Portfolio Optimisation under Transaction Costs

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joint work with
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We fix a strictly positive càdlàg stock price process $S = (S_t)_{0 \leq t \leq T}$.

For $0 < \lambda < 1$ we consider the bid-ask spread $[(1 - \lambda)S, S]$.

A self-financing trading strategy is a càglàd finite variation process $\varphi = (\varphi_t^0, \varphi_t^1)_{0 \leq t \leq T}$ such that

$$d\varphi_t^0 \leq -S_t(d\varphi_t^1)_+ + (1 - \lambda)S_t(d\varphi_t^1)_-$$

$\varphi$ is called admissible if, for some $M > 0$,

$$\varphi_t^0 + (1 - \lambda)S_t(\varphi_t^1)_+ - S_t(\varphi_t^1)_- \geq -M$$
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A consistent-price system is a pair \((\tilde{S}, Q)\) such that \(Q \sim P\), the process \(\tilde{S}\) takes its value in \([(1 - \lambda)S, S]\), and \(\tilde{S}\) is a \(Q\)-martingale.

Identifying \(Q\) with its density process

\[
Z^0_t = \mathbb{E} \left[ \frac{dQ}{dP} \bigg| \mathcal{F}_t \right], \quad 0 \leq t \leq T
\]

we may identify \((\tilde{S}, Q)\) with the \(\mathbb{R}^2\)-valued martingale

\(Z = (Z^0_t, Z^1_t)_{0 \leq t \leq T}\) such that

\[
\tilde{S} := \frac{Z^1_t}{Z^0_t} \in [(1 - \lambda)S, S].
\]

For \(0 < \lambda < 1\), we say that \(S\) satisfies \((CPS^\lambda)\) if there is a consistent price system for transaction costs \(\lambda\).
Definition [Jouini-Kallal (’95), Cvitanic-Karatzas (’96), Kabanov-Stricker (’02),...]

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The set of non-negative claims attainable at price $x$ is

$$C(x) = \left\{ X_T \in L^0_+ : \text{there is an admissible } \varphi = (\varphi^0_t, \varphi^1_t)_{0 \leq t \leq T} \right.\\ \text{starting at } (\varphi^0_0, \varphi^1_0) = (x, 0) \text{ and ending at } (\varphi^0_T, \varphi^1_T) = (X_T, 0) \left. \right\}$$

Given a utility function $U : \mathbb{R}_+ \to \mathbb{R}$ define

$$u(x) = \sup \{ \mathbb{E}[U(X_T) : X_T \in C(x)] \}.$$
Question 1

What are conditions ensuring that $C(x)$ is closed in $L^0_+(\mathbb{P})$. (w.r. to convergence in measure)?

Theorem [Cvitanic-Karatzas ('96), Campi-S. ('06)]:

Suppose that $(CPS^\mu)$ is satisfied, for all $\mu > 0$, and fix $\lambda > 0$. Then $C(x) = C^\lambda(x)$ is closed in $L^0$.

Remark [Guasoni, Rasonyi, S. ('08)]

If the process $S = (S_t)_{0 \leq t \leq T}$ is continuous and has conditional full support, then $(CPS^\mu)$ is satisfied, for all $\mu > 0$. For example, exponential fractional Brownian motion verifies this property.
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The dual objects

Definition

We denote by $D(y)$ the convex subset of $L^0_+(\mathbb{P})$

$$D(y) = \{ yZ^0_T = y\frac{dQ}{d\mathbb{P}}, \text{for some consistent price system } (\tilde{S}, Q) \}$$

and

$$D(y) = \overline{\text{sol} (D(y))}$$

the closure of the solid hull of $D(y)$ taken with respect to convergence in measure.
Definition [Kramkov-S. ('99), Karatzas-Kardaras ('06), Campi-Owen ('11), ...]

We call a process $Z = (Z^0_t, Z^1_t)_{0 \leq t \leq T}$ a super-martingale deflator if $Z^0_0 = 1$, $\frac{Z^1_t}{Z^0_t} \in [(1 - \lambda)S, S]$, and for each $x$-admissible, self-financing $\varphi$ the value process

$$(\varphi^0_t + x)Z^0_t + \varphi^1_t Z^1_t = Z^0_t (\varphi^0_t + x + \varphi^1_t \frac{Z^1_t}{Z^0_t})$$

is a super-martingale.

Proposition

$D(y) = \{yZ^0_T : Z = (Z^0_t, Z^1_t)_{0 \leq t \leq T} a super-martingale deflator\}$
Definition [Kramkov-S. ('99), Karatzas-Kardaras ('06), Campi-Owen ('11),...]

We call a process \( Z = (Z_t^0, Z_t^1)_{0 \leq t \leq T} \) a super-martingale deflator if \( Z_0^0 = 1, \frac{Z_1^1}{Z_0^0} \in [(1 - \lambda)S, S] \), and for each \( x \)-admissible, self-financing \( \varphi \) the value process

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\( \mathcal{D}(y) = \{yZ_T^0 : Z = (Z_t^0, Z_t^1)_{0 \leq t \leq T} \text{ a super- martingale deflator} \} \)
Theorem (Czichowsky, Muhle-Karbe, S. ('12))

Let $S$ be a càdlàg process, $0 < \lambda < 1$, suppose that $(CPS^\mu)$ holds true, for each $\mu > 0$, suppose that $U$ has reasonable asymptotic elasticity and $u(x) < U(\infty)$, for $x < \infty$. Then $C(x)$ and $D(y)$ are polar sets:

\[
X_T \in C(x) \text{ iff } \langle X_T, Y_T \rangle \leq xy, \quad \text{for } Y_T \in D(y) \\
Y_T \in D(y) \text{ iff } \langle X_T, Y_T \rangle \leq xy, \quad \text{for } X_T \in C(y)
\]

Therefore by the abstract results from [Kramkov-S. ('99)] the duality theory for the portfolio optimisation problem works as nicely as in the frictionless case: for $x > 0$ and $y = u'(x)$ we have
(i) There is a unique primal optimiser $\hat{X}_T(x) = \hat{\phi}_T^0$ which is the terminal value of an optimal $(\hat{\phi}_t^0, \hat{\phi}_t^1)_{0 \leq t \leq T}$.

(i') There is a unique dual optimiser $\hat{Y}_T(y) = \hat{Z}_T^0$ which is the terminal value of an optimal super-martingale deflator $(\hat{Z}_t^0, \hat{Z}_t^1)_{0 \leq t \leq T}$.

(ii) $U'(\hat{X}_T(x)) = \hat{Z}_t^0(y)$, $-V'(\hat{Z}_T(y)) = \hat{X}_T(x)$

(iii) The process $(\hat{\phi}_t^0 \hat{Z}_t^0 + \hat{\phi}_t^1 \hat{Z}_t^1)_{0 \leq t \leq T}$ is a martingale, and therefore

$\{d\hat{\phi}_t^0 > 0\} \subseteq \{\frac{\hat{Z}_t^1}{\hat{Z}_t^0} = (1 - \lambda)S_t\}$,

$\{d\hat{\phi}_t^0 < 0\} \subseteq \{\frac{\hat{Z}_t^1}{\hat{Z}_t^0} = S_t\}$,

etc. etc.
Theorem [Cvitanic-Karatzas ('96)]

In the setting of the above theorem *suppose* that \((\hat{Z}_t)_{0 \leq t \leq T}\) is a local martingale. Then \(\hat{S} = \frac{\hat{Z}^1}{\hat{Z}^0}\) is a *shadow price*, i.e. the optimal portfolio for the frictionless market \(\hat{S}\) and for the market \(S\) under transaction costs \(\lambda\) coincide.

**Sketch of Proof**

Suppose (w.l.g.) that \((\hat{Z}_t)_{0 \leq t \leq T}\) is a true martingale. Then \(\frac{d\hat{Q}}{d\hat{P}} = \hat{Z}^0_T\) defines a *probability measure* under which the process \(\hat{S} = \frac{\hat{Z}^1}{\hat{Z}^0}\) is a martingale. Hence we may apply the frictionless theory to \((\hat{S}, \hat{Q})\). \(\hat{Z}^0_T\) is (a fortiori) the dual optimizer for \(\hat{S}\).

As \(\hat{X}_T\) and \(\hat{Z}^0_T\) satisfy the first order condition

\[ U'(\hat{X}_T) = \hat{Z}^0_T, \]

\(\hat{X}_T\) must be the optimizer for the frictionless market \(\hat{S}\) too.
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As \(\hat{X}_T\) and \(\hat{Z}_0^T\) satisfy the first order condition
\[ U'(\hat{X}_T) = \hat{Z}_0^T, \]
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Question
When is the dual optimizer $\hat{Z}$ a local martingale? Are there cases when it only is a super-martingale?
Theorem [Czichowsky-S. (’12)]

Suppose that $S$ is continuous and satisfies ($NFLVR$), and suppose that $U$ has reasonable asymptotic elasticity. Fix $0 < \lambda < 1$ and suppose that $u(x) < U(\infty)$, for $x < \infty$.

Then the dual optimizer $\hat{Z}$ is a local martingale. Therefore $\hat{S} = \frac{\hat{Z}^1}{\hat{Z}^0}$ is a shadow price.

Remark

The condition ($NFLVR$) cannot be replaced by requiring ($CPS^\lambda$), for each $\lambda > 0$. 
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Examples

Frictionless Example [Kramkov-S. ('99)]

Let \( U(x) = \log(x) \). The stock price \( S = (S_t)_{t=0,1} \) is given by

\[
\begin{align*}
1 & \quad \rho \quad 2 \\
1 & \quad \varepsilon_1 \quad 1 \\
1 & \quad \varepsilon_n \quad \frac{1}{n}
\end{align*}
\]

...
Here $\sum_{n=1}^{\infty} \varepsilon_n = 1 - p \ll 1$.

For $x = 1$ the optimal strategy is to buy one stock at time 0 i.e. $\hat{\varphi}_1^1 = 1$.

Let $A_n = \{S_1 = \frac{1}{n}\}$ and consider $A_\infty = \{S_1 = 0\}$ so that $\mathbb{P}[A_n] = \varepsilon_n > 0$, for $n \in \mathbb{N}$, while $\mathbb{P}[A_\infty] = 0$.

Intuitively speaking, the constraint $\hat{\varphi}_1^1 \leq 1$ comes from the null-set $A_\infty$ rather than from any of the $A_n$'s. It turns out that the dual optimizer $\hat{Z}$ verifies $\mathbb{E}[\hat{Z}_1] < 1$, i.e. only is a super-martingale. Intuitively speaking, the optimal measure $\hat{Q}$ gives positive mass to the $\mathbb{P}$-null set $A_\infty$ (compare Cvitanic-Schachermayer-Wang ('01), Campi-Owen ('11)).
Discontinuous Example under transaction costs $\lambda$
(Czichowsky, Muhle-Karbe, S. ('12), compare also Benedetti, Campi, Kallsen, Muhle-Karbe ('11)).

For $x = 1$ it is optimal to buy $\frac{1}{1+\lambda}$ many stocks at time 0. Again, the constraint comes from the $\mathbb{P}$-null set $A_\infty = \{S_1 = 1\}$.

There is no shadow-price. The intuitive reason is again that the binding constraint on the optimal strategy comes from the $\mathbb{P}$-null set $A_\infty = \{S_1 = 1\}$.
Continuous Example under Transaction Costs [Czichowsky-S. (’12)]

Let \((W_t)_{t \geq 0}\) be a Brownian motion, starting at \(W_0 = w > 0\), and
\[
\tau = \inf \{ t : W_t - t \leq 0 \}
\]

Define the stock price process
\[
S_t = e^{t \wedge \tau}, \quad t \geq 0.
\]

\(S\) does not satisfy \((NFLVR)\), but it does satisfy \((CPS^\lambda)\), for all \(\lambda > 0\).

Fix \(U(x) = \log(x)\), transaction costs \(0 < \lambda < 1\), and the initial endowment \((\varphi_0^0, \varphi_0^1) = (1, 0)\).
For the trade at time \(t = 0\), we find three regimes determined by thresholds \(0 < \underline{w} < \bar{w} < \infty\).
(i) if \( w \leq \bar{w} \) we have \((\hat{\phi}_0^0, \hat{\phi}_0^1) = (1, 0)\), i.e. no trade.

(ii) if \( \underline{w} < w < \bar{w} \) we have \((\hat{\phi}_0^0, \hat{\phi}_0^1) = (1 - a, a)\), for some \( 0 < a < \frac{1}{\lambda} \).

(iii) if \( w \geq \bar{w} \), we have \((\hat{\phi}_0^0, \hat{\phi}_0^1) = (1 - \frac{1}{\lambda}, \frac{1}{\lambda})\), so that the liquidation value is zero (maximal leverage).
We now choose $W_0 = w$ with $w > \bar{w}$.

Note that the optimal strategy $\hat{\varphi}$ continues to increase the position in stock, as long as $W_t - t \geq \bar{w}$.

If there were a shadow price $\hat{S}$, we therefore necessarily would have

$$\hat{S}_t = e^t, \quad \text{for } 0 \leq t \leq \inf\{u : W_u - u \leq \bar{w}\}.$$ 

But this is absurd, as $\hat{S}$ clearly does not allow for an e.m.m.
Let \((B_t^H)_{0 \leq t \leq T}\) be a fractional Brownian motion with Hurst index \(H \in (0, 1]\). Let \(S = \exp(B_t^H)\), and fix \(\lambda > 0\) and \(U(x) = \log(x)\).

Is the dual optimiser a local martingale or only a super-martingale? Equivalently, is there a shadow price \(\hat{S}\)?
Αγαπητέ Γιάννη,
Σε ευχαριστώ για τη φιλία σου και σου εύχομαι Χρόνια Πολλά για τα Γενέθλιά σου!

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