Information Percolation in Segmented Markets

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Probability, Control, and Finance
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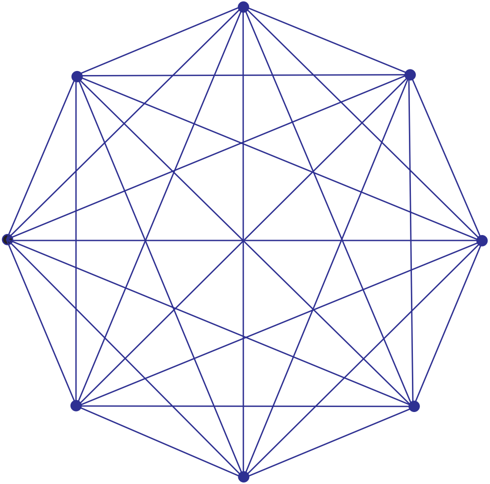


Figure: An over-the-counter market.

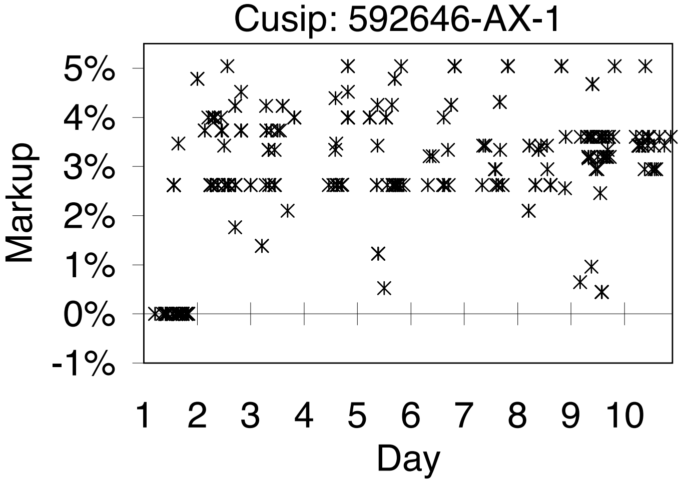


Figure: Transaction price dispersion in muni market. Source: Green, Hollifield, and Schürhoff (2007). See, also, Goldstein and Hotchkiss (2007).

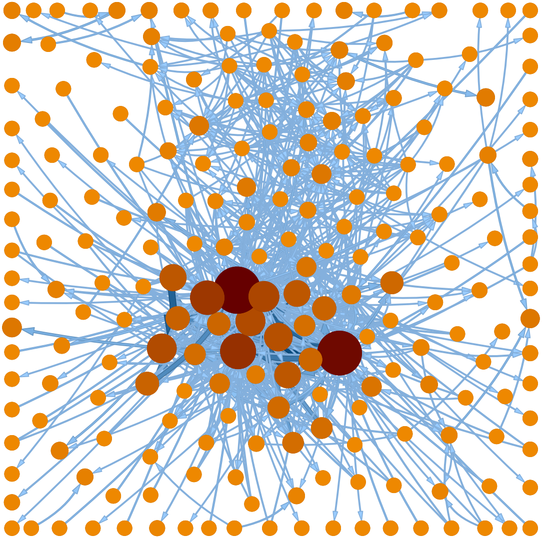


Figure: Daily trade in the federal funds Market. Source: Bech and Atalay (2012).

Information Transmission in Markets

Informational Role of Prices: Hayek (1945), Grossman (1976), Grossman and Stiglitz (1981).

- Centralized exchanges:
  - Wilson (1977), Townsend (1978), Milgrom (1981), Vives (1993), Pesendorfer and Swinkels (1997), and Reny and Perry (2006).
- Over-the-counter markets:
  - Wolinsky (1990), Blouin and Serrano (2002), Golosov, Lorenzoni, and Tsyvinski (2009).
  - Duffie and Manso (2007), Duffie, Giroux, and Manso (2008), Duffie, Malamud, and Manso (2010).

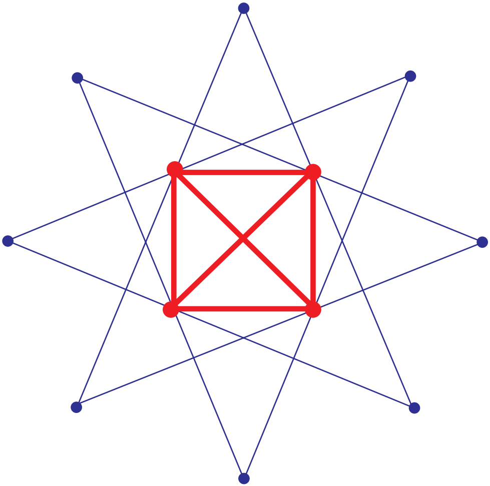


Figure: Many OTC markets are dealer-intermediated.

Model Primitives

- Agents: a non-atomic measure space $(G, \mathcal{G}, \gamma)$.
- Uncertainty: a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.
- An asset has a random payoff X with outcomes H and L.
- Agent i is initially endowed with a finite set $S_i = \{s_1, \ldots, s_n\}$ of $\{0, 1\}$-signals.
- Agents have disjoint sets of signals.
- The measurable subsets of $\Omega \times G$ are enriched from the product σ-algebra enough to allow signals to be essentially pairwise X-conditionally independent, and to allow Fubini, and thus the exact law of large numbers (ELLN). (Sun, JET, 2006).

Information Types

After observing signals $S = \{s_1, \ldots, s_n\}$, the logarithm of the likelihood ratio between states $X = H$ and $X = L$ is by Bayes' rule:

$$\log \frac{\mathbb{P}(X = H \mid s_1, \ldots, s_n)}{\mathbb{P}(X = L \mid s_1, \ldots, s_n)} = \log \frac{\mathbb{P}(X = H)}{\mathbb{P}(X = L)} + \sum_{i=1}^{n} \log \frac{p_i(s_i \mid H)}{p_i(s_i \mid L)},$$

where $p_i(s \mid k) = \mathbb{P}(s_i = s \mid X = k)$. We say that the "type" θ associated with this set of signals is

$$\theta = \sum_{i=1}^{n} \log \frac{p_i(s_i \mid H)}{p_i(s_i \mid L)}.$$

ELLN for Cross-Sectional Type Density

- The ELLN implies that, on the event $\{X = H\}$, the fraction of agents whose initial type is no larger than some given number y is almost surely

$$F^H(y) = \int_G 1_{\{\theta_\alpha \leq y\}} \, d\gamma(\alpha) = \int_G \mathbb{P}(\theta_\alpha \leq y \,|\, X = H) \, d\gamma(\alpha),$$

  where θ_α is the initial type of agent α.
- On the event $\{X = L\}$, the cross-sectional distribution function F^L of types is likewise defined and characterized.
- We suppose that F^H and F^L have densities, denoted $g^H(\,\cdot\,, 0)$ and $g^L(\,\cdot\,, 0)$ respectively.
- We write $g(x, 0)$ for the random variable whose outcome is $g^H(x, 0)$ on the event $\{X = H\}$ and $g^L(x, 0)$ on the event $\{X = L\}$.

Information is Additive in Type

Proposition

Let $S = \{s_1, \ldots, s_n\}$ and $R = \{r_1, \ldots, r_m\}$ be disjoint sets of signals, with associated types θ and ϕ. If two agents with types θ and ϕ reveal their types to each other, then both agents achieve the posterior type $\theta + \phi$.

This follows from Bayes' rule, by which

$$\log \frac{\mathbb{P}(X = H \,|\, S, R, \theta + \phi)}{\mathbb{P}(X = L \,|\, S, R, \theta + \phi)} = \log \frac{\mathbb{P}(H = H)}{\mathbb{P}(X = L)} + \theta + \phi,$$

$$= \log \frac{\mathbb{P}(X = H \,|\, \theta + \phi)}{\mathbb{P}(X = L \,|\, \theta + \phi)}$$

Dynamics of Cross-Sectional Density of Types

Each period, each agent is matched, with probability λ, to a randomly chosen agent (uniformly distributed). They share their posteriors on X.

Duffie and Sun (AAP 2007, JET 2012): With essential-pairwise-independent random matching of agents,

$$g(x, t+1) = (1-\lambda) g(x,t) + \int_{-\infty}^{+\infty} \lambda g(y,t) g(x-y,t)\, dy, \quad x \in \mathbb{R}, \quad \text{a.s.}$$

which can be written more compactly as

$$g(t+1) = (1-\lambda) g(t) + \lambda g(t) * g(t),$$

where $*$ denotes convolution.

Solution of Cross-Sectional Distribution Types

- The Fourier transform of $g(\cdot, t)$ is

$$\hat{g}(z,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-izx} g(x,t)\, dx.$$

- From (11), for each z in $\mathbb{R}$,

$$\frac{d}{dt}\, \hat{g}(z,t) = -\lambda \hat{g}(z,t) + \lambda \hat{g}^2(z,t), \tag{1}$$

- Thus, the differential equation for the transform is solved by

$$\hat{g}(z,t) = \frac{\hat{g}(z,0)}{e^{\lambda t}(1 - \hat{g}(z,0)) + \hat{g}(z,0)}. \tag{2}$$

Solution of Cross-Sectional Distribution Types

Proposition

The unique solution of the dynamic equation (11) for the cross-sectional type density is the Wild sum

$$g(\theta,t) = \sum_{n\geq 1} e^{-\lambda t}(1-e^{-\lambda t})^{n-1} g^{*n}(\theta,0), \tag{3}$$

where $g^{*n}(\,\cdot\,,0)$ is the n-fold convolution of $g(\,\cdot\,,0)$ with itself.

The solution (3) is justified by noting that the Fourier transform $\hat{g}(z,t)$ can be expanded from (2) as

$$\hat{g}(z,t) = \sum_{n\geq 1} e^{-\lambda t}(1-e^{-\lambda t})^{n-1} \hat{g}(z,0)^n,$$

which is the transform of the proposed solution for $g(\,\cdot\,,t)$.

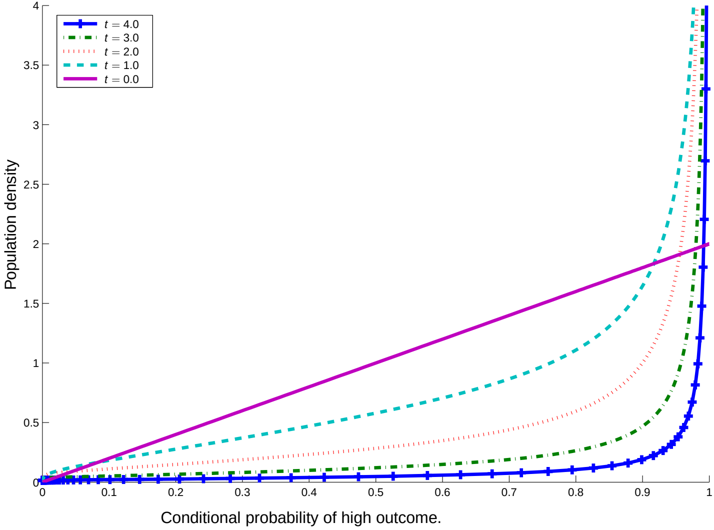
Numerical Example

- Let $\lambda = 1$ and $\mathbb{P}(X = H) = 1/2$.
- Agent α initially observes s_α, with

$$\mathbb{P}(s_\alpha = 1 \,|\, X = H) + \mathbb{P}(s_\alpha = 1 \,|\, X = L) = 1.$$

- $\mathbb{P}(s_\alpha = 1 \,|\, X = H)$ has a cross-sectional distribution over investors that is uniform over the interval $[1/2, 1]$.
- On the event $\{X = H\}$ of a high outcome, this initial allocation of signals induces an initial cross-sectional density of $f(p) = 2p$ for the likelihood $\mathbb{P}(X = H \,|\, s_\alpha)$ of a high state.

On the event $\{X = H\}$, the evolution of the cross-sectional population density of posterior probabilities of the event $\{X = H\}$.



Multi-Agent Meetings

The Boltzmann equation for the cross-sectional distribution μ_t of types is

$$\frac{d}{dt}\mu_t = -\lambda\,\mu_t + \lambda\,\mu_t^{*m}.$$

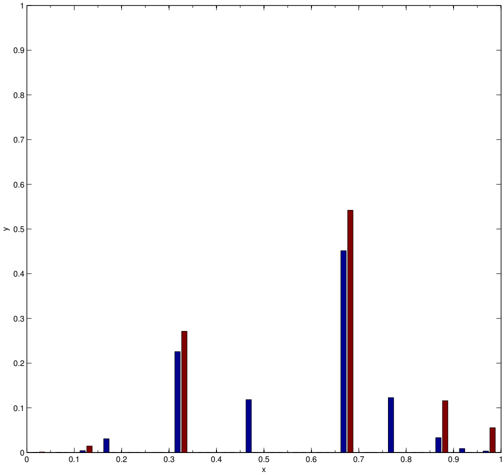
We obtain the ODE,

$$\frac{d}{dt}\hat{\mu}_t = -\lambda\,\hat{\mu}_t + \lambda\,\hat{\mu}_t^{m}\,,$$

whose solution satisfies

$$\hat{\mu}_t^{m-1} = \frac{\hat{\mu}_0^{m-1}}{e^{(m-1)\lambda t}(1-\hat{\mu}_0^{m-1}) + \hat{\mu}_0^{m-1}}\,. \tag{4}$$

Groups of 2 (blue) versus Groups of 3 (red)



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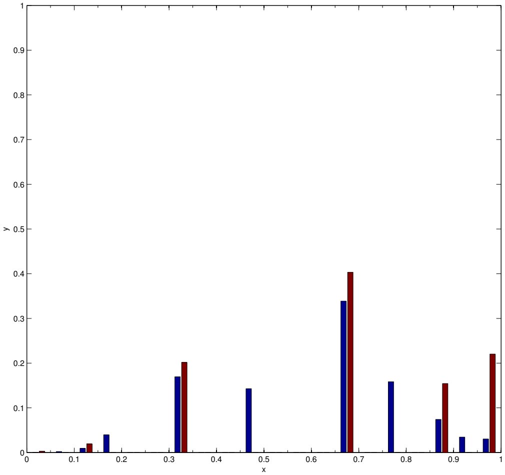

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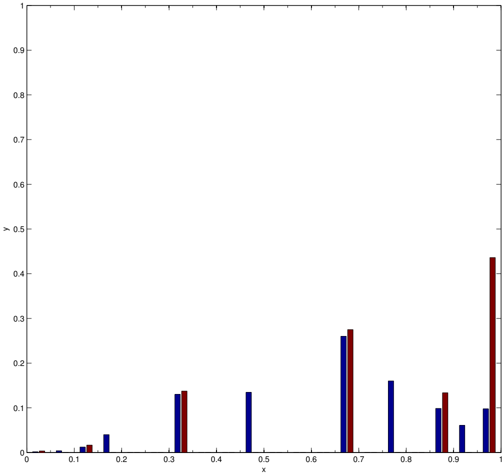

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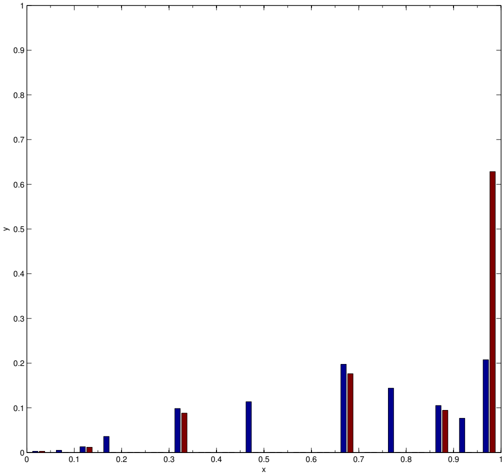

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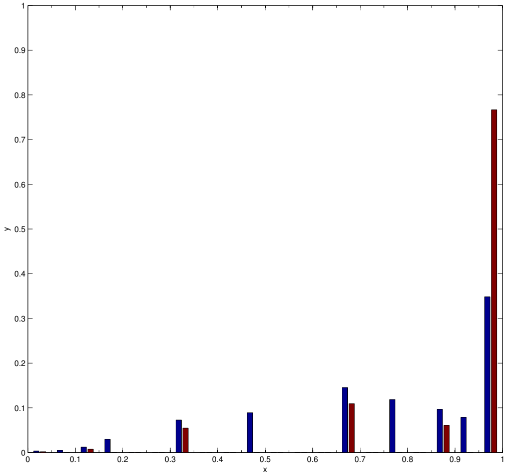

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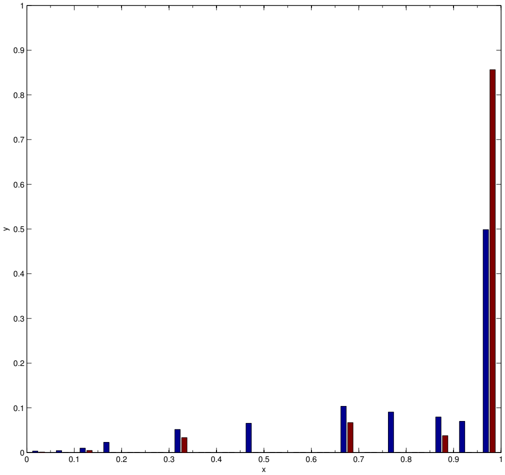

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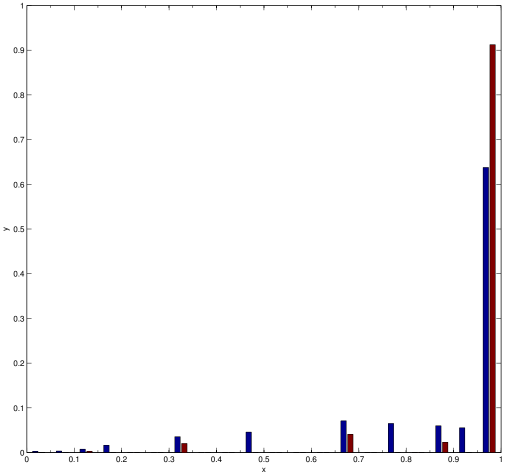

Other Extensions

- Privately gathered information.
- Public information releases (such as tweets or transaction announcements).
  - Duffie, Malamud, and Manso (2010).
- Endogenous search intensity
  - Duffie, Malamud, and Manso (2009).

A Segmented OTC Market

- Agents of class $i \in \{1, \ldots, M\}$ have matching probability λ_i.
- Upon meeting, the probability that a class-j agent is selected as a counterparty is κ_{ij}.
- At some time T, the economy ends, X is revealed, and the utility realized by an agent of class i for each additional unit of the asset is

$$U_i = v_i 1_{\{X=L\}} + v^H 1_{\{X=H\}},$$

for strictly positive v^H and $v_i < v^H$.

Trade by Seller's Price Double Auction

- If $v_i = v_j$, there is no trade (Milgrom and Stokey, 1982; Serrano-Padial, 2008).
- Upon a meeting with gains from trade, say $v_i < v_j$, the counterparties participate in a seller's price double auction.
- That is, if the buyer's bid β exceeds the seller's ask σ, trade occurs at the price σ.
- The class of one's counterparty is common knowledge.

Equilibrium

The prices (σ, β) constitute an equilibrium for a seller of class i and a buyer of class j provided that, fixing β, the offer σ maximizes the seller's conditional expected gain,

$$E\left[(\sigma - E(U_i \,|\, \mathcal{F}_S \cup \{\beta\}))1_{\{\sigma<\beta\}} \,|\, \mathcal{F}_S\right],$$

and fixing σ, the bid β maximizes the buyer's conditional expected gain

$$E\left[(E(U_j \,|\, \mathcal{F}_B \cup \{\sigma\}) - \sigma)1_{\{\sigma<\beta\}} \,|\, \mathcal{F}_B\right].$$

We look for equilibria that are completely revealing, of the form $(B(\theta), S(\phi))$, for a buyer and seller of types θ and ϕ, for strictly increasing $B(\,\cdot\,)$ and $S(\,\cdot\,)$.

Technical Conditions

Definition: A function $g(\cdot)$ on the real line is of exponential type α at $-\infty$ if, for some constants $c > 0$ and $\gamma > -1$,

$$\lim_{x \to -\infty} \frac{g(x)}{|x|^{\gamma} e^{\alpha x}} = c. \tag{5}$$

In this case, we write $g(x) \sim \mathrm{Exp}_{-\infty}(c, \gamma, \alpha)$. We use the notation $g(x) \sim \mathrm{Exp}_{+\infty}(c, \gamma, \alpha)$ analogously for the case of $x \to +\infty$.

Condition: For all i, g_{i0} is C^1 and strictly positive. For some $\alpha_- \geq 2.4$ and $\alpha_+ > 0$

$$\frac{d}{dx} g_{i0}^H(x) \sim \mathrm{Exp}_{-\infty}(c_{i,-}, \gamma_{i,-}, \alpha_-)$$

and

$$\frac{d}{dx} g_{i0}^H(x) \sim \mathrm{Exp}_{+\infty}(c_{i,+}, \gamma_{i,+}, \alpha_+)$$

for some $c_{i,\pm} > 0$ and some $\gamma_{i,\pm} \geq 0$.

Equilibrium Bidding Strategies

- We provide an ODE for the equilibrium type $\Phi(b)$ of a prospective buyer whose equilibrium bid is b. The ODE is the first-order condition for maximizing the probability of a trade multiplied by the expected profit given a trade.
- A prospective buyer of type ϕ bids $B(\phi) = \Phi^{-1}(\phi)$.
- A prospective seller of type θ offers $S(\theta) = \Theta^{-1}(\theta)$, where

$$\Theta(v) = \log \frac{v - v_i}{v^H - b} - \Phi(v), \quad v \in (v_i, v^H).$$

The ODE for the Buyer's Type

Lemma: For any initial condition $\phi_0 \in \mathbb{R}$, there exists a unique solution $\Phi(\,\cdot\,)$ on $[v_i, v^H)$ to the ODE

$$\Phi'(b) \;=\; \frac{1}{v_i - v_j}\left(\frac{b - v_i}{v^H - b}\,\frac{1}{h_{it}^H(\Phi(b))} + \frac{1}{h_{it}^L(\Phi(b))}\right), \quad \Phi(v_i) \;=\; \phi_0.$$

This solution, also denoted $\Phi(\phi_0, b)$, is monotone increasing in both b and ϕ_0. Further, $\lim_{b \to v^H} \Phi(b) = +\infty$.

The limit $\Phi(-\infty, b) \;=\; \lim_{\phi_0 \to -\infty} \Phi(\phi_0, b)$ exists and is strictly monotone and continuously differentiable with respect to b.

Bidding Strategies

Proposition

Suppose that (S, B) *is a continuous equilibrium such that* $S(\theta) \leq v^H$ *for all* $\theta \in \mathbb{R}$. *Let* $\phi_0 = B^{-1}(v_i) \geq -\infty$. *Then,*

$$B(\phi) \ = \ \Phi^{-1}(\phi), \quad \phi \ > \ \phi_0,$$

Further, $\lim_{\theta \to -\infty} S(\theta) \ = \ v_i$ *and* $\lim_{\theta \to -\infty} S(\theta) \ = \ v^H$, *and for any* θ, *we have* $S(\theta) \ = \ \Theta^{-1}(\theta)$. *Any buyer of type* $\phi < \phi_0$ *does not trade, and has a bidding policy* B *that is not uniquely determined at types below* ϕ_0.

The unique welfare maximizing equilibrium is that associated with $\lim_{\phi_0 \to -\infty} \Phi(\phi_0, b)$. *This equilibrium exists and is fully revealing.*

Evolution of Type Distribution

Dynamics for the distribution of types of agents of class i:

$$g_{i,t+1} \;=\; (1-\lambda_i)\, g_{it} \;+\; \lambda_i\, g_{it} * \sum_{j=1}^{M} \kappa_{ij}\, g_{jt}, \quad i \in \{1,\ldots,M\}\,.$$

Taking Fourier transforms:

$$\hat{g}_{i,t+1} \;=\; (1-\lambda_i)\, \hat{g}_{it} \;+\; \lambda_i\, \hat{g}_{it} \sum_{j=1}^{M} \kappa_{ij}\, \hat{g}_{jt}, \quad i \in \{1,\ldots,M\}.$$

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Special Case: $N = 2$ and $\lambda_1 = \lambda_2$

Proposition: Suppose $N = 2$ and $\lambda_1 = \lambda_2 = \lambda$. Then

$$\hat{\psi}_1 = \frac{e^{-\lambda t}\,(\hat{\psi}_{20} - \hat{\psi}_{10})}{\hat{\psi}_{20} e^{-\hat{\psi}_{20}(1-e^{-\lambda t})} - \hat{\psi}_{10} e^{-\hat{\psi}_{10}(1-e^{-\lambda t})}}\,\hat{\psi}_{10}\,e^{-\hat{\psi}_{10}(1-e^{-\lambda t})}$$

$$\hat{\psi}_2 = \frac{e^{-\lambda t}\,(\hat{\psi}_{20} - \hat{\psi}_{10})}{\hat{\psi}_{20} e^{-\hat{\psi}_{20}(1-e^{-\lambda t})} - \hat{\psi}_{10} e^{-\hat{\psi}_{10}(1-e^{-\lambda t})}}\,\hat{\psi}_{20}\,e^{-\hat{\psi}_{20}(1-e^{-\lambda t})}.$$

General Case: Wild Sum Representation

Theorem: There is a unique solution of the evolution equation, given by

$$\psi_{it} \; = \; \sum_{k \in \mathbb{Z}_+^N} a_{it}(k) \, \psi_{10}^{*k_1} * \cdots * \psi_{N0}^{*k_N},$$

where ψ_{i0}^{*n} denotes n-fold convolution,

$$a'_{it} \; = \; -\lambda_i \, a_{it} \; + \; \lambda_i \, a_{it} * \sum_{j=1}^{N} \kappa_{ij} \, a_{jt}, \quad a_{i0} = \delta_{e_i},$$

$$(a_{it} * a_{jt})(k_1, \ldots, k_N) \; = \; \sum_{l=(l_1,\ldots,l_N) \in \mathbb{Z}_+^N \, , \, l<k} a_{it}(l) \, a_{jt}(k-l),$$

and

$$a_{it}(e_i) \; = \; e^{-\lambda_i t} \, a_{i0}(e_i).$$

Endogenous Information Acquisition

- A signal packet is a set of signals with type density f, satisfying the technical conditions.
- Agents are endowed with $N_{\min}$ signal packets, and can acquire up to $\overline{n}$ more, at a cost of π each.
- Agents conjecture the packet quantity choices $N = (N_1, \ldots, N_M)$ of the M classes.
- An agent of class i who initially acquires n signal packets, and as a result has information-type $\Theta_{n,N,t}$ at time t, has initial expected utility

$$u_{i,n,N} = E\left(-\pi n + \sum_{t=1}^{T} \lambda_i \sum_{j} \kappa_{ij}\, v_{ijt}(\Theta_{n,N,t}; B_{ijt}, S_{ijt})\right). \quad (6)$$

Information Acquisition Equilibrium

Some $(N_i$, (S_{ijt}, B_{ijt}), $g_{it})$ is a pure-strategy rational expectations equilibrium if

- The cross-sectional type density g_{it} satisfies the evolution equation with initial condition the $(N_{\min} + N_i)$-fold convolution of the packet type density f.
- The bid and ask functions (S_{ijt}, B_{ijt}) are revealing double-auction equilibria.
- The number N_i of signal packets acquired by class i solves $\max_{n \in \{0,\dots,\overline{n}\}} u_{i,n,N}$.

3-Class Incentives

- We take the case of two equal-mass seller classes with matching probabilities λ_1 and $\lambda_2 > \lambda_1$.
- The buyer class has matching probability $(\lambda_1 + \lambda_2)/2$.
- $\frac{d}{dx} f^H(x) \sim \mathrm{Exp}_{-\infty}(c_0, 0, \alpha + 1)$ and $\frac{d}{dx} f^H(x) \sim \mathrm{Exp}_{+\infty}(c_0, 0, -\alpha)$ for some $\alpha \geq 1.4$ for some $c_0 > 0$.

Proposition

For $\frac{v_b - v_s}{v^H - v_b}$ and T large enough, information acquisition is a strategic complement. By contrast, for smaller T, there exist counterexamples to strategic complementarity.

Information Acquisition Incentives

- If T is not too great, increasing λ_2 of the more active class-2 sellers *lowers* the incentive of the less active class-1 sellers to gather more information. This can be explained as follows.
- As class-2 sellers become more active, buyers learn at a faster rate. The impact of this on the incentive of the "slower" class-1 sellers to gather information is determined by a "learning effect" and an opposing "pricing effect."
- The learning effect is that, knowing that buyers will learn faster as λ_2 is raised, a less connected seller is prone to acquire more information in order to avoid being at an informational disadvantage when facing buyers.
- The pricing effect is that, in order to avoid missing unconditional private-value expected gains from trade with better-informed buyers, sellers find it optimal to reduce their ask prices.
- The learning effect dominates the pricing effect if and only if there are sufficiently many trading rounds.

- We show cases in which increasing λ_2 leads to a full collapse of information acquisition (meaning that, in any equilibrium, the fraction of agents that acquire signals is zero).
- Compare with the case of a static double auction, corresponding to $T = 0$. With only one round of trade, the learning effect is absent and the expected gain from acquiring information for class-1 sellers is proportional to λ_1 and does not depend on λ_2. Similarly, the gain from information acquisition for buyers is linear and increasing in λ_2. Consequently, in the static case, an increase in λ_2 always leads to more information acquisition.