi_c(\eta, .)$ are the pure adapted strategies.
Mixed-strategy equilibrium

For every \( \nu \in \mathcal{P}(Y) \), denote

\[
M(\nu) := \inf_{\pi \in \Pi_c(\eta, \nu)} \mathbb{E}^{\pi} [c(x, y) + V[\nu](y)]
\]

**Definition**

An element \( \pi \in \Pi_c(\eta, .) \) is called a **mixed-strategy equilibrium** if

- \( \pi \) attains \( M(\nu) \),
- where \( \nu = p_2 \# \pi \), i.e., \( \pi \in \Pi_c(\eta, \nu) \).

**Remark.** Mixed-strategy equilibria are solutions to causal transport problems: if \( \pi^* \) m-s equilibrium, with \( p_2 \# \pi^* = \nu^* \), then it attains

\[
\inf_{\pi \in \Pi_c(\eta, \nu^*)} \mathbb{E}^{\pi} [c(x, y)].
\]

Analogously, pure equilibria = solutions to CT pbs over Monge maps
From the remark, we always have \( \text{equilibrium} \Rightarrow \text{optimal transport} \)

For **potential games**, we will have “\( \iff \)” in some sense

**Assumption**

There exists \( \mathcal{E} : \mathcal{P}(\mathcal{Y}) \rightarrow \mathbb{R} \) such that \( V \) is the **first variation** of \( \mathcal{E} \):

\[
\lim_{\epsilon \rightarrow 0^+} \frac{\mathcal{E}(\nu + \epsilon(\mu - \nu)) - \mathcal{E}(\nu)}{\epsilon} = \int_{\mathcal{Y}} V[\nu](y)(\mu - \nu)(dy), \quad \forall \nu, \mu \in \mathcal{P}(\mathcal{Y})
\]

E.g. \( V = V_c + V_a \) (repulsive+attractive effect) is the first variation of

\[
\mathcal{E}(\nu) = \int_{\mathcal{Y}} F\left(y, \frac{d\nu}{dm}(y)\right) m(dy) + \frac{1}{2} \int_{\mathcal{Y} \times \mathcal{Y}} \phi(y, z) \nu(dz) \nu(dy),
\]

where \( F(y, u) = \int_0^u f(y, s)ds \).
Consider the variational problem

\[
\begin{align*}
\inf_{\nu \in \mathcal{P}(Y)} & \left\{ \inf_{\pi \in \Pi_{c}(\eta, \nu)} \mathbb{E}^{\pi}[c(x, y)] + \mathcal{E}[\nu] \right\} \\
& = \text{CT}(\eta, \nu)
\end{align*}
\]

**Theorem**

*Let \( \mathcal{E} \) be convex, then the following are equivalent:*

(i) \( \pi^* \) is a mixed-strategy equilibrium, with \( p_2 \# \pi^* = \nu^* \);

(ii) \( \nu^* \) solves \((VP)\), and \( \pi^* \) solves \( \text{CT}(\eta, \nu^*) \).

**Remarks.** 1. Convexity only needed for “(i) \( \Rightarrow \) (ii)”
2. Convexity satisfied in the congestion case \((V = V_c)\)
3. Alternatively: displacement convexity can be used
Potential games

Corollary (uniqueness)

If $\mathcal{E}$ strictly convex $\Rightarrow$ all m-s equilibria have same second marginal $\nu^*$, i.e., unique optimal distribution of actions.

Indeed, $\nu \mapsto CT(\eta, \nu)$ convex, hence $\mathcal{E}$ strictly convex implies unique solution $\nu^*$ for (VP). Then apply theorem.

Corollary (existence)

For $V = V_c$ and growth condition on $f$ $\Rightarrow$ $\exists$ m-s equilibrium.

Indeed, the growth condition ensures existence of a solution $\nu^*$ for (VP), and $CT(\eta, \nu^*)$ admits a solution $\pi^*$ since $c$ is bounded below and l.s.c. Then apply theorem.
Example

Let $\mathbb{T} = \{0, 1, ..., T\}$, and $X = Y = \mathbb{R}^{T+1}$. If

- $\eta$ has independent increments, and
- $c(x, y) = c_0(x_0, y_0) + \sum_{t=1}^{T} c_t(x_t - x_{t-1}, y_t - y_{t-1})$, with $c_t(u, v) = k_t(u - v)$ and $k_t$ convex,

Then:

- m-s equilibria (if $\exists$) are determined by the second marginal
- m-s equilibria are the Knothe-Rosenblatt rearrangements
- if moreover $\eta$ has a density, all m-s equilibria are in fact pure
The Knothe-Rosenblatt map
The Knothe-Rosenblatt map

\[ T_1(x_1) \quad T_2(x_2|x_1) \]

\[ X_1 \quad X_2 \]
Actions as controls on dynamics

- The previous result describes a specific situation where optimal actions are increasing with the type.
- When these conditions not satisfied, which form of CT/equilibria?

**Example.** Let actions = controls on dynamics:

\[ X_t = (k_t^1 X_{t-1} + k_t^2 \alpha_t) + \epsilon_t, \quad t = 1, \ldots, T, \quad X_0 = x_0, \]

with associated cost \( f_t(X_t, \alpha_t, \nu_t) \) at time \( t \). As \( X_t = fct(\epsilon_i, \alpha_i, i \leq t) \),

\[ f_t(X_t, \alpha_t, \nu_t) = c_t(\epsilon_{0:t}, \alpha_{0:t}, \nu_t), \]

hence total cost \( = \mathbf{E}[\sum_{t=0}^{T} c_t(\epsilon_{0:t}, \alpha_{0:t}, \nu_t)]. \)

\[ \rightarrow \] Fits into previous framework, by reading “noises as types”.
With the above example in mind, we will consider

**McKean-Vlasov control problem:**

\[
\inf_{\alpha} E^P \left[ \int_0^T \tilde{f}_t \left( X_t, \alpha_t, \mathbb{P} \circ (X_t, \alpha_t)^{-1} \right) dt + \tilde{g} \left( X_T, \mathbb{P} \circ X_T^{-1} \right) \right]
\]

subject to

\[
dX_t = b_t \left( X_t, \alpha_t, \mathbb{P} \circ X_t^{-1} \right) dt + dW_t
\]

Let us first mention **connections to large systems of interacting controlled state processes**
The **private state process** $X^i$ of player $i$ is given by the solution to

$$dX^i_t = b_t(X^i_t, \alpha^i_t, \bar{\mu}^N_t)dt + dW^i_t$$

- $W^1, ..., W^N$ independent Wiener processes
- $\alpha^1, ..., \alpha^N$ controls of the $N$ players
- $\bar{\mu}^N_t = \frac{1}{N-1} \sum_{j \neq i} \delta_X^j_t$ empirical distrib. of states of the other players

The **objective** of player $i$ is to choose a $\alpha^i$ in order to minimize

$$\mathbb{E} \left[ \int_0^T \tilde{f}_t(X^i_t, \alpha^i_t, \tilde{\nu}^N_t) dt + \tilde{g}(X^i_T, \bar{\mu}^N_T) \right]$$

- $\tilde{\nu}^N_t = \frac{1}{N-1} \sum_{j \neq i} \delta_{(X^j_t, \alpha^j_t)}$ empirical joint distrib. of states and controls of the other players

Statistically identical players: same functions $b_t, \tilde{f}_t, \tilde{g}$
From N-player game to McKean-Vlasov control problem

Approximation by asymptotic arguments:
- first optimization then limit for $N \to \infty$, or
- viceversa, first limit for $N \to \infty$ and then optimization

SDE State Dynamics
for N players
\[ \lim_{N \to \infty} \]
optimization
Nash equilibrium
for N players
\[ \lim_{N \to \infty} \]
Mean-Field Game

McKean-Vlasov dynamics
optimization
controlled McK-V dyn

(Carmona-Delarue-Lachapelle 2012)
McKean-Vlasov control problem

Back to the McKean-Vlasov control problem.

For simplicity:
- no terminal cost: \( \tilde{g} = 0 \)
- separable costs: \( \tilde{f}_t(x, a, \nu) = f_t(x, a) + K_t(\nu) \)

Therefore

\[
\inf_{\alpha} \mathbb{E}^\mathbb{P} \left[ \int_0^T \{ f_t(X_t, \alpha_t) + K_t \left( \mathbb{P} \circ (X_t, \alpha_t)^{-1} \right) \} dt \right]
\]

\[
dX_t = b_t \left( X_t, \alpha_t, \mathbb{P} \circ X_t^{-1} \right) dt + dW_t,
\]

with \( f_t : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \), \( K_t : \mathcal{P}(\mathbb{R} \times \mathbb{R}) \to \mathbb{R} \), \( b_t : \mathbb{R} \times \mathbb{R} \times \mathcal{P}(\mathbb{R}) \to \mathbb{R} \)
**McKean-Vlasov control problem**

**Definition.** A weak solution to the McKean-Vlasov control problem is a tuple \((\Omega, (\mathcal{F}_t)_{t \in [0,T]}, \mathbb{P}, W, X, \alpha)\) such that:

(i) \((\Omega, (\mathcal{F}_t)_{t \in [0,T]}, \mathbb{P})\) supports \(X\) and a BM \(W\), \(\alpha\) is \(\mathcal{F}^X\)-progress. measurable and \(\mathbb{E}^\mathbb{P}[\int_0^T |\alpha_t|^2] < \infty\)

(ii) the state equation \(dX_t = b_t \left(X_t, \alpha_t, \mathbb{P} \circ X_t^{-1}\right) dt + dW_t\) holds

(iii) if \((\Omega', (\mathcal{F}'_t)_{t \in [0,T]}, \mathbb{P}', W', X', \alpha')\) is another tuple s.t. (i)-(ii) hold,
\[
\mathbb{E}^\mathbb{P}\left[\int_0^T \left\{f_t(X_t, \alpha_t) + K_t \left(\mathbb{P} \circ (X_t, \alpha_t)^{-1}\right)\right\} dt\right] \leq \mathbb{E}^{\mathbb{P}'}\left[\int_0^T \left\{f_t(X'_t, \alpha'_t) + K_t \left(\mathbb{P}' \circ (X'_t, \alpha'_t)^{-1}\right)\right\} dt\right].
\]
→ We need some **technical assumptions**.

→ In the case of linear drift:

\[ dX_t = (c_1^t X_t + c_2^t \alpha_t + c_3^t \mathbb{E}[X_t])dt + dW_t, \]

\( c_i^t \in \mathbb{R}, c_2^t > 0 \), the assumptions reduce to:

- \( f_t(x, .) \) convex (and \( f_t(., y) \) at least quadratic growth)
- \( K_t \) is \( <c \)-monotone

**Example.**

- \( f_t(x, a) = d_1^t x + d_2^t a + d_3^t x^2 + d_4^t a^2, \quad d_i^t \in \mathbb{R}, d_4^t > 0 \)
- \( K_t(\zeta) = F_t(\bar{\zeta}_1, \bar{\zeta}_2), \) any \( F_t, \bar{\zeta}_i := \int yd(p_i \# \zeta)(y) \)
Characterization via non-anticipative optimal transport

- formulate a transport problem in the path space $C([0, T])$
- denote by $\gamma$ the Wiener measure on $C([0, T])$
- $(\omega, \overline{\omega})$ generic element on $C([0, T]) \times C([0, T])$
- “move noises into states”

Theorem

*Under the mentioned assumptions, the weak MKV problem is equivalent to the variational problem*

$$\inf_{\mu \ll \gamma} \inf_{\pi \in \Pi_{bc}(\gamma, \mu)} \left\{ \mathbb{E}^{\pi} \left[ \int_{0}^{T} f_t (\overline{\omega}_t, u_t(\omega, \overline{\omega}, \mu)) \, dt \right] + \int_{0}^{T} K_t \left( (p_2, u_t(\omega, \overline{\omega}, \mu)) \# \pi_t \right) \, dt \right\}$$

where $u_t(\omega, \overline{\omega}, \mu) = b_t^{-1}(\overline{\omega}_t, \mu_t)((\overline{\omega} - \omega)_t)$.

$$\Pi_{bc}(\gamma, \mu) = \left\{ \pi \in \Pi_c(\gamma, \mu) : \ell \# \pi \in \Pi_c(\mu, \gamma) \right\}, \text{ where } \ell(x, y) = (y, x)$$
Characterization via non-anticipative optimal transport

Remarks.

- The optimization over $\Pi_{bc}(\gamma, \mu)$ is not a standard optimal transport problem $\Rightarrow$ new analysis for existence/duality.
- When mean-field cost is $K_t(P \circ X_t^{-1})$ $\Rightarrow$ standard causal transport problem (A.-Backhoff-Zalashko 2016)

Example.

- state dynamics: $dX_t = \alpha_t dt + dW_t$
- cost: $\mathbb{E}^P \left[ \frac{1}{2} \int_0^T (X_t^2 + \alpha_t^2) dt \right] + \int_0^T K_t(P \circ X_t^{-1}) dt$

$\Rightarrow$ in the variational problem we have causal optimal transport w.r.t. Cameron-Martin distance:

$$\inf_{\pi \in \Pi_{bc}(\gamma, \mu)} \mathbb{E}^\pi [||\omega - \omega||_H^2] = \mathcal{H}(\mu|\gamma),$$

hence we are left with

$$\inf_{\mu \ll \gamma} \{\mathcal{H}(\mu|\gamma) + P(\mu)\}, \quad P(\mu) \text{ penalty term}$$
Conclusions

In the case with **hidden/no dynamics**:  
- **characterization** of equilibrium via non-anticipative transport  
- **existence** and **uniqueness** results  
- characterization of causal optimal transport (≠ KR)...

In the case with **state dynamics**:  
- **characterization** of weak McKean-Vlasov solutions via non-anticipative transport  
- existence and uniqueness... 
- characterization of causal optimal transport...
Bibliography et al.


Thank you for your attention and

Buon compleanno Ioannis! :)