Some Martingale Aspects of Financial Bubbles

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Probability, Control and Finance: A Conference in Honor of

Ioannis Karatzas

Columbia University, June 4-8, 2012

Encounters with Ioannis:

- Bad Honnef (near Bonn)
  Conference on Stochastic Differential Systems (1985 ?)

- Columbia
  e.g. late 90's ("insider trading"
  But: $K\# = Q$)

- Berlin 2001

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always inspiring
always joyful

Scale of "Respectability" for Applications of Probability.

$\approx$ mid-60's

- Mathematical Physics (Statistical Mechanics, Quantum Field Theory, ..)
- Mathematical Biology
- Mathematical Economics

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- Actuarial Mathematics

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Financial Mathematics ?

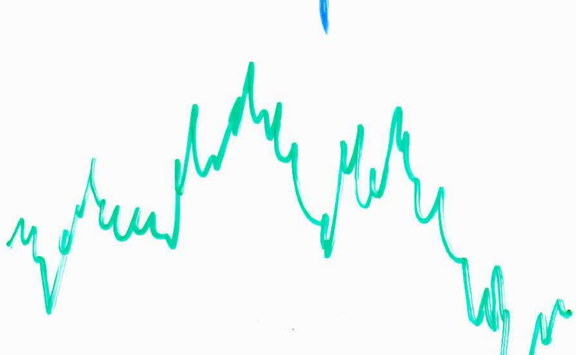
Since mid-60's:

- systematic use of probabilistic methods in finance (pioneered at MIT)

- rediscovery of "Théorie de la Spéculation" Louis Bachelier (1900):

<u>Brownian motion</u>

as a model for price fluctuation of a liquid asset



Since mid-eighties:

"Mathematical Finance" /
"Financial Mathematics"

- emerged as a new frontier within Mathematics, i.e., a source of new research problems of intrinsic mathematical interest
- leading probabilists entered the field, among them

Ioannis Karatzas

(and others in this room e.g. Nicole El Karoui, Mark Davis, ..)

Language / Notation of Mathematics

more precisely:

mathematical language of randomness

= theory of <u>probability</u>

has entered (shaped the discourse in <u>finance</u> (academia / industry)

Ioannis Karatzas:

Lectures on the Mathematics of Finance (1996)

Ioannis Karatzas, Steven Shreve:

Brownian Motion and Stochastic Calculus (1991)

Methods of Mathematical Finance (1998)

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a (parallel) world of crystalline clarity and great intellectual beauty

"Model Platonism"

(H. Albert 1967,
MPI Collective Goods
J. Inst. Economics 2012)

or rather

"general theory"

(cf. L. Foldes, Imp. College)

"Spotlights on Reality"

(Alain Connes):

create

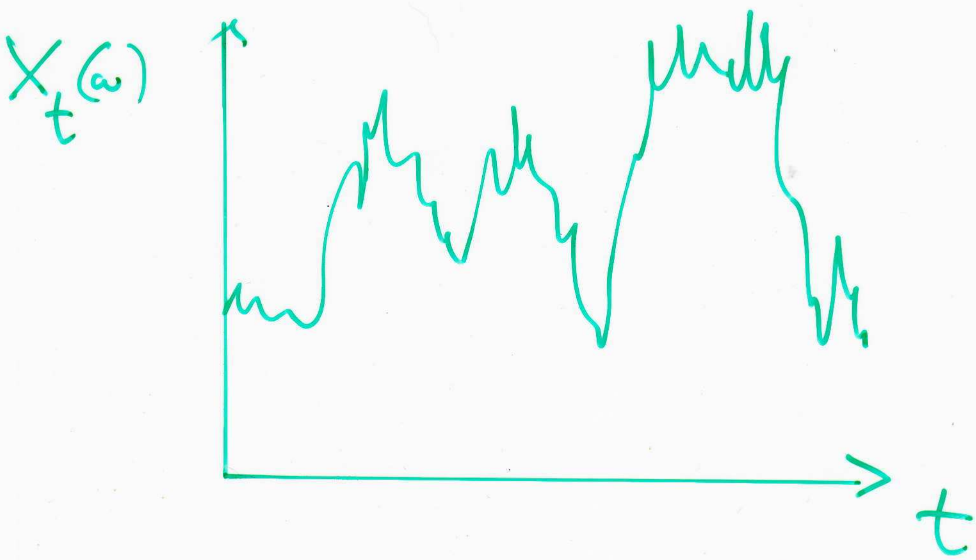
- great clarity (locally)
- dark shadows

Conceptual framework
for

pricing derivatives

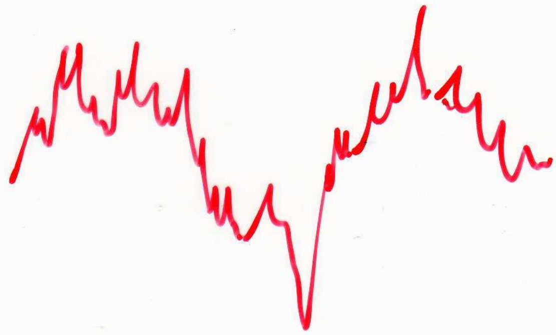
"contingent claims"

= (non-linear) functionals
of some underlying
financial asset with
given price fluctuation



Broad interdisciplinary consensus:

Price fluctuation of a (liquid) financial asset



should be viewed as a stochastic process

$$X_t(\omega), \quad 0 \leq t \leq T$$

on some probability space

$$(\Omega, \mathcal{F}, P)$$

typically

"objectivist"

interpretation / intuition:

- P exists
- P can be identified (partially) by statistical/ econometric methods
- P should satisfy certain a priori constraints ("market efficiency"/ "absence of arbitrage")

Basic theoretical argument:

$\nexists$ "free lunch"

"$\Longleftrightarrow$" Kreps - Harrison

Delbaen - Schachermayer

$\exists$ "martingale measure"

$$P^* \approx P$$

i.e.

$$E^*[X_{t+h} - X_t \mid \mathcal{F}_t] = 0$$

up to localization

Thus: "no free lunch"

$$\Updownarrow$$

$\mathcal{P}^* :=$ all equivalent martingale measures $P^* \approx P$

$\neq \emptyset$

$$\Downarrow$$ Jacod, Yor, ...

(X_t) = "semimartingale"

= stochastic integrator

Dellacherie (→ Itô calculus)

= Brownian motion up to a random time change

(W. Doeblin, ..., L. Dubins, ..., I. Monroe)
1940 50's 1972

In this context:

paradigm of perfect (super) replication

via

Itô/Malliavin calculus,
Itô representation,
optional decomposition,
...

role of Probability:

<u>not</u> prediction,

but:

a sophisticated
<u>consistency check</u>
for pricing / hedging

– at the level of
P^* !

or, more radically,
on its support,
without probability:

Prediction

in terms of P
(drift, ...) :

does not intervene/
matter !

standard setting in Financial Mathematics is probabilistic:

$$\mathcal{P} \quad \text{vs.} \quad \mathcal{P}^*$$

$\uparrow$ "objective" / "real world" probability measure

$\uparrow$ pricing measure (the market's "belief")

$$\mathcal{P} = \,?$$

de Finetti, Keynes: $\nexists \, \mathcal{P}$

looking forward:

de Finetti (1931/37):

"Probability (P) does not exist"

but:

prices (P^*) do!

via financial bets, based on subjective degrees of belief:

"You"

= the financial market

Interplay between

$\mathcal{P}$ and $\mathcal{P}^*$ ($\mathcal{Q}^*$)

"historical"/ "real world" measure | "martingale"/ "risk neutral" measure

as a prediction scheme, looking forward ?

? | much more is known :

at each time t:

$$\mathcal{P}^*_t := \mathcal{P}^*[\,\cdot\,|\,\mathcal{F}_t]$$

$\approx$ (traded claims with maturities $> t$)

= the market's implicit <u>prediction scheme</u> at time t

via prices of "plain vanilla" claims ($\to$ marginals $\mathcal{D}$ spine) and more complex derivatives ($\to$ joint distributions)

— <u>consistent</u> (via arbitrage)

- across claims
- across future times $s > t$

<u>But</u>:

Dynamics of p_t^* ?

involves

market microstructure,
i.e., many heterogeneous agents with interacting preferences / expectations
"Herding Behavior"
bubbles / crashes

toy models: ✓
serious prediction: not in sight!

Bubbles

(in "model platonism")

$\underbrace{\text{"nice" martingales}}_{\text{uniformly integrable}}$ vs.

$\underbrace{\text{"not so nice" (local) martingales}}_{\text{not uniformly integrable}}$

e.g. strict local martingale

~ "singular" behavior in a measure theoretic sense:

mass disappears but where does it go?

c.f. Walter's talk:

finitely additive measure (Oletiviev ~ 1970)

with some topological assumptions:

Bubbles

(in "model platonism")

"nice" martingales vs.

uniformly integrable

"not so nice" (local) martingales

not uniformly integrable

e.g. strict local martingale

~ "singular" behavior in a measure theoretic sense:

mass disappears but where does it go?

c.f. Walter's talk:

finitely additive measure (Metivier ~ 1970)

with some topological assumptions

(avoiding projective limit space):

X supermartingale ≥ 0

$\sim$ "exit measure" P^X

F. 1972, 1973, 2006

on $\Omega \times (0,\infty]$
predictable σ-field

$$E_P[X_t; A_t] = P^X[A_t \times (t,\infty]]$$

"life time" $\zeta(\omega,t) := t$

X "not nice" martingale
(local)

$$\iff P^X \not\ll P \otimes \delta_\infty$$

pure local martingale $\iff$
$\zeta < \infty$ P^X-a.s., = "explosion time"
(predictable)

I. Bubbles

"in the eye of the beholder"

F. Biagini, H.F., S. Nedelcu (in progress)
following Jarrow, Protter, Shimbo (2006, 2011)

Consider an asset with
dividend process $0 \le D_t \uparrow D_\infty$
price process $S_t \ge 0$
wealth process

$$W_t := S_t + D_t$$

— a (local) martingale under each $P^* \in \mathcal{P}^*$

Assume: $W_\infty = D_\infty$, i.e., $S_\infty = 0$
(w.l.o.g.)

Note: Each $P^* \in \mathcal{P}^*$ provides a legitimate (market consistent) view of the future, and (S_t) is supported by each such $P^* \in \mathcal{P}^*$:

$$S_t = \underset{\tau \geq t}{\text{ess.sup}}\; E_{P^*}[D_\tau - D_t + S_\tau \mid \mathcal{F}_t]$$

since

$$\omega_t \underset{\text{Doob}}{=} \underset{\tau \geq t}{\text{ess.sup}}\; E_{P^*}[\omega_\tau \mid \mathcal{F}_t]$$

cf. Harrison-Kreps: Speculative Investor Behavior in a Stock Market with Heterogeneous Expectations (1978)

$$= \underset{\substack{\tau \geq t \\ P^* \in \mathcal{P}^*}}{\text{ess.sup}}\; E_{P^*}[\omega_\tau \mid \mathcal{F}_t]$$