Some Martingale Aspects of Financial Bubbles

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Probability, Control and Finance: A Conference in Honor of

Ioannis Karatzas
Columbia University, June 4-8, 2012
Encounters with Ioannis:

- Bad Honnef (near Bonn)
  Conference on Stochastic Differential Systems (1985?)

- Columbia
  e.g. late 90's ("insider trading"
  But: $K^f = 2$

- BerCiu 2001

always inspiring
always joyful
Scale of "Respectability" for Applications of Probability
≈ mid-60's

Mathematical Physics
(Statistical Mechanics, Quantum Field Theory, ...)

Mathematical Biology

Mathematical Economics

Actuarial Mathematics

Financial Mathematics
Since mid-60's:

- systematic use of probabilistic methods in finance (pioneered at MIT)

- rediscovery of "Théorie de La Spéculation" Louis Bachelier (1900):

  Brownian motion

  as a model for price fluctuation of a liquid asset
Since mid-eighties:

"Mathematical Finance"

"Financial Mathematics"

- emerged as a new frontier within Mathematics, i.e.,
  a source of new research problems of intrinsic mathematical interest

- Leading probabilists entered the field, among them
  Ioannis Karatzas

(and others in this room, e.g., Nicole El Karoui, Mark Davis...
Language/Notation of Mathematics

more precisely:

mathematical Language of randomness

= theory of probability

has entered/-shaped the discourse in finance (academia/industry)
Ioannis Karatzas:


Ioannis Karatzas, Steven Shreve:

*Brownian Motion and Stochastic Calculus* (1991)


a (parallel) world of crystalline clarity and great intellectual beauty
"Model Rationism"


or rather

"general theory"

(cf. L. Foldes, Imp. College)

"Spotlights on Reality"

(Alain Coomes)

create

- great clarity (locally)
- dark shadows
Conceptual framework for pricing derivatives

"contingent claims"

\[ = (\text{non-linear}) \text{ functionals of some underlying financial asset with given price fluctuation} \]

\[ X_t(\omega) \]
Broad interdisciplinary consensus:

Price fluctuation of a (liquid) financial asset should be viewed as a stochastic process

\[ X_t(\omega), \ 0 \leq t \leq T \]
on some probability space 

\( (\Omega, \mathcal{F}, P) \)
typically

"objectivist"

interpretation / intuition:

- P exists
- P can be identified (partially) by statistical/econometric methods
- P should satisfy certain a priori constraints ("market efficiency"/"absence of arbitrage")
Basic theoretical argument:

\[ \exists \text{ "free lunch" } \]

\[ \iff \]

Kreps - Harrison

\[ \Downarrow \]

Delbaen - Schachermayer

\[ \equiv \text{ "martingale measure" } \]

\[ \mathbb{P}^* \approx \mathbb{P} \]

i.e.

\[ \mathbb{E}^* \left[ X_{t+h} - X_t \mid \mathcal{F}_t \right] = 0 \]

up to localization
Thus: "no free lunch"

\[ \mathfrak{P}^* := \text{all equivalent martingale measures} \text{, } \mathbb{P}^* \sim \mathbb{P} \]

\[ \mathfrak{P}^* \neq \emptyset \]

\[ \Rightarrow \text{ Jacod, Yor, ...} \]

\[ (X_t) = \text{"semimartingale"} \]
\[ = \text{stochastic integrator} \]
\[ = \text{Dellacherie} \quad (\rightarrow \text{ into calculus}) \]
\[ = \text{Brownian motion up to a random time change} \]

(\text{W. Doeblin, ...}, \text{L. Dobinski, ...}, \text{I. Monroe, ...})

1940, 50's, 1972
In this context:

paradigm of perfect (super) replication via

Itô/Malliavin calculus, Itô representation, optional decomposition, ...

role of Probability: not prediction, but:

a sophisticated consistency check for pricing/hedging

- at the level of $P^*$

or, more radically, on its support, without probability:
Prediction

in terms of \( P \)

drift, ...)

does not intervene

matter!
Standard setting in Financial Mathematics is probabilistic:

\[ P \quad \text{vs.} \quad P^* \]

\[ \uparrow \quad \uparrow \]

"objective"/"real world" probability measure

pricing measure (the market's "belief")

\[ P = ? \]

d'\text{e} \text{ Finetti, Keynes: } \exists P \]
Looking forward:
de Finetti (1931/37):

"Probability (P) does not exist"

but:

prices (P*) do!

via financial bets, based on subjective degrees of belief:

"You"

= the financial market
Interplay between $P$ and $P^*$ ($\mathbb{P}^*$)

\[ \uparrow \]

"historical" / "real world" measure  |  "martingale" / "risk neutral" measure

as a prediction scheme, looking forward ?

?  |  much more is known:
at each time $t$:

$$P^*_{t} = P^*_{[\Delta t]}$$

$\theta$ (traded claims with maturities $> t$)

= the market's implicit prediction scheme at time $t$

via prices of "plain vanilla" claims ($\rightarrow$ marginal supplies)
and more complex derivatives ($\rightarrow$ joint distributions)

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consistent (via arbitrage)

- across claims
- across future times $s > t$
Dynamics of \( P^* \) at \( t \) involves market microstructure, i.e., many heterogeneous agents with interacting preferences / expectations. "Feeding behavior" and "bubbles / crashes". Toy models: \( \checkmark \) serious prediction: not in sight!
**Bubbles**

(c.f. "model platonism")

"nice" martingales vs.

uniformaly integrable

"not so nice" martingales

not uniformaly integrable

e.g. strict local martingale

≈ "singular" behavior

in a measure theoretic sense:

mass disappears

but where does it go?

c.f. Walter's talk:

finitely additive measure

(heurist n (1970))

with some topological assumption)
Bubbles

(in "model platonism")

"nice" martingales vs.
uniformly integrable

(not so nice" martingales
not uniformly integrable
e.g. strict local martingale

"singular" behavior
in a measure theoretic sense: mass disappears but where does it go?

c.f. Walter's talk:

finitely additive measure
(Elterrier & 1970)
with some topological assumptions
(avoiding projective limit space):

\[ X \supseteq \text{super martingale} \ni 0 \]

\[ \sim \]

"exit measure" \[ \mathcal{P} X \]

on \[ \Omega \times (0, \infty) \]

predictable \[ \mathcal{F} \] field

\[ \mathbb{E}_\mathcal{P} [X_t; \mathcal{F}_t] = \mathcal{P} X [X_t \times (t, \infty)] \]

"life time" \[ S(\omega, t) := t \]

\[ X \ni \text{"not nice" martingale} \]

\[ \iff \]

\[ \mathcal{P} X \nleq \mathcal{P} \circ S_0 \]

pure \[ \text{local martingale} \iff \]

\[ S < \infty \text{ \( \mathcal{P} \)}} \text{-a.s.} \]

= "explosion time" 

(predictable)
I. **Bubbles**

"in the eye of the beholder"

F. Biagini, H.F., S. Nedelec (in progress)
following Carmona, Potters, Shimbo (2006, 2011)

Consider an asset with

- dividend process \( 0 \leq D_t \leq D_0 \)
- price process \( S_t \geq 0 \)
- wealth process

\[
\omega_t = S_t + D_t
\]

- a (local) martingale
under each \( P^* \in \Omega^* \)

**Assume:** \( \omega_\infty = D_\infty \), i.e., \( S_\infty = 0 \)

(a.e.o.g.)
Note: Each \( P^* \in \mathcal{P}^* \) provides a legitimate (market consistent) view of the future, and \( (S_t) \) is supported by each such \( P^* \in \mathcal{P}^* \):

\[
S_t = \underset{\omega \geq t}{\text{ess. sup}} \mathbb{E}_{P^*} \left[ \sum_{\alpha=0}^{\infty} \omega_{\alpha} \frac{I_{\alpha}}{S_t} \right]
\]

since

\[
\omega_t = \underset{\omega \geq t}{\text{ess. sup}} \mathbb{E}_{P^*} \left[ \omega_{\alpha} \right]_{\alpha \geq t}
\]

\( \omega \in \mathcal{O}, \quad \alpha \geq t \)

\text{cf. Harrison-Kreps: Speculative Investor Behavior in a Stock Market with Heterogeneous Expectations (1978)}

\[
= \underset{\alpha \geq t}{\text{ess. sup}} \mathbb{E}_{P^*} \left[ \omega_{\alpha} \mid S_t \right]
\]

\( P^* \in \mathcal{P}^* \)