\[(\Omega_t) \quad \text{"nice" (uniformly integrable)} \quad \text{under } \mathbb{P}^* \in \mathcal{Q}^* \ni \Omega \]

\[\omega_t = \mathbb{E}_{\mathbb{P}^*}[\omega_0 \mid \mathcal{F}_t]\]

hence

\[s_t = \mathbb{E}_{\mathbb{P}^*}[D_0 - D_t \mid \mathcal{F}_t]\]

= potential generated

= "fundamental" price

as seen under \(\mathbb{P}^*\)

"not nice" under \(\mathbb{P}^* \in \mathcal{Q}^* \ni \Omega \)

\[\Rightarrow \quad \omega_t > \mathbb{E}_{\mathbb{P}^*}[\omega_0 \mid \mathcal{F}_t]\]

hence

\[s_t > \mathbb{E}_{\mathbb{P}^*}[D_0 - D_t \mid \mathcal{F}_t]\]

- Both views may coexist

(cf. Examples in Jawor-Potter)
"Bubble" as seen under $P^*$:

$$
\beta^*_t := S_t - \mathbb{E}^* [\mathbb{E}_t \left[ \omega_t \mid \mathbb{F}_t \right]]
$$

fundamental value as seen under $P^*$

$$
= \omega_t - \mathbb{E}^* [\omega_t \mid \mathbb{F}_t]
$$

$$
= 0 \quad \text{if } P^* \in \mathbb{OU}
$$

$$
> 0, \quad \text{(local) martingale, } \not\rightarrow 0 \quad \text{not nice (not } \gamma \text{-weak integral)}
$$

In general:

$$
S_t = \beta^*_t + \mathbb{E}^* [\beta_{20} - \beta_t \mid \mathbb{F}_t]
$$

local martingale potential of claim

Riesz decomposition
Is bubble?

depends on choice of

i) market-consistent valuation $P_t^*$

to determine the fundamental value

ii) information structure

(cf. T., Proffter 2011)

Birth of a bubble?

~ dynamics in $O^*$
Fix  
\[ P_1^* \in OUI \quad \text{"nice"} \]
\[ P_2^* \in OUI \quad \text{"not nice"} \]

Garrow-Putter: sudden birth of a bubble by sudden regime change.

Biagini, F., Niedelos:
slow birth by slow shift from \( P_1^* \) to \( P_2^* \):

- optimistic ("exuberant") view
- pessimistic ("pessimistic"?) view
$$P^*_t[\mathcal{F}_t] = \frac{1}{t^2} P^*_t[\mathcal{F}_t] + (1-t)P^*_t[\mathcal{F}_t]$$

$$0 \leq t \leq 1, \quad \lambda_0 = 0, \quad \lambda_0 \geq 1$$

$$\beta_t := S_t - E_t[\mathbb{E}_0 \mathbb{D}_t \mid \mathcal{F}_t]$$

*Fundamental value as seen at $t$ (shifting view)*

**Theorem:** $(\beta_t)$ is a *submartingale* in the "moving frame" $(\mathbb{P}^*_t)$ up to

$$\tau := \inf \{ t > 0 / \lambda_t = 13 \}$$

then a *local martingale* (supermartingale).

- and under $\mathbb{P}^*$, under an additional condition
The Turner Review:

A regulatory response to the global banking crisis

(FSA = Financial Services Authority
March 2009)

1.1. (iv), p. 22:

"Misplaced reliance on sophisticated maths"
1.4 (iii), p. 44-45:

"mathematically modelable risk"

vs.

"Knightian uncertainty"

Frank Knight:

"Risk, Uncertainty, and Profit"

(Dissertation, Yale, 1916/21)
Risk = "known uncertain" \sim P \\

(Knightian) Uncertainty = "unknown uncertain" \sim model ambiguity \\
[a whole class of "plausible" probabilistic models]
Knightian Uncertainty:

a rich source of new mathematical problems!

Possible approaches:

1. Replace \( P \) by its "support"

2. Replace \( P \) by \( P' \)
"plain vanilla" options:
via "Calcul d'ito sans probabilite"
(A.F., 2001, 2nd ECM)

"exotic" options:
R. Cout et al. (2009, 2011)
via strictly pathwise Malliavin calculus

In the same spirit:
Mark Davis, Jan Ocko

in particular:

\[ Ito - S\alpha n\acute{a} via \]
\[ \text{strictly pathwise construction of local time} \]

*Footnote:* The mystery person is a woman, and her name is Monika Wörmel.

\[ \text{Diplomarbeit ETHZ (1983)} \]
\[ "\text{Lokalzeiten für stetige Martingale}" \]
II. \textbf{Bubbles in the dynamic of convex risk measures}

B. Acciaio, A.F., I. Penner
\textit{Finance and Stochastics} (2012)

A.F., I. Penner

\underline{Convex / coherent risk measures}

Artzner, Delbaen, Eber, Heath (1999)
Frittelli, Rosazza-Gianin (2002)
F., Schied (2002)
Monetary risk as a capital requirement:

(Regulatory perspective, cf. Basel II, III)

\[ g(x) = \inf \{ \mu \mid x + \mu \in \mathcal{A} \} \]

↑

"acceptable positions"

A convex (diversification is not penalized)

↓

Fenchel–Moreau + "monetary"
\[ g(X) = \sup_{\omega \in \Omega} \left( E[\omega - X] - \alpha(\omega) \right) \]

where

\[ \Omega := \text{a class of probability measures on } (\Omega, \mathcal{F}) \]

\[ \alpha(\omega) := \sup_{X \in \mathcal{X}} E[\omega - X] \]

— an explicit formalization of model uncertainty ("Knightian" uncertainty)
"robust" view:

no probability measure is given a priori, but:

probability measures do come in as "stress tests"!
Risk measure / monetary valuation

\[ U = -g \]

applied to (discounted) cash flows (in discrete time)

\[ C_t, \ t=0,1,\ldots \]

adapted on \((\Omega, \mathcal{F}), (\mathcal{F}_t)_{t=0,1}\ldots\)

(under Knightian uncertainty)

such that

\[ X_t = \sum_{k=0}^{t} C_k, \ t=0,1,\ldots \]

is a bounded measurable function on

\[ \Omega := \Omega \times \{0,1,\ldots\} \]

\[ \mathcal{F} := \mathcal{B} \text{ (adapted processes)} \]

"optional" \(\mathcal{F}\)-field

i.e. \(X \in \mathcal{F}\), even \(e \in \mathcal{F}\)
$g : \overline{\mathbb{E}} \rightarrow \mathbb{R}^r$

convex risk measure

\[ g(\mathbb{E}) = \sup_{\overline{Q} \in \mathcal{M}_{1,\mathbb{F}}(\mathbb{F},\mathbb{E})} (\mathbb{E}_{\overline{Q}} [-\mathbb{X}] - \alpha(\overline{Q})) \]

= ?

(cf. Banach Limits et al.)
For any "relevant" $\tilde{\Omega} \in \mathcal{F}_{\text{w}, \text{u}, \text{r}}$ with $\alpha(\tilde{\Omega}) < \infty$

exists $\sigma$-additive probability measure $\mathcal{Q}$ on extended space $\tilde{\Omega}^*: = \Omega \times \{0, \ldots, \infty\}$

such that

$\mathbb{E}_{\mathcal{Q}}[X] = \mathbb{E}_{\mathcal{Q}}[\tilde{X}]$

$\forall X \in \tilde{\mathcal{F}}_{\infty}$

$X(\infty) := \lim_{t \to \infty} X(t)$
via Itô-Watanabe factorization:

\[ \tilde{\mathcal{Q}} = \mathcal{Q} \otimes \gamma \]

\( \mathcal{Q} = \) probability measure on \((\Omega, \mathcal{F})\)

\( \gamma = \) optional random measure on \(\{0, \ldots, \infty\}, \leq 0 \)

\[ \sum_{t=0}^{\infty} \gamma_t(\omega) + \gamma_\infty(\omega) = 1 \quad \mathcal{Q}\text{-a.s.} \]

\[ g(\bar{x}) = \sup_{\mathcal{Q}} \sup_{\gamma} \sup_{\alpha} \]

\[ E_{\mathcal{Q}} \left[ \sum_{t=0}^{\infty} (-x_t) \gamma_t - x_\infty \gamma_\infty \right] - \alpha(\mathcal{Q}, \gamma) \]

(Cf. also Kandaras, Ann. Appl. Probab.)

for a related decomposition.
for \( (\text{via integration by parts}) \)

\[
C_t := X_t - X_{t-1}, \quad D_t := \sum_{s=t}^{\infty} \delta_s
\]

\( \text{predictable discounting} \)

\[
g(\bar{\alpha}) = \sup_{\bar{Q}} \sup_{\bar{D}} \left( \alpha(\bar{Q}, \bar{D}) \right)
\]

- combines model ambiguity with discounting ambiguity under Knightian uncertainty
translated into

\textbf{monetary valuation}

\textbf{dynamic version:}

\[ U_t (C_{t+1}, C_{t+2}, \ldots) = \]

\[ \text{ess. inf} \left( E_{\mathcal{Q}} \left[ \frac{\infty}{s_{t+1}} \frac{D_s}{S_t} C_s \bigg| \mathcal{F}_t \right] + \alpha_t (\mathcal{Q}, D) \right) \]

\textbf{Time consistency} \iff \[ \forall \mathcal{Q}, D \text{ with } \alpha_0 (\mathcal{Q}, D) < \infty, \]

\[ \alpha_t := \alpha_t (\mathcal{Q}, D) \text{ satisfies} \]

\[ \int_{t}^{\infty} \alpha_s = \int_{t}^{\infty} \alpha_{t, t+1} + E_{\mathcal{Q}} \left[ D_{t+1} \bigg| \mathcal{F}_t \right] \text{ one-step penalties } \geq 0 \]

\[ \text{i.e. } \alpha \text{ - supermartingale} \]

\[ + \text{ explicit Doob decomposition} \]

\[ \Rightarrow \text{ Riesz decomposition into } \textbf{potential} + \textbf{bubble} \]
Asymptotic safety

\[ \forall \mathcal{Q}, \mathcal{D} : x_0(\mathcal{Q}, \mathcal{D}) \leq \infty \]

\[ U_t := U_t(\mathcal{C}_{t+1}, \ldots) , \quad \alpha_t := \alpha_t(\mathcal{Q}, \mathcal{D}) \]

satisfy

\[ \frac{\partial U_t}{\partial t} + \sum_{s=0}^{t} D_s C_s \quad \text{a.-a.s., in } L^{\infty}(\mathcal{Q}) \]

\[ \sum_{s=0}^{\infty} D_s C_s \quad \text{a.-a.s., actual final outcome} \]

**Assessment of future cash flow**

**Theorem:** Asymptotic safety holds iff \( \forall \mathcal{Q}, \mathcal{D} \) with \( x_0(\mathcal{Q}, \mathcal{D}) < \infty \)

\[ \frac{\partial x_t}{\partial t} \quad \text{is a } \mathcal{Q}-\text{potential} \]

i.e.

\[ \text{A bubble in the penalization} \]

i.e.

**no excessive neglect of relevant models!**
In contrast to Turner Review:

"Knightian Uncertainty"
is not orthogonal to
"sophisticated maths",
but a

rich source

of mathematical problems!
for example

Ioannis Karatzas
Daniel Fernholz,

"Optimal Arbitrage under model uncertainty"
(2011)
Back to Ioannis

60

cf. Financial Times x 3 weeks go

I am curious,

and I wish you (to the benefit of all of us)
many happy returns
and
Alles Gute,
Ioannis!