Stochastic Control at Warp Speed *

Mike Harrison
Graduate School of Business
Stanford University

June 7, 2012

* Based on current work by Peter DeMarzo, which extends in certain ways the model and analysis of P. DeMarzo and Y. Sannikov, Optimal Security Design and Dynamic Capital Structure in a Continuous-Time Agency Model, *J. of Finance*, Vol. 61 (2006), 2681-2724.

Warp Drive (Star Trek)

From Wikipedia, the free encyclopedia

Warp Drive is a faster-than-light (FTL) propulsion system in the setting of many science fiction works, most notably *Star Trek*. A spacecraft equipped with a warp drive may travel at velocities greater than that of light by many orders of magnitude, while circumventing the relativistic problem of time dilation.

Outline

Baseline problem

Modified problem with $u < \infty$

Formal analysis with $u = \infty$

Open questions

Baseline problem

- State space for the controlled process X is the finite interval $[R,S]$.
- An *admissible control* is a pair of adapted processes $C = (C_t)$ and $\beta = (\beta_t)$ such that C is non-negative and non-decreasing and $\ell \leq \beta_t \leq u$ for all t.
- Dynamics of X specified by the differential relationship

$$dX_t = \gamma X_t\, dt + \beta_t\, dZ_t - dC_t, \quad 0 \leq t \leq \tau, \ \text{where} \ \tau = \inf\{t \geq 0: X_t \leq R\}.$$

Baseline problem

- Data are constants $L, R, X_0, S, \ell, u, r, \gamma, \mu > 0$ such that $R < X_0 < S$, $\ell < u$ and $r < \gamma$.

- $Z = (Z_t\,, t \geq 0)$ is standard Brownian motion on $(\Omega, \mathcal{F}, P)$ and $(\mathcal{F}_t)$ is the filtration generated by Z.

- State space for the controlled process X is the finite interval $[R, S]$.

- An *admissible control* is a pair of adapted processes $C = (C_t)$ and $\beta = (\beta_t)$ such that C is non-negative and non-decreasing and $\ell \leq \beta_t \leq u$ for all t.

- Dynamics of X specified by the differential relationship

$$dX_t = \gamma X_t\, dt + \beta_t\, dZ_t - dC_t\,, \quad 0 \leq t \leq \tau\,, \text{ where } \tau = \inf\{t \geq 0\colon X_t \leq R\}.$$

Baseline problem

- Data are constants $L, R, X_0, S, \ell, u, r, \gamma, \mu > 0$ such that $R < X_0 < S$, $\ell < u$ and $r < \gamma$.

- $Z = (Z_t\,, t \geq 0)$ is standard Brownian motion on $(\Omega, \mathcal{F}, P)$ and $(\mathcal{F}_t)$ is the filtration generated by Z.

- State space for the controlled process X is the finite interval $[R, S]$.

- An *admissible control* is a pair of adapted processes $C = (C_t)$ and $\beta = (\beta_t)$ such that C is non-negative and non-decreasing and $\ell \leq \beta_t \leq u$ for all t.

- Dynamics of X specified by the differential relationship

$$dX_t = \gamma X_t \, dt + \beta_t \, dZ_t - dC_t\,, \quad 0 \leq t \leq \tau\,, \text{ where } \tau = \inf\,\{t \geq 0 \colon X_t \leq R\}.$$

- Controller's objective is to

$$\text{maximize } E\left(\int_0^{\tau} e^{-rt}(\mu\, dt - dC_t) + L e^{-r\tau}\right) .$$

Story behind the baseline problem

1. The owner of a business employs an agent for the firm's day-to-day management. The owner's problem is to design a performance-based compensation scheme, hereafter called a *contract*, for the agent (see 7 below).

Story behind the baseline problem

1. The owner of a business employs an agent for the firm's day-to-day management. The owner's problem is to design a performance-based compensation scheme, hereafter called a *contract*, for the agent (see 7 below).

2. The firm's cumulative earnings are modeled by a Brownian motion $Y_t = \mu t + \sigma Z_t$, $t \geq 0$. Assume for the moment that the agent and the owner both observe Y.

Story behind the baseline problem

1. The owner of a business employs an agent for the firm's day-to-day management. The owner's problem is to design a performance-based compensation scheme, hereafter called a *contract*, for the agent (see 7 below).

2. The firm's cumulative earnings are modeled by a Brownian motion $Y_t = \mu t + \sigma Z_t$, $t \geq 0$. Assume for the moment that the agent and the owner both observe Y.

3. The owner commits to $(C_t, 0 \leq t \leq \tau)$ as the agent's cumulative compensation process, based on observed earnings; τ is the agent's *termination date*. Upon termination the agent will accept outside employment; from the agent's perspective, the income stream associated with that outside employment is equivalent in value to a one-time payout of R.

Story behind the baseline problem

1. The owner of a business employs an agent for the firm's day-to-day management. The owner's problem is to design a performance-based compensation scheme, hereafter called a *contract*, for the agent (see 7 below).

2. The firm's cumulative earnings are modeled by a Brownian motion $Y_t = \mu t + \sigma Z_t$, $t \geq 0$. Assume for the moment that the agent and the owner both observe Y.

3. The owner commits to $(C_t, 0 \leq t \leq \tau)$ as the agent's cumulative compensation process, based on observed earnings; τ is the agent's *termination date*. Upon termination the agent will accept outside employment; from the agent's perspective, the income stream associated with that outside employment is equivalent in value to a one-time payout of R.

4. The agent is risk neutral and discounts at interest rate $\gamma > 0$. We denote by X_t the agent's *continuation value* at time t. That is, X_t is the conditional expected present value, as of time t, of the agent's income from that point onward, including income from later outside employment, given the observed earnings $(Y_s, 0 \leq s \leq t)$.

5. To keep the agent from defecting, the contract $(C_t, 0 \le t \le \tau)$ must be designed so that $X_t \ge R$ for $0 \le t \le \tau$. To avoid trivial complications we also require $X_t \le S$ for $0 \le t \le \tau$, where S is some large constant.

5. To keep the agent from defecting, the contract $(C_t,\ 0 \le t \le \tau)$ must be designed so that $X_t \ge R$ for $0 \le t \le \tau$. To avoid trivial complications we also require $X_t \le S$ for $0 \le t \le \tau$, where S is some large constant.

6. It follows from the martingale representation property of Brownian motion that $(X_t,\ 0 \le t \le \tau)$ can be represented in the form $dX = \gamma X\ dt - dC + \beta\ dZ$ for some suitable integrand β.

5. To keep the agent from defecting, the contract $(C_t,\ 0 \le t \le \tau)$ must be designed so that $X_t \ge R$ for $0 \le t \le \tau$. To avoid trivial complications we also require $X_t \le S$ for $0 \le t \le \tau$, where S is some large constant.

6. It follows from the martingale representation property of Brownian motion that $(X_t,\ 0 \le t \le \tau)$ can be represented in the form $dX = \gamma X\ dt - dC + \beta\ dZ$ for some suitable integrand β.

7. In truth the owner does *not* observe the earnings process Y, but rather is dependent on earnings reports by the agent. Payments to the agent are necessarily based on *reported* earnings, and there is a threat that the agent will under-report earnings, keeping the difference for himself. To motivate truthful reporting by the agent, the contract $(C_t,\ 0 \le t \le \tau)$ must be designed so that $\beta_t \ge \ell$ for $0 \le t \le \tau$, where $\ell > 0$ is a given problem parameter.

5. To keep the agent from defecting, the contract $(C_t,\ 0 \le t \le \tau)$ must be designed so that $X_t \ge R$ for $0 \le t \le \tau$. To avoid trivial complications we also require $X_t \le S$ for $0 \le t \le \tau$, where S is some large constant.

6. It follows from the martingale representation property of Brownian motion that $(X_t,\ 0 \le t \le \tau)$ can be represented in the form $dX = \gamma X\ dt - dC + \beta\ dZ$ for some suitable integrand β.

7. In truth the owner does *not* observe the earnings process Y, but rather is dependent on earnings reports by the agent. Payments to the agent are necessarily based on *reported* earnings, and there is a threat that the agent will under-report earnings, keeping the difference for himself. To motivate truthful reporting by the agent, the contract $(C_t,\ 0 \le t \le \tau)$ must be designed so that $\beta_t \ge \ell$ for $0 \le t \le \tau$, where $\ell > 0$ is a given problem parameter.

8. The upper bound $\beta_t \le u$ is artificial, imposed for the sake of tractability. We will let $u \uparrow \infty$ later.

5. To keep the agent from defecting, the contract $(C_t,\ 0 \le t \le \tau)$ must be designed so that $X_t \ge R$ for $0 \le t \le \tau$. To avoid trivial complications we also require $X_t \le S$ for $0 \le t \le \tau$, where S is some large constant.

6. It follows from the martingale representation property of Brownian motion that $(X_t,\ 0 \le t \le \tau)$ can be represented in the form $dX = \gamma X\ dt - dC + \beta\ dZ$ for some suitable integrand β.

7. In truth the owner does *not* observe the earnings process Y, but rather is dependent on earnings reports by the agent. Payments to the agent are necessarily based on *reported* earnings, and there is a threat that the agent will under-report earnings, keeping the difference for himself. To motivate truthful reporting by the agent, the contract $(C_t,\ 0 \le t \le \tau)$ must be designed so that $\beta_t \ge \ell$ for $0 \le t \le \tau$, where $\ell > 0$ is a given problem parameter.

8. The upper bound $\beta_t \le u$ is artificial, imposed for the sake of tractability. We will let $u \uparrow \infty$ later.

9. The owner is risk neutral, discounts at rate $r > 0$, earns at expected rate μ over the interval $(0,\tau)$, and receives liquidation value $L > 0$ upon termination.

5. To keep the agent from defecting, the contract $(C_t, 0 \le t \le \tau)$ must be designed so that $X_t \ge R$ for $0 \le t \le \tau$. To avoid trivial complications we also require $X_t \le S$ for $0 \le t \le \tau$, where S is some large constant.

6. It follows from the martingale representation property of Brownian motion that $(X_t, 0 \le t \le \tau)$ can be represented in the form $dX = \gamma X\, dt - dC + \beta\, dZ$ for some suitable integrand β.

7. In truth the owner does *not* observe the earnings process Y, but rather is dependent on earnings reports by the agent. Payments to the agent are necessarily based on *reported* earnings, and there is a threat that the agent will under-report earnings, keeping the difference for himself. To motivate truthful reporting by the agent, the contract $(C_t, 0 \le t \le \tau)$ must be designed so that $\beta_t \ge \ell$ for $0 \le t \le \tau$, where $\ell > 0$ is a given problem parameter.

8. The upper bound $\beta_t \le u$ is artificial, imposed for the sake of tractability. We will let $u \uparrow \infty$ later.

9. The owner is risk neutral, discounts at rate $r > 0$, earns at expected rate μ over the interval $(0,\tau)$, and receives liquidation value $L > 0$ upon termination.

10. We will initially treat X_0 (the total value to the agent of the contract that is offered) as a given constant, and will eventually choose X_0 to maximize expected value to owner.

Baseline problem (again)

- Data are constants $L, R, X_0, S, \ell, u, r, \gamma, \mu > 0$ such that $R < X_0 < S$, $\ell < u$ and $r < \gamma$.

- $Z = (Z_t\,, t \geq 0)$ is standard Brownian motion on $(\Omega, \mathcal{F}, P)$ and $(\mathcal{F}_t)$ is the filtration generated by Z.

- State space for the controlled process X is the finite interval $[R, S]$.

- An *admissible control* is a pair of adapted processes $C = (C_t)$ and $\beta = (\beta_t)$ such that C is non-negative and non-decreasing and $\ell \leq \beta_t \leq u$ for all t.

- Dynamics of X specified by the differential relationship

$$dX_t = \gamma X_t\, dt + \beta_t\, dZ_t - dC_t\,, \quad 0 \leq t \leq \tau\,, \text{ where } \tau = \inf\{t \geq 0: X_t \leq R\}.$$

- Controller's objective is to

$$\text{maximize } E\left(\int_0^\tau e^{-rt}(\mu\, dt - dC_t) + Le^{-r\tau}\right).$$

Solution of the baseline problem

For $x \in [R,S]$ let $V(x)$ be the maximum objective value that the controller can achieve when using an admissible control and starting from state $X_0 = x$. A standard heuristic argument suggests that $V(\cdot)$ must satisfy the HJB equation

$$\max_{\substack{c \geq 0 \\ \ell \leq \beta \leq u}} \left\{ (\mu - c) - rV(x) - cV'(x) + \frac{1}{2}\beta^2 V''(x) \right\} = 0 \quad \text{for } R \leq x \leq S, \tag{1}$$

with $V(R) = L$.

Solution of the baseline problem

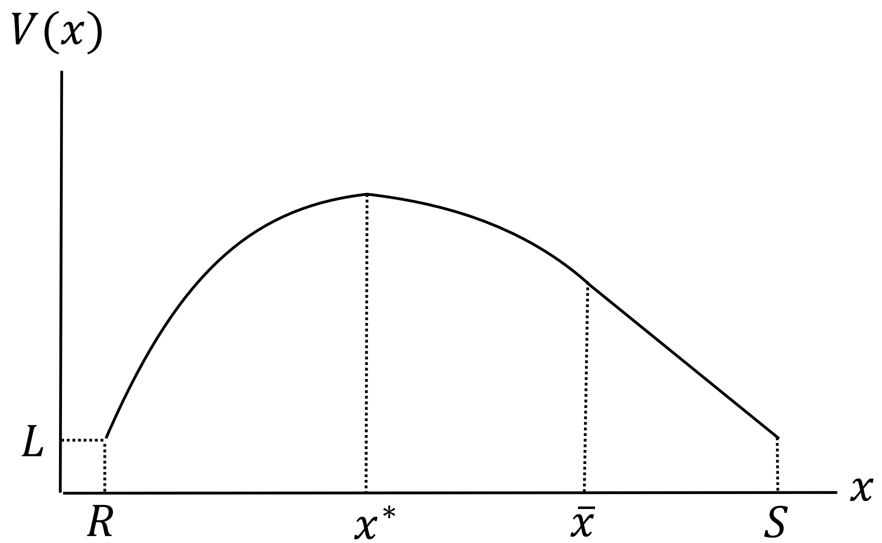
For $x \in [R,S]$ let $V(x)$ be the maximum objective value that the controller can achieve when using an admissible control and starting from state $X_0 = x$. A standard heuristic argument suggests that $V(\cdot)$ must satisfy the HJB equation

$$(1) \qquad \max_{\substack{c \geq 0 \\ \ell \leq \beta \leq u}} \{(\mu - c) - rV(x) - cV'(x) + \tfrac{1}{2}\beta^2 V''(x)\} = 0 \ \text{ for } \ R \leq x \leq S,$$

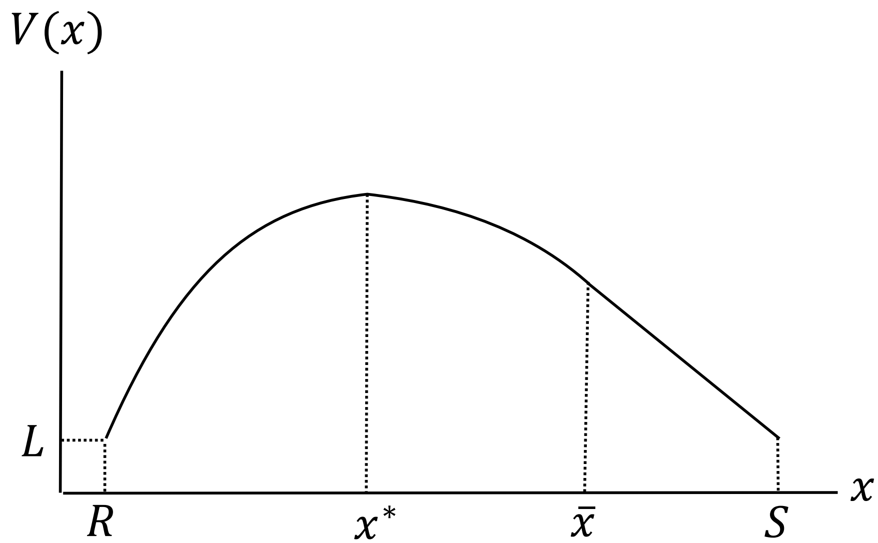
with $V(R) = L$. Of course, we can re-express this as

$$(1)' \qquad \mu - rV(x) - \min_{c \geq 0} \{c[1 + V'(x)]\} + \tfrac{1}{2} \max_{\ell \leq \beta \leq u} \{\beta^2 V''(x)\} = 0, \quad R \leq x \leq S.$$

Proposition 1 For any choice of $S > R$, equation (1) has a unique C^2 solution V, and for all S sufficiently large the structure of that solution is as follows: there exist constants x^* and $\bar{x}$, not depending on S, such that $R < x^* < \bar{x} < S$, V is strictly concave on $[S, \bar{x}]$, V reaches its maximum value at x^*, and $V'(\cdot) = -1$ on $[\bar{x}, S]$.

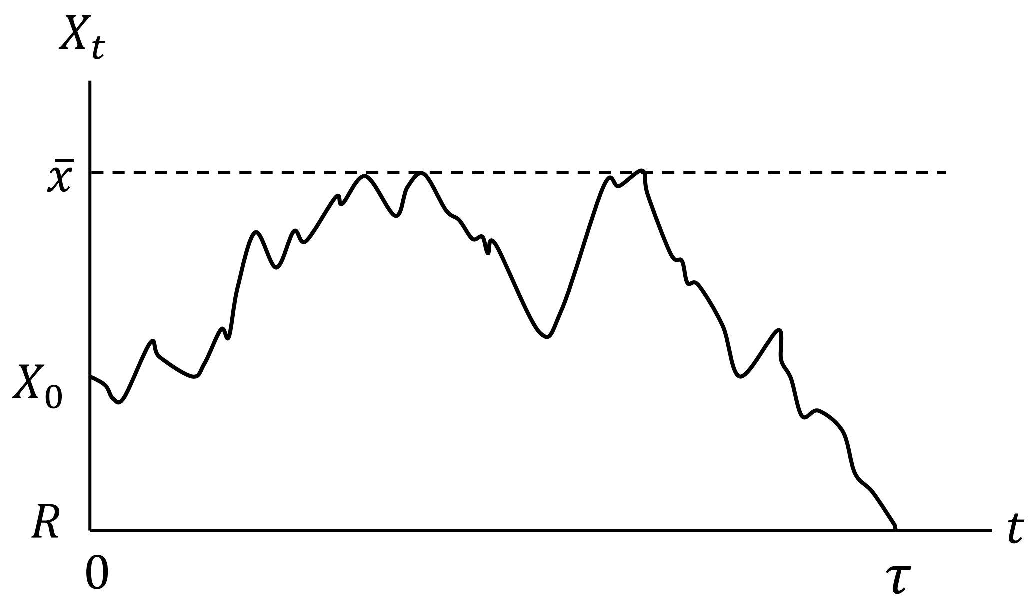


Proposition 1 For any choice of $S > R$, equation (1) has a unique C^2 solution V, and for all S sufficiently large the structure of that solution is as follows: there exist constants x^* and $\bar{x}$, not depending on S, such that $R < x^* < \bar{x} < S$, V is strictly concave on $[S, \bar{x}]$, V reaches its maximum value at x^*, and $V'(\cdot) = -1$ on $[\bar{x}, S]$.

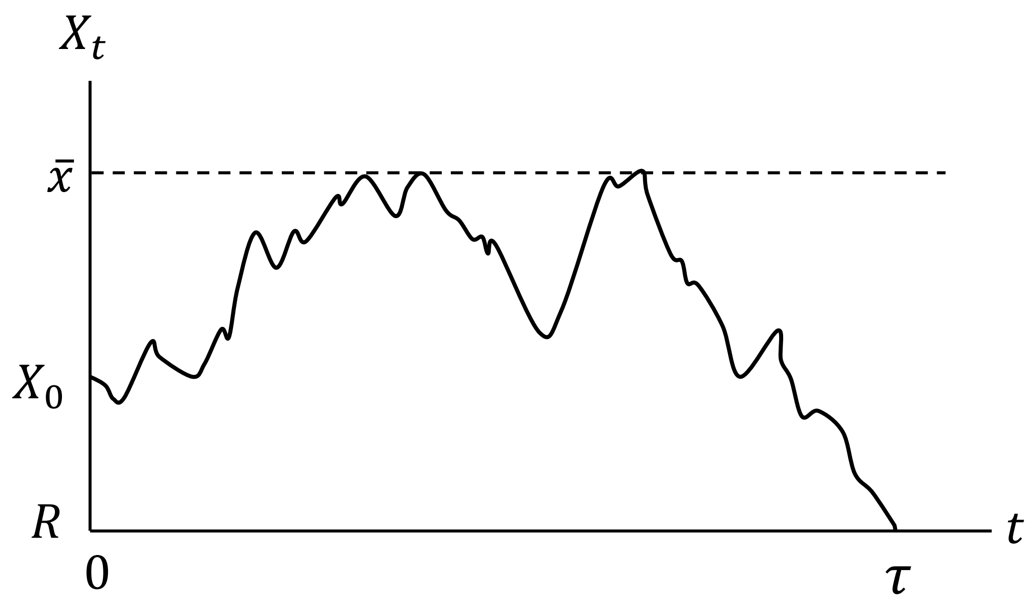


Remark The optimal contract (from the owner's perspective) delivers value $X_0 = x^*$ to the agent.

Proposition 2 For any choice of S sufficiently large, $V(X_0)$ is an upper bound on the objective value achievable with an admissible control, and that bound can be achieved as follows: set $\beta_t \equiv \ell$ and let C be the non-decreasing adapted process that enforces an upper reflecting barrier at level $\bar{x}$.



Proposition 2 For any choice of S sufficiently large, $V(X_0)$ is an upper bound on the objective value achievable with an admissible control, and that bound can be achieved as follows: set $\beta_t \equiv \ell$ and let C be the non-decreasing adapted process that enforces an upper reflecting barrier at level $\bar{x}$.



This is the main result of DeMarzo and Sannikov (2006).

Modified problem formulation

- New data are constants $b \in (R, S)$ and $k > 0$.

- The owner now must pay monitoring costs at rate $K(X_t)$ over the time interval $[0,\tau]$, where $K(x) = k$ for $R \leq x \leq b$, and $K(x) = 0$ otherwise. Everything else is the same as before.

Modified problem formulation

- New data are constants $b \in (R, S)$ and $k > 0$.

- The owner now must pay monitoring costs at rate $K(X_t)$ over the time interval $[0,\tau]$, where $K(x) = k$ for $R \le x \le b$, and $K(x) = 0$ otherwise. Everything else is the same as before.

Story behind the modified formulation

When his continuation value X falls below the critical level b, the agent is prone toward risky behavior that could have disastrous consequences for the firm; to prevent such behavior the owner must intensify monitoring of the agent, which incurs an added cost.

Modified problem formulation

- New data are constants $b \in (R, S)$ and $k > 0$.

- The owner now must pay monitoring costs at rate $K(X_t)$ over the time interval $[0,\tau]$, where $K(x) = k$ for $R \le x \le b$, and $K(x) = 0$ otherwise. Everything else is the same as before.

Story behind the modified formulation

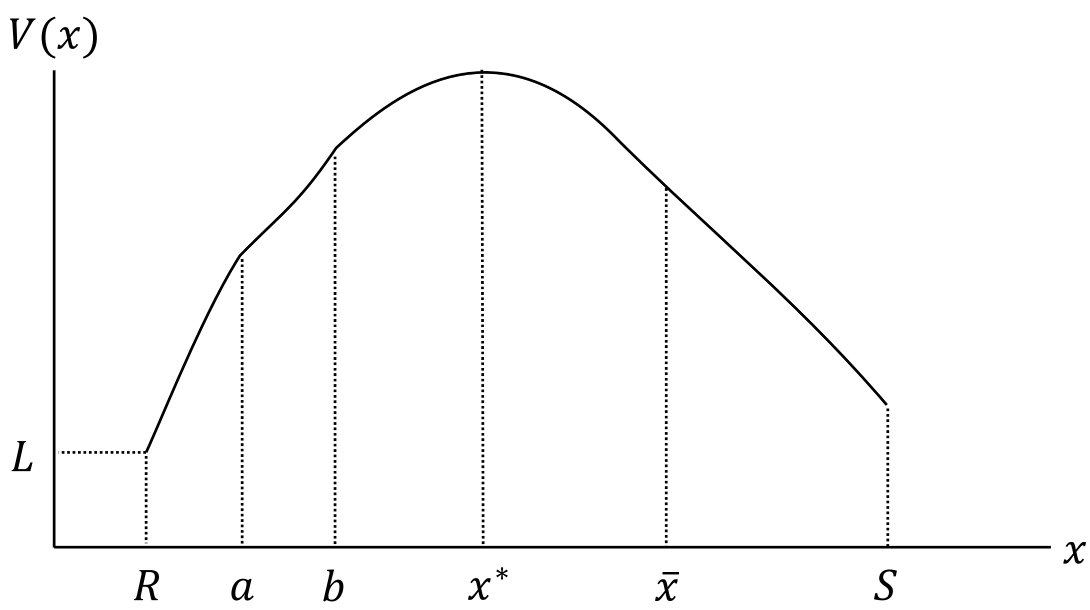
When his continuation value X falls below the critical level b, the agent is prone toward risky behavior that could have disastrous consequences for the firm; to prevent such behavior the owner must intensify monitoring of the agent, which incurs an added cost.

Modified HJB equation

(2) $$\mu - rV(x) - K(x) - \min_{c \ge 0} \{c[1 + V'(x)]\} + \frac{1}{2} \max_{\ell \le \beta \le u} \{\beta^2 V''(x)\} = 0, \quad R \le x \le S.$$

Example

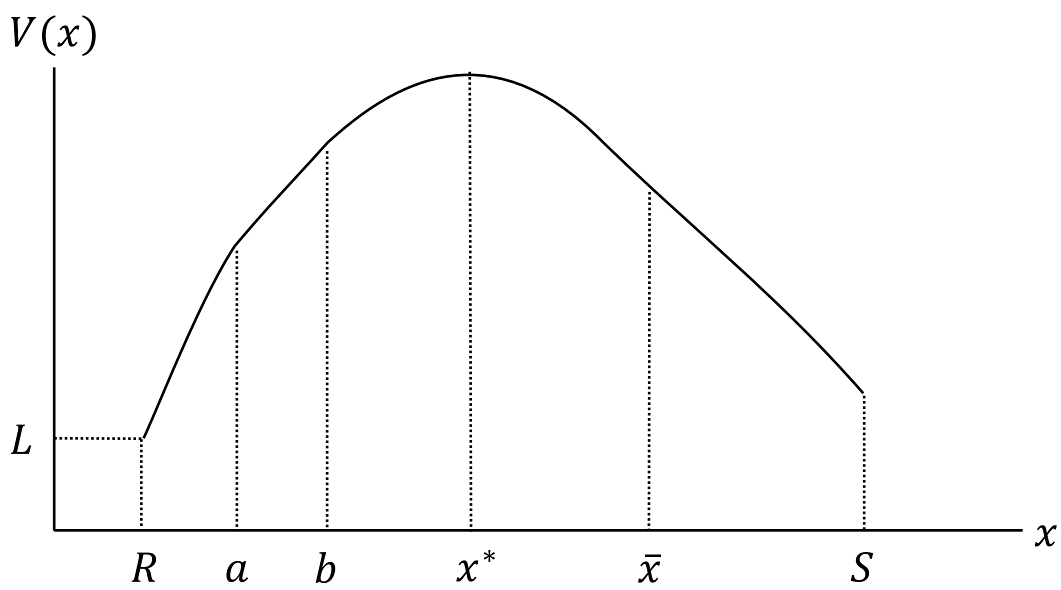
We now consider a certain numerical example that includes a large value for the artificial upper bound u. (Other specifics of the example would tell you nothing.) For this particular numerical example, equation (2) has a unique C^2 solution V for any choice of $S > R$, and for all S sufficiently large that solution has the structure pictured below. The maximizing value of c in the HJB equation (2) is $c = 0$ on $[0, \bar{x})$ and $c = \infty$ on $[\bar{x}, S]$. The maximizing value of β is $\beta = \ell$ on $[0, a]$, $\beta = u$ on $[a, b]$, and $\beta = \ell$ again on $[b, \bar{x}]$.



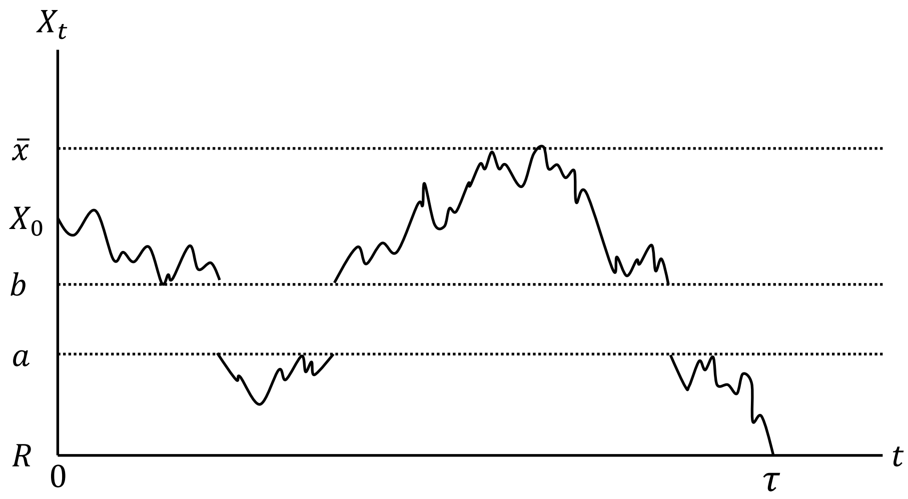
Formal analysis with $u = \infty$

(3) $$\mu - rV(x) - K(x) - \min_{c \geq 0}\{c[1 + V'(x)]\} + \tfrac{1}{2}\max_{\beta \geq \ell}\{\beta^2 V''(x)\} = 0, \quad R \leq x \leq S.$$

For the specific example referred to above, equation (3) has a C^1 solution V of the form pictured below: it is strictly concave on $[R,a)$, linear on $[a,b]$, strictly concave on $(b, \bar{x})$ and linear with $V'(\cdot) = -1$ on $[\bar{x}, S]$. The constants a and $\bar{x}$ do not depend on S, assuming S is sufficiently large.



Probabilistic realization of the formal solution



Probabilistic realization (continued)

Let $(\mathcal{G}_t)$ be the filtration generated by X. It is straight-forward to show that

$$X_t = E\left(\int_t^\tau e^{-\gamma(s-t)} dC_s + Re^{-\gamma(\tau-t)} \,\middle|\, \mathcal{G}_t\right), \quad 0 \le t \le \tau,$$

$$V(x) = E\left(\int_0^\tau e^{-rt}(\mu\, dt - dC_t) + Le^{-r\tau} \,\middle|\, X_0 = x\right) \quad \text{for} \quad x \in [0, a] \cup [b, \bar{x}]$$

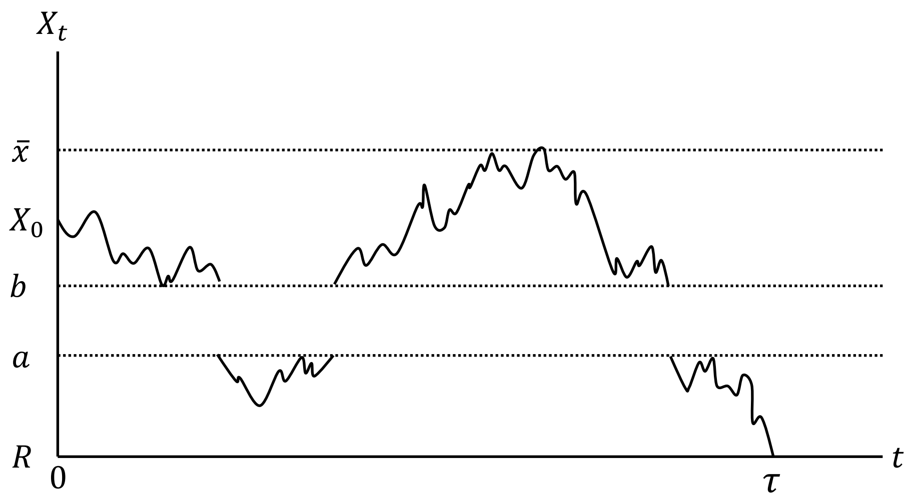
$$V(x) = \left(\frac{b-x}{b-a}\right) V(a) + \left(\frac{x-a}{b-a}\right) V(b) \quad \text{for} \quad x \in (a, b).$$

Probabilistic realization of the formal solution

Let $N_a(t)$ and $N_b(t)$ be two Poisson processes, each with unit intensity, defined on the same probability space as Z, independent of Z and of each other. Let $\delta = b - a > 0$ and X be the unique process satisfying

$$X_t = X_0 + \int_0^t \gamma X_s ds + \ell Z_t - [A_t - \delta N_a(\delta^{-1} A_t)] + [B_t - \delta N_b(\delta^{-1} B_t)] - C_t, \ \ 0 \le t \le \tau ,$$

where A is the local time of X at level a, and B is the local time of X at level b; as before, C is the increasing process that enforces an upper reflecting barrier at level $\bar{x}$, and τ is the first time at which X hits level 0.



Probabilistic realization (continued)

Let $(\mathcal{G}_t)$ be the filtration generated by X. It is straight-forward to show that

$$X_t = E\left(\int_t^\tau e^{-\gamma(s-t)}\, dC_s + R e^{-\gamma(\tau-t)} \,\middle|\, \mathcal{G}_t\right), \quad 0 \le t \le \tau\,,$$

$$V(x) = E\left(\int_0^\tau e^{-rt}(\mu\, dt - dC_t) + L e^{-r\tau} \,\middle|\, X_0 = x\right) \;\text{ for }\; x \in [0,a] \cup [b,\bar{x}]$$

$$V(x) = \left(\frac{b-x}{b-a}\right) V(a) + \left(\frac{x-a}{b-a}\right) V(b) \;\text{ for }\; x \in (a,b).$$

It follows easily from the martingale representation property of Brownian motion that *X* is *not* adapted to the filtration $(\mathcal{F}_t)$ generated by Z alone.

Open questions

1. How to define an admissible control for the relaxed example with $u = \infty$. It should be that

   (i) $V(X_0)$ is an upper bound on the value achievable using any admissible control, and

   (ii) the control described above is admissible, hence optimal (because it achieves the bound).

Open questions

1. How to define an admissible control for the relaxed example with $u = \infty$. It should be that

   (i) $V(X_0)$ is an upper bound on the value achievable using any admissible control, and

   (ii) the control described above is admissible, hence optimal (because it achieves the bound).

2. How to extend the analysis to allow an arbitrary piecewise-continuous cost function $K(\cdot)$ on $[R,S]$.

Open questions

1. How to define an admissible control for the relaxed example with $u = \infty$. It should be that

   (i) $V(X_0)$ is an upper bound on the value achievable using any admissible control, and

   (ii) the control described above is admissible, hence optimal (because it achieves the bound).

2. How to extend the analysis to allow an arbitrary piecewise-continuous cost function $K(\cdot)$ on $[R,S]$.

3. How to formulate an attractive general problem on a compact interval $[R, S]$, without the special structure of this particular application.

Open questions

1. How to define an admissible control for the relaxed example with $u = \infty$. It should be that

   (i) $V(X_0)$ is an upper bound on the value achievable using any admissible control, and

   (ii) the control described above is admissible, hence optimal (because it achieves the bound).

2. How to extend the analysis to allow an arbitrary piecewise-continuous cost function $K(\cdot)$ on $[R,S]$.

3. How to formulate an attractive general problem on a compact interval $[R, S]$, without the special structure of this particular application.

The End