Probability, Control and Finance

In honor for Ioannis Karatzas

Columbia University, June 6, 2012

Monique Jeanblanc,
Université d'Évry-Val-D'Essonne

Random times and Azéma supermartingales

Joint work with S. Song

Financial support from Fédération Bancaire Française

Problem

Motivation: In credit risk, in mathematical finance, one works with a random time which represents the default time (in a single default context). Many studies are based on the intensity process: starting with a reference filtration $\mathbb{F}$, the intensity process of τ is the $\mathbb{F}$ predictable increasing process Λ such that

$$\mathbb{1}_{\tau \leq t} - \Lambda_{t \wedge \tau}$$

is a $\mathbb{G}$-martingale, where $\mathcal{G}_t = \cap_{\epsilon > 0} \mathcal{F}_{t+\epsilon} \vee \sigma(\tau \wedge (t + \epsilon))$.

Then, the problem is : **given Λ, construct a random time τ which admits Λ as intensity**.

A classical construction is: extend the probability space $(\Omega, \mathbb{F}, \mathbb{P})$ so that there exists a random variable Θ, with exponential law, independent of $\mathcal{F}_\infty$ and define

$$\tau := \inf\{t \,:\, \Lambda_t \geq \Theta\}$$

Our goal is to provide other constructions.

One starts with noting that, in general,

$$Z_t = \mathbb{P}(\tau > t|\mathcal{F}_t)$$

is a supermartingale (called the Azéma supermartingale) with multiplicative decomposition $Z_t = N_t D_t$, where N is a local martingale and D a decreasing predictable process.

Assuming that Z does not vanishes, we set $D_t = e^{-\Lambda_t}$. We shall now assume **that Λ is continuous**, and that $Z_0 = 1$. Then, one proves that Λ is the intensity of τ.

Problem $(\star)$: let $(\Omega, \mathbb{F}, \mathbb{P})$ be a filtered probability space, Λ an increasing continuous process, N a non-negative local martingale such that

$$0 < N_t e^{-\Lambda_t} \leq 1$$

Construct, on the canonical extended space $(\Omega \times [0, \infty])$, a probability $\mathbb{Q}$ such that

1. **restriction condition** $\mathbb{Q}|_{\mathcal{F}_\infty} = \mathbb{P}|_{\mathcal{F}_\infty}$
2. **projection condition** $\mathbb{Q}[\tau > t|\mathcal{F}_t] = N_t e^{-\Lambda_t}$

Here, τ is the canonical map. We shall note $\mathbb{P}(X) := \mathbb{E}_{\mathbb{P}}(X)$.

Particular case: $Z = e^{-\Lambda}$.

In that case a solution (the Cox solution) is

$$\tau = \inf\{t \,:\, \Lambda_t \geq \Theta\}$$

where Θ is a random variable with unit exponential law, independent of $\mathcal{F}_\infty$, or in other words $\mathbb{Q} = \mathbb{Q}^C$ where, for $A \in \mathcal{F}_\infty$:

$$\mathbb{Q}^C(A \cap \{s < \tau \leq t\}) = \mathbb{P}\left(1\!\!1_A \int_s^t e^{-\Lambda_u} d\Lambda_u\right)$$

so that

$$\mathbb{Q}^C(\tau > \theta | \mathcal{F}_t) = e^{-\Lambda_\theta}, \text{ for } t \geq \theta$$

Outline of the talk

- Increasing families of martingales
- Semi-martingale decompositions
- Predictable Representation Theorem
- Exemple

The link between the supermartingale Z and the conditional law $\mathbb{Q}(\tau \in du|\mathcal{F}_t)$ for $u \leq t$ is: Let $M_t^u = \mathbb{Q}(\tau \leq u|\mathcal{F}_t)$, then M is increasing w.r.t. u and

$$\begin{aligned} M_u^u &= 1 - Z_u \\ M_t^u &\leq M_t^t = 1 - Z_t \end{aligned}$$

(Note that, for $t < u$, $M_t^u = \mathbb{E}(1 - Z_u|\mathcal{F}_t)$).

Solving the problem $(\star)$ is equivalent to find a family M^u

Family iM_Z

An increasing family of positive martingales bounded by $1 - Z$ (in short iM_Z) is a family of processes $(M^u : 0 < u < \infty)$ satisfying the following conditions:

1. Each M^u is a càdlàg $\mathbb{P}$-$\mathbb{F}$ **martingale** on $[u, \infty]$.
2. For any u, the martingale M^u is positive and closed by $M^u_\infty = \lim_{t\to\infty} M^u_t$.
3. For each fixed t, $0 < t \leq \infty$, $u \in [0, t] \to M^u_t$ **is a right continuous increasing map**.
4. $M^u_u = 1 - Z_u$ and $M^u_t \leq M^t_t = 1 - Z_t$ for $u \leq t \leq \infty$.

Given an iM_Z, let $d_u M_\infty^u$ be the random measure on $(0, \infty)$ associated with the increasing map $u \to M_\infty^u$. The following probability measure $\mathbb{Q}$ is a solution of the problem $(\star)$

$$\mathbb{Q}(F) := \mathbb{P}\left(\int_{[0,\infty]} F(u, \cdot)\big(M_\infty^0 \delta_0(du) + d_u M_\infty^u + (1 - M_\infty^\infty)\delta_\infty(du)\big)\right)$$

The two properties for $\mathbb{Q}$:

- **Restriction condition**: For $B \in \mathcal{F}_\infty$,

$$\mathbb{Q}(B) = \mathbb{P}\left(\mathbb{I}_B \int_{[0,\infty]} (M_\infty^0 \delta_0(du) + d_u M_\infty^u + (1 - M_\infty^\infty)\delta_\infty(du))\right) = \mathbb{P}[B]$$

- **Projection condition:** For $0 \leq t < \infty$, $A \in \mathcal{F}_t$,

$$\mathbb{Q}[A \cap \{\tau \leq t\}] = \mathbb{P}[\mathbb{I}_A M_\infty^t] = \mathbb{P}[\mathbb{I}_A M_t^t] = \mathbb{Q}[\mathbb{I}_A (1 - Z_t)]$$

are satisfied.

Constructions of iM_Z

Hypothesis (✠) For all $0 < t < \infty$, $0 \leq Z_t < 1, 0 \leq Z_{t-} < 1$.

The simplest iM_Z

Assume conditions (✠). The family

$$M_t^u := (1 - Z_t) \exp\left(-\int_u^t \frac{Z_s}{1 - Z_s} d\Lambda_s\right), \ 0 < u < \infty, u \leq t \leq \infty,$$

defines an iM_Z, called **basic solution**. We note that

$$dM_t^u = -M_{t-}^u \frac{e^{-\Lambda_t}}{1 - Z_{t-}} dN_t, \ 0 < u \leq t < \infty.$$

Other solutions To construct an iM_Z, we have to check four constraints :

$i.\ M_u^u = (1 - Z_u)$

$ii.\ 0 \leq M^u$

$iii.\ M^u \leq 1 - Z$

$iv.\ M^u \leq M^v$ for $u < v$

These constraints are easy to handle if M^u are solutions of a SDE:

The constraint i indicates the initial condition;
the constraint ii means that we must take an exponential SDE;
the constraint iv is a comparison theorem for one dimensional SDE,
the constraint iii can be handled by local time as described in the following result :

Let m be a $(\mathbb{P}, \mathbb{F})$-local martingale such that $m_u \leq 1 - Z_u$. Then, $m_t \leq (1 - Z_t)$ on $t \in [u, \infty)$ if and only if the local time at zero of $m - (1 - Z)$ on $[u, \infty)$ is identically null.

Other solutions To construct an iM_Z, we have to check four constraints :

$i.$ $M^u_u = (1 - Z_u)$

$ii.$ $0 \leq M^u$

$iii.$ $M^u \leq 1 - Z$

$iv.$ $M^u \leq M^v$ for $u < v$

These constraints are easy to handle if M^u are solutions of a SDE:
The constraint i indicates the initial condition;
the constraint ii means that we must take an exponential SDE;
the constraint iv is a comparison theorem for one dimensional SDE,
the constraint iii can be handled by local time as described in the following result :

Let m be a $(\mathbb{P}, \mathbb{F})$-local martingale such that $m_u \leq 1 - Z_u$. Then, $m_t \leq (1 - Z_t)$ on $t \in [u, \infty)$ if and only if the local time at zero of $m - (1 - Z)$ on $[u, \infty)$ is identically null.

Other solutions To construct an iM_Z, we have to check four constraints :

$i.$ $M^u_u = (1 - Z_u)$

$ii.$ $0 \leq M^u$

$iii.$ $M^u \leq 1 - Z$

$iv.$ $M^u \leq M^v$ for $u < v$

These constraints are easy to handle if M^u are solutions of a SDE:
The constraint i indicates the initial condition;
the constraint ii means that we must take an exponential SDE;
the constraint iv is a comparison theorem for one dimensional SDE,
the constraint iii can be handled by local time as described in the following result :

Let m be a $(\mathbb{P}, \mathbb{F})$-local martingale such that $m_u \leq 1 - Z_u$. Then, $m_t \leq (1 - Z_t)$ on $t \in [u, \infty)$ if and only if the local time at zero of $m - (1 - Z)$ on $[u, \infty)$ is identically null.

Generating equation when $1 - Z > 0$

Hypothesis (✠✠):

1. For all $0 < t < \infty$, $0 \leq Z_t < 1, 0 \leq Z_{t-} < 1$.
2. All $\mathbb{P}$-$\mathbb{F}$ martingales are continuous.

Assume (✠✠). Let Y be a $(\mathbb{P}, \mathbb{F})$ local martingale and f be a (bounded) Lipschitz function with $f(0) = 0$. For any $0 \leq u < \infty$, we consider the equation

$$(\star_u) \left\{ \begin{array}{lll} dX_t & = & X_t \left(-\dfrac{e^{-\Lambda_t}}{1 - Z_t} dN_t + f(X_t - (1 - Z_t)) dY_t \right), \ u \leq t < \infty \\ X_u & = & x \end{array} \right.$$

Generating equation when $1 - Z > 0$

Hypothesis (✠✠):

1. For all $0 < t < \infty$, $0 \leq Z_t < 1, 0 \leq Z_{t-} < 1$.
2. All $\mathbb{P}$-$\mathbb{F}$ martingales are continuous.

Assume (✠✠). Let Y be a $(\mathbb{P}, \mathbb{F})$ local martingale and f be a bounded Lipschitz function with $f(0) = 0$. For any $0 \leq u < \infty$, we consider the equation

$$(\star_u) \left\{ \begin{array}{rcl} dX_t &=& X_t \left(-\frac{e^{-\Lambda_t}}{1 - Z_t} dN_t + f(X_t - (1 - Z_t)) dY_t \right), \quad u \leq t < \infty \\ X_u &=& x \end{array} \right.$$

Let M^u be the solution on $[u, \infty)$ of the equation $(\star_u)$ with initial condition $M_u^u = 1 - Z_u$. Then, $(M^u, u \leq t < \infty)$ defines an iM_Z.

Particular case:

in the case of a Brownian filtration, for $N = 1$ (so that $Z_t = e^{-\Lambda_t}$ and $f(x) = x$,

$$\begin{cases} dM_t^u &= M_t^u \left(M_t^u - (1 - Z_t)\right) dB_t, \; u \leq t < \infty \\ M_u^u &= 1 - Z_u \end{cases}$$

In that case, one can check that $M_\infty^u = \mathbb{1}_{\tau \leq u}$. The fact that Z is decreasing show that τ is a pseudo-stopping time (i.e., times such that, for any BOUNDED $\mathbb{F}$ martingale m, one has

$$\mathbb{E}(m_\tau) = m_0$$

hence, for any $\mathbb{F}$ martingale X, the stopped process x^τ is a $\mathbb{G}$ martingale.

Balayage formula when $1 - Z$ can reach zero

We introduce $\mathcal{Z} = \{s : 1 - Z_s = 0\}$ and, for $t \in (0, \infty)$, the random time

$$g_t \quad := \quad \sup\{0 \le s \le t : s \in \mathcal{Z}\}$$

Hypothesis($\mathcal{Z}$) The set $\mathcal{Z}$ is not empty and is closed.
The measure $d\Lambda$ has a decomposition $d\Lambda_s = dV_s + dA_s$ where V, A are continuous increasing processes such that dV charges only $\mathcal{Z}$ while dA charges its complementary $\mathcal{Z}^c$.
Moreover, we suppose

$$\mathbb{I}_{\{g_t \le u < t\}} \int_u^t \frac{Z_s}{1 - Z_s} dA_s < \infty$$

for any $0 < u < t < \infty$.

We suppose that $\mathbf{Hy}(\mathcal{Z})$. The family

$$M_t^u = (1 - Z_u) - \int_u^t \mathbb{I}_{\{g_s \le u\}} \exp\left(-\int_u^s \frac{Z_v}{1 - Z_v} dA_v\right) e^{-\Lambda_s} dN_s$$

defines an iM_Z.

Note that

$$M_t^u = \mathbb{I}_{\{g_t \le u\}} \exp\left(-\int_u^t \frac{Z_s}{1 - Z_s} dA_s\right) (1 - Z_t), \; 0 < u < \infty, u \le t \le \infty.$$

Particular case:

If $\mathbb{F}$ is a Brownian filtration and

$$Z_t = N_t \left(\sup_{s \leq t} N_s\right)^{-1}$$

where $N_t \rightarrow_{t \rightarrow \infty} 0$, then $\tau = \sup\{t : N_t = \sup_{s \leq t} N_s\}$ which is an honest time. (See Nikghebali and Yor)

Assuming that some $\mathbb{F}$-adapted asset is traded (as well as a savings account with null interest rate, so that the associated market is complete and arbitrage free), it is easy to check that the martingale N is the value of a self-financing portfolio with initial value 1, admissible, such that $N_\tau > 1$, therefore there exists arbitrage opportunities before τ. One can also prove that there do not exist e.m.m. on the enlarged filtration $\mathbb{G}$ (a simple one would be $1/N_{\cdot \wedge \tau}$ which is a strict local $\mathbb{G}$-martingale). The same argument holds also after τ (see also Imkeller, Platen, Kardaras, Swierb)

Particular case:

If $\mathbb{F}$ is a Brownian filtration and

$$Z_t = N_t \,(\sup_{s\leq t} N_s)^{-1}$$

where $N_t \to_{t\to\infty} 0$, then $\tau = \sup\{t : N_t = \sup_{s\leq t} N_s\}$ which is an honest time. (See Nikghebali and Yor)

Assuming that some $\mathbb{F}$-adapted asset is traded (as well as a savings account with null interest rate, so that the associated market is complete and arbitrage free), it is easy to check that the martingale N is the value of a self-financing portfolio with initial value 1, admissible, such that $N_\tau > 1$, therefore there exists arbitrage opportunities before τ.One can also prove that there do not exist e.m.m. on the enlarged filtration $\mathbb{G}$ (a simple one would be $1/N_{\cdot\wedge\tau}$ which is a strict local $\mathbb{G}$-martingale). The same argument holds also after τ (see also Imkeller, Platen, Kardaras, Swierb)

Particular case:

If $\mathbb{F}$ is a Brownian filtration and

$$Z_t = N_t \, (\sup_{s \le t} N_s)^{-1}$$

where $N_t \to_{t \to \infty} 0$, then $\tau = \sup\{t : N_t = \sup_{s \le t} N_s\}$ which is an honest time. (See Nikghebali and Yor)

Assuming that some $\mathbb{F}$-adapted asset is traded (as well as a savings account with null interest rate, so that the associated market is complete and arbitrage free), it is easy to check that the martingale N is the value of a self-financing portfolio with initial value 1, admissible, such that $N_\tau > 1$, therefore there exists arbitrage opportunities before τ. One can also prove that there do not exist e.m.m. on the enlarged filtration $\mathbb{G}$ (a simple one would be $1/N_{\cdot \wedge \tau}$ which is a strict local $\mathbb{G}$-martingale). The same argument holds also after τ (see also Imkeller, Platen, Kardaras, Swierb)

Enlargement of filtration problem solved by SDE

Here we study in particular the enlargement of filtration problem.

- $\mathbb{G}$ is a progressive enlargement of $\mathbb{F}$.
- The $\mathbb{F}$-local martingales remain always $\mathbb{G}$-semimartingales on the interval $[0, \tau]$. whose semimartingale decomposition formula is given in Jeulin.
- The $\mathbb{F}$-local martingales' behaviour on the interval $[\tau, \infty)$ in the filtration $\mathbb{G}$ depends on the model.

Semimartingale decomposition formula for the $\star_u$, in the case $1 - Z > 0$

We suppose

• $\mathbf{Hy}(\maltese\maltese)$ and $Z_\infty = 0$

• for each $0 \le t \le \infty$, the map $u \to M_t^u$ is continuous on $[0, t]$, where M^u is solution of the generating equation $(\star)$: $0 \le u < \infty$,

$$(\star_u) \begin{cases} dM_t &= M_t \left(-\frac{e^{-\Lambda_t}}{1-Z_t} dN_t + f(M_t - (1 - Z_t))dY_t \right), \ u \le t < \infty \\ M_u &= 1 - Z_u \end{cases}$$

We prove that, for our models, the hypothesis $(\mathcal{H}')$ holds between $\mathbb{F}$ and $\mathbb{G}$ and we obtain semimartingale decomposition formula.

Let $\mathbb{Q}$ be the probability on the product space $[0,\infty]\otimes\Omega$ associated with the iM_Z

Let X be a $\mathbb{P}$-$\mathbb{F}$ local martingale. Then the process

$$\begin{aligned}\widetilde{X}_t &= X_t - \int_0^t \mathbb{1}_{\{s\leq\tau\}}\frac{e^{-\Lambda_s}}{Z_s}d\langle N,X\rangle_s + \int_0^t \mathbb{1}_{\{\tau<s\}}\frac{e^{-\Lambda_s}}{1-Z_s}d\langle N,X\rangle_s \\ &\quad - \int_0^t \mathbb{1}_{\{\tau<s\}}(f(M_s^\tau-(1-Z_s))+M_s^\tau f'(M_s^\tau-(1-Z_s)))d\langle Y,X\rangle_s\end{aligned}$$

is a $\mathbb{Q}$-$\mathbb{G}$-local martingale.

Semimartingale decomposition formula in case of zeros of $1 - Z$

We suppose $\mathbf{Hy}(\mathcal{Z})$. We consider the iM_Z constructed above and its associated probability measure $\mathbb{Q}$ on $[0, \infty] \times \Omega$. Let $g = lim_{t\to\infty} g_t$.

Let X be a $(\mathbb{P}, \mathbb{F})$-local martingale. Then

$$X_t - \int_0^t 1\!\!1_{\{s \le g\vee\tau\}} \frac{e^{-\Lambda_s}}{Z_{s-}} d\langle N, X\rangle_s + \int_0^t 1\!\!1_{\{g\vee\tau < s\}} \frac{e^{-\Lambda_s}\ d\langle N, X\rangle_s}{1 - Z_{s-}},\ 0 \le t < \infty,$$

is a $(\mathbb{Q}, \mathbb{G})$-local martingale.

It is noted that the above formula has the same form as the formula for honest time, whilst $g \vee \tau$ is not a honest time in the filtration $\mathbb{F}$.

Predictable Representation Property

Assume ✠✠ and that

1. there exists an $(\mathbb{P}, \mathbb{F})$-martingale m which admits the $(\mathbb{P}, \mathbb{F})$-Predictable Representation Property
2. The martingales N and Y are orthogonal

Let $\tilde{m}$ be the $(\mathbb{P}, \mathbb{G})$-martingale part of the $(\mathbb{P}, \mathbb{G})$-semimartingale m.
Then, $(\tilde{m}, M)$ enjoys the $(\mathbb{Q}, \mathbb{G})$-Predictable Representation Property where
$M_t = 1\!\!1_{\tau \leq t} - \Lambda_{t \wedge \tau}$.

Jeanblanc, M. and Song, S. (2010)
Explicit Model of Default Time with given Survival Probability. *Stochastic Processes and their Applications*
Default times with given survival probability and their $\mathbb{F}$-martingale decomposition formula. *Stochastic Processes and their Applications*

Nikeghbali, A. and Yor, M. (2006) Doob's maximal identity, multiplicative decompositions and enlargements of filtrations, *Illinois Journal of Mathematics*, 50, 791-814.

Li, L. and Rutkowski, M. (2010) Constructing Random Times Through Multiplicative Systems, SPA, 2012.

In that paper, the authors give a solution to the problem ($\star$), based on **Meyer, P.A. (1967)**: On the multiplicative decomposition of positive supermartingales. In: *Markov Processes and Potential Theory*, J. Chover, ed., J. Wiley, New York, pp. 103–116.

Thank you for your attention

HAPPY BIRTHDAY IOANNIS

Santorini -The naked child

Bend if you can to the dark sea forgetting
the flute's sound on naked feet
that trod your sleep in the other, the sunken life.
Write if you can on your last shell
the day the place the name
and fling it into the sea so that it sinks.

Giorgios SEFERIS

Föllmer's measure

One may think that a solution of the problem $(\star)$ is given by the Föllmer measure associated with Z, defined as

$$\mathbb{Q}^{\mathsf{F}}[F] = \mathbb{P}[\int_0^\infty F(s,\cdot) Z_s d\Lambda_s], \; F \in \mathcal{B}[0,\infty] \otimes \mathcal{F}_\infty.$$

which satisfies the projection condition.

In order to be a solution of the problem $(\star)$, $\mathbb{Q}^{\mathsf{F}}$ must be an extension of $\mathbb{P}$, i.e.,

$$\mathbb{P}[A] = \mathbb{Q}^{\mathsf{F}}[A] = \mathbb{P}[\mathbb{I}_A \int_0^\infty Z_s d\Lambda_s], \; A \in \mathcal{F}_\infty.$$

This is equivalent to the condition: $\int_0^\infty Z_s d\Lambda_s \equiv 1$. The last condition combined with the assumption $Z_\infty = 0$ implies, from the Doob-Meyer decomposition of Z written in differential form as $dZ_t = e^{-\Lambda_t} dN_t - Z_t d\Lambda_t$:

$$Z_t = \mathbb{P}[\int_0^\infty Z_s d\Lambda_s | \mathcal{F}_t] - \int_0^t Z_s d\Lambda_s = 1 - \int_0^t Z_s d\Lambda_s$$

i.e., $Z_t = e^{-\Lambda_t}$

Proofs

Proof of properties of M^u

• Inequality $M^u \leq 1 - Z$ on $[u, \infty)$ is satisfied if the local time of $\Delta = M^u - (1-Z)$ at zero is null. This is the consequence of the following estimation:

$$
\begin{aligned}
d\langle \Delta \rangle_t &= \Delta_t^2 \left(\frac{e^{-\Lambda_t}}{1 - Z_t} \right)^2 d\langle N \rangle_t + M_t^2 f^2(\Delta_t) d\langle Y \rangle_t - 2\Delta_t \frac{e^{-\Lambda_t}}{1 - Z_t} M_t f(\Delta_t) d\langle N, Y \rangle_t \\
&\leq 2\Delta_t^2 \left(\frac{e^{-\Lambda_t}}{1 - Z_t} \right)^2 d\langle N \rangle_t + 2M_t^2 f^2(\Delta_t) d\langle Y \rangle_t \\
&\leq 2\Delta_t^2 \left(\frac{e^{-\Lambda_t}}{1 - Z_t} \right)^2 d\langle N \rangle_t + 2M_t^2 K^2 \Delta_t^2 d\langle Y \rangle_t
\end{aligned}
$$

From this, we can write

$$
\int_0^t \mathbb{I}_{\{0 < \Delta_s < \epsilon\}} \frac{1}{\Delta_s^2} d\langle \Delta \rangle_s < \infty, \; 0 < \epsilon, 0 < t < \infty
$$

and get the result according to Revuz-Yor.

• Inequality $M^u \leq M^v$ on $[v, \infty)$ when $u < v$. The comparison theorem holds for SDE($\natural$). We note also that M^u and M^v satisfy the same SDE($\natural$) on $[v, \infty)$. So, since $M^u_v \leq (1 - Z_v) = M^v_v$, $M^u_t \leq M^v_t$ for all $t \in [v, \infty)$.

Balayage Formula

Let Y be a continuous semi-martingale and define

$$g_t = \sup\{s \leq t : Y_s = 0\},$$

with the convention $\sup\{\emptyset\} = 0$. Then

$$h_{g_t} Y_t = h_0 Y_0 + \int_0^t h_{g_s} dY_s$$

for every predictable, locally bounded process h.

We need only to prove that each M^u satisfies the above equation, and therefore, that M^u is a local $\mathbb{P}$-$\mathbb{F}$ martingale. Let

$$E_t^u = \exp\left(-\int_u^t \frac{Z_s}{1-Z_s} dA_s\right)$$

Then,

$$d\left(E_t^u(1-Z_t)\right) = E_t^u\left(-e^{-\Lambda_t} dN_t + Z_t dV_t\right)$$

We apply the balayage formula and we obtain

$$\begin{aligned}
M_t^u &= \mathbb{I}_{\{g_t \le u\}} E_t^u (1-Z_t) \\
&= \mathbb{I}_{\{g_t \le u\}} (1-Z_u) + \int_u^t \mathbb{I}_{\{g_s \le u\}} E_s^u \left(-e^{-\Lambda_s} dN_s + Z_s dV_s\right) \\
&= (1-Z_u) - \int_u^t \mathbb{I}_{\{g_s \le u\}} E_s^u e^{-\Lambda_s} dN_s
\end{aligned}$$

Semi martingale

The theorem can be proved in quite the same way as in the preceding theorem, except some precaution on the zeros of $1 - Z$. Recall that the elements in iM_Z satisfy the equation:

$$M_t^u = (1 - Z_u) - \int_u^t \mathbb{I}_{\{g_s \le u\}} E_s^u e^{-\Lambda_s} dN_s, \ u \le t < \infty.$$

Let $0 \le a < b \le s < t$ and $A \in \mathcal{F}_s$. Put aside the integrability question. We have

$$\begin{aligned}
& \mathbb{Q}[1\!\!1_A 1\!\!1_{\{a < g \vee \tau \le b\}}(X_t - X_s)] = \mathbb{Q}[1\!\!1_A (M_\infty^b - M_\infty^a)(X_t - X_s)] \\
= \ & \mathbb{Q}[1\!\!1_A \int_s^t 1\!\!1_{\{g_r \le b\}} E_r^b (-e^{-\Lambda_r}) d\langle N, X\rangle_r] - \mathbb{Q}[1\!\!1_A \int_s^t 1\!\!1_{\{g_r \le a\}} E_r^a (-e^{-\Lambda_r}) d\langle N, X\rangle_r] \\
= \ & \mathbb{Q}[1\!\!1_A 1\!\!1_{\{g \vee \tau \le b\}} \int_s^t \frac{(-e^{-\Lambda_r})}{1 - Z_r} d\langle N, X\rangle_r] - \mathbb{Q}[1\!\!1_A 1\!\!1_{\{g \vee \tau \le a\}} \int_s^t \frac{(-e^{-\Lambda_r})}{1 - Z_r} d\langle N, X\rangle_r] \\
= \ & \mathbb{Q}[1\!\!1_A 1\!\!1_{\{a < g \vee \tau \le b\}} \int_s^t \frac{(-e^{-\Lambda_r})}{1 - Z_r} d\langle N, X\rangle_r]
\end{aligned}$$

Another Method

Gapeev, P. V., Jeanblanc, M., Li, L., and Rutkowski, M. (2009): Constructing Random Times with Given Survival Processes and Applications to Valuation of Credit Derivatives. Forthcoming in: *Contemporary Quantitative Finance* Springer-Verlag 2010.

In that paper, the probability $\mathbb{Q}$ is constructed as a probability measure equivalent to the solution of Cox model $\mathbb{Q}^C$ on $[0,\infty] \times \Omega$ associated with Λ. Define

$$d\mathbb{Q}|_{\mathcal{G}_t} = L_t d\mathbb{Q}^C|_{\mathcal{G}_t},\ 0 \leq t < \infty$$

where $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(\tau \wedge t)$ and $L_t = \ell_t \mathbb{1}_{t<\tau} + L_t(\tau)\mathbb{1}_{\tau \leq t}$. If L satisfies

$$\begin{aligned} \ell_t &= N_t, \\ (\mathtt{L}):\quad N_t e^{-\Lambda_t} + \int_0^t L_t(s) e^{-\Lambda_s} d\Lambda_s &= 1,\ 0 \leq t < \infty. \end{aligned}$$

where, for any s, the process $(L_t(s), t \geq s)$ **is an $\mathbb{F}$-martingale** satisfying $L_s(s) = N_s$, then, $\mathbb{Q}$ is a solution of the problem $(\star)$.

Conditions: **find** $L_t = \ell_t 1\!\!1_{t<\tau} + L_t(\tau,\cdot)1\!\!1_{\tau\le t}$ **such that**

$$\begin{aligned} \ell_t &= N_t, \\ (\mathtt{L}): \quad N_t e^{-\Lambda_t} + \int_0^t L_t(s,\cdot)e^{-\Lambda_s}d\Lambda_s &= 1, \ 0 \le t < \infty. \end{aligned}$$

where $(L_t(s), t \ge s)$ **is an** $\mathbb{F}$**-martingale satisfying** $L_s(s) = N_s$.

• The form of L is a general form for $\mathbb{G}$-adapted processes

• The condition on martingality of $L_t(s), t \ge s$ is to ensure that L is a $\mathbb{G}$-martingale

• The condition $L_t 1\!\!1_{t<\tau} = N_t 1\!\!1_{t<\tau}$ is stated to satisfy the **projection condition**

• The condition $(\mathtt{L})$ is needed to satisfy **the restriction condition** (and implies that L is a $\mathbb{G}$-martingale).

In fact, the process $L_t = \ell_t 1\!\!1_{t<\tau} + L_t(\tau,\cdot)1\!\!1_{\tau\le t}$ is a $\mathbb{G}$ local martingale iff $L_t(s), t \ge s$ and $\mathbb{E}(L_t|\mathcal{F}_t)$ are $\mathbb{F}$-martingales.

To solve $(\mathtt{L})$ the idea is to find X and Y so that $L_t(s) = X_t Y_s$ and $N_t = X_t Y_t$.

Conditions: **find** $L_t = \ell_t 1\!\!1_{t<\tau} + L_t(\tau,\cdot)1\!\!1_{\tau\le t}$ **such that**

$$\begin{aligned} \ell_t &= N_t, \\ (\mathtt{L}): \quad N_t e^{-\Lambda_t} + \int_0^t L_t(s,\cdot)e^{-\Lambda_s}d\Lambda_s &= 1,\ 0 \le t < \infty. \end{aligned}$$

where $(L_t(s), t \ge s)$ **is an** $\mathbb{F}$**-martingale satisfying** $L_s(s) = N_s$.

• The form of L is a general form for $\mathbb{G}$-adapted processes

• The condition on martingality of $L_t(s), t \ge s$ is to ensure that L is a $\mathbb{G}$-martingale

• The condition $L_t 1\!\!1_{t<\tau} = N_t 1\!\!1_{t<\tau}$ is stated to satisfy the **projection condition**

• The condition $(\mathtt{L})$ is needed to satisfy **the restriction condition** (and implies that L is a $\mathbb{G}$-martingale).

In fact, the process $L_t = \ell_t 1\!\!1_{t<\tau} + L_t(\tau,\cdot)1\!\!1_{\tau\le t}$ is a $\mathbb{G}$ local martingale iff $\mathbb{E}(L_t|\mathcal{F}_t)$ and for any s, $L_t(s), t \ge s$ are $\mathbb{F}$-martingales.

To solve $(\mathtt{L})$ the idea is to find X and Y so that $L_t(s) = X_t Y_s$ and $N_t = X_t Y_t$.

Conditions: **find** $L_t = \ell_t \mathbb{1}_{t<\tau} + L_t(\tau,\cdot)\mathbb{1}_{\tau \le t}$ **such that**

$$\begin{aligned} \ell_t &= N_t, \\ (\mathtt{L}): \quad N_t e^{-\Lambda_t} + \int_0^t L_t(s,\cdot) e^{-\Lambda_s} d\Lambda_s &= 1, \; 0 \le t < \infty. \end{aligned}$$

where $(L_t(s), t \ge s)$ **is an** $\mathbb{F}$**-martingale satisfying** $L_s(s) = N_s$.

• The form of L is a general form for $\mathbb{G}$-adapted processes

• The condition on martingality of $L_t(s), t \ge s$ is to ensure that L is a $\mathbb{G}$-martingale

• The condition $L_t \mathbb{1}_{t<\tau} = N_t \mathbb{1}_{t<\tau}$ is stated to satisfy the **projection condition**

• The condition (L) is needed to satisfy **the restriction condition** (and implies that L is a $\mathbb{G}$-martingale).

In fact, the process $L_t = \ell_t \mathbb{1}_{t<\tau} + L_t(\tau,\cdot)\mathbb{1}_{\tau \le t}$ is a $\mathbb{G}$ local martingale iff $\mathbb{E}(L_t|\mathcal{F}_t)$ and for any s, $L_t(s), t \ge s$ are $\mathbb{F}$-martingales.

To solve (L) the idea is to find X and Y so that $L_t(s) = X_t Y_s$ and $N_t = X_t Y_t$.

Example

Let φ is the standard Gaussian density and Φ the Gaussian cumulative function, $\mathbb{F}$ generated by a Brownian motion B.

Let $X = \int_0^\infty f(s)dB_s$ where f is a deterministic, square-integrable function and $Y = \psi(X)$ where ψ is a positive and strictly increasing function. Then,

$$\mathbb{P}(Y \leq u|\mathcal{F}_t) = \mathbb{P}\Big(\int_t^\infty f(s)dB_s \leq \psi^{-1}(u) - m_t|\mathcal{F}_t\Big)$$

where $m_t = \int_0^t f(s)dB_s$ is $\mathcal{F}_t$-measurable. It follows that

$$M_t^u := \mathbb{P}(Y \leq u|\mathcal{F}_t) = \Phi\Big(\frac{\psi^{-1}(u) - m_t}{\sigma(t)}\Big)$$

The family M_t^u is then a family of iM_Z martingales which satisfies

$$dM_t^u = -\varphi\left(\Phi^{-1}(M_t^u)\right)\frac{f(t)}{\sigma(t)}dB_t$$

The multiplicative decomposition of $Z_t = N_t \exp\left(-\int_0^t \lambda_s ds\right)$ where

$$
\begin{aligned}
dN_t &= N_t \frac{\varphi(Y_t)}{\sigma(t)\Phi(Y_t)} dm_t, \quad \lambda_t = \frac{h'(t)\,\varphi(Y_t)}{\sigma(t)\,\Phi(Y_t)} \\
Y_t &= \frac{m_t - \psi^{-1}(t)}{\sigma(t)}
\end{aligned}
$$

The basic martingale satisfies

$$
dM_t^u = -M_t^u \frac{f(t)\varphi(Y_t)}{\sigma(t)\Phi(-Y_t)} dB_t.
$$