On the Multi-Dimensional Controller and Stopper Games

Erhan Bayraktar

Joint work with Yu-Jui Huang
University of Michigan, Ann Arbor

June 7, 2012

Outline

Consider a zero-sum controller-and-stopper game:

- Two players: the "controller" and the "stopper".
- A state process X^α: can be manipulated by the controller through the selection of α.
- Given a time horizon $T > 0$. The stopper has
  - the right to choose the duration of the game, in the form of a stopping time τ in $[0, T]$ a.s.
  - the obligation to pay the controller the running reward $f(s, X_s^\alpha, \alpha_s)$ at every moment $0 \le s < \tau$, and the terminal reward $g(X_\tau^\alpha)$ at time τ.
- Instantaneous discount rate: $c(s, X_s^\alpha)$, $0 \le s \le T$.

Value Functions

Define the lower value function of the game

$$V(t,x) := \sup_{\alpha \in \mathcal{A}_t} \inf_{\tau \in \mathcal{T}^t_{t,T}} \mathbb{E}\Bigg[\int_t^\tau e^{-\int_t^s c(u, X_u^{t,x,\alpha})du} f(s, X_s^{t,x,\alpha}, \alpha_s) ds + e^{-\int_t^\tau c(u, X_u^{t,x,\alpha})du} g(X_\tau^{t,x,\alpha}) \Bigg],$$

- $\mathcal{A}_t :=$ {admissible controls indep. of $\mathcal{F}_t$},
- $\mathcal{T}^t_{t,T} :=$ {stopping times in $[t, T]$ a.s. & indep. of $\mathcal{F}_t$}.

Note: the upper value function is defined similarly: $U(t,x) := \inf_\tau \sup_\alpha \mathbb{E}[\cdots]$. We say the game has a value if these two functions coincide.

Related Work

The game of control and stopping is closely related to some common problems in mathematical finance:

- Karatzas & Kou [1998]; Karatzas & Zamfirescu; [2005], B. & Young [2010]; B., Karatzas, and Yao (2010),
- More recently, in the context of 2BSDEs (Soner, Touzi, Zhang) and G-expectations (Peng).

Related Work (continued)

One-dimensional case: Karatzas and Sudderth [2001] study the case where X^α moves along a given interval on $\mathbb{R}$. Under appropriate conditions, they

- show that the game has a value;
- construct explicitly a saddle-point of optimal strategies (α^*, τ^*).

Difficult to extend their results to multi-dimensional cases (their techniques rely heavily on optimal stopping theorems for one-dimensional diffusions).

Related Work (continued)

Multi-dimensional case: Karatzas and Zamfirescu [2008] develop a martingale approach to deal with this. Again, it is shown that the game has a value and a saddle point of optimal strategies is constructed,

- the volatility coefficient of X^α has to be nondegenerate.
- the volatility coefficient of X^α cannot be controlled.

Our Goal

We intend to investigate a much more general multi-dimensional controller-and-stopper game in which both the drift and the volatility coefficients of X^α can be controlled, and the volatility coefficient can be degenerate.

Main Result: The game has a value (i.e. $U = V$) and the value function is the unique viscosity solution to an obstacle problem of an HJB equation.

One can then construct a numerical scheme to compute the value function, see e.g. B. and Fahim [2011] for a stochastic numerical method.

Methodology

- Show: V_* is a viscosity supersolution
  - prove continuity of an optimal stopping problem.
  - derive a weak DPP for V, from which the supersolution property follows.
- Show: U^* is a viscosity subsolution
  - prove continuity of an optimal control problem.
  - derive a weak DPP for U, from which the subsolution property follows.
- Prove a comparison result. Then $U^* \leq V_*$. Since $U^* \geq U \geq V \geq V_*$, we have $U = V$, i.e. the game has a value!!

Outline

Consider a fixed time horizon $T > 0$.

- $\Omega := C([0, T]; \mathbb{R}^d)$.
- $W = \{W_t\}_{t \in [0,T]}$: the canonical process, i.e. $W_t(\omega) = \omega_t$.
- $\mathbb{P}$: the Wiener measure defined on Ω.
- $\mathbb{F} = \{\mathcal{F}_t\}_{t \in [0,T]}$: the $\mathbb{P}$-augmentation of $\sigma(W_s, s \in [0, T])$.

For each $t \in [0, T]$, consider

- $\mathbb{F}^t$: the $\mathbb{P}$-augmentation of $\sigma(W_{t \vee s} - W_t, s \in [0, T])$.
- $\mathcal{T}^t$:={$\mathbb{F}^t$-stopping times valued in $[0, T]$ $\mathbb{P}$-a.s.}.
- $\mathcal{A}_t$:={$\mathbb{F}^t$-progressively measurable M-valued processes}, where M is a separable metric space.
- Given $\mathbb{F}$-stopping times τ_1, τ_2 with $\tau_1 \le \tau_2$ $\mathbb{P}$-a.s., define $\mathcal{T}^t_{\tau_1,\tau_2}$:=$\{\tau \in \mathcal{T}^t$ valued in $[\tau_1, \tau_2]$ $\mathbb{P}$-a.s.$\}$.

Concatenation

Given $\omega, \omega' \in \Omega$ and $\theta \in \mathcal{T}$, we define the concatenation of ω and ω' at time θ as

$$(\omega \otimes_\theta \omega')_s := \omega_r 1_{[0,\theta(\omega)]}(s) + (\omega'_s - \omega'_{\theta(\omega)} + \omega_{\theta(\omega)}) 1_{(\theta(\omega),T]}(s), \; s \in [0, T].$$

For each $\alpha \in \mathcal{A}$ and $\tau \in \mathcal{T}$, we define the shifted versions:

$$\begin{aligned} \alpha^{\theta,\omega}(\omega') &:= \alpha(\omega \otimes_\theta \omega') \\ \tau^{\theta,\omega}(\omega') &:= \tau(\omega \otimes_\theta \omega'). \end{aligned}$$

ASSUMPTIONS ON b AND σ

Given $\tau \in \mathcal{T}$, $\xi \in \mathcal{L}_d^p$ which is $\mathcal{F}_\tau$-measurable, and $\alpha \in \mathcal{A}$, let $X^{\tau,\xi,\alpha}$ denote a $\mathbb{R}^d$-valued process satisfying the SDE:

$$dX_t^{\tau,\xi,\alpha} = b(t, X_t^{\tau,\xi,\alpha}, \alpha_t)dt + \sigma(t, X_t^{\tau,\xi,\alpha}, \alpha_t)dW_t, \tag{1}$$

with the initial condition $X_\tau^{\tau,\xi,\alpha} = \xi$ a.s.

Assume: $b(t, x, u)$ and $\sigma(t, x, u)$ are deterministic Borel functions, and continuous in (x, u); moreover, $\exists\ K > 0$ s.t. for $t \in [0, T]$, $x, y \in \mathbb{R}^d$, and $u \in M$

$$\begin{aligned} &|b(t,x,u) - b(t,y,u)| + |\sigma(t,x,u) - \sigma(t,y,u)| \leq K|x-y|, \\ &|b(t,x,u)| + |\sigma(t,x,u)| \leq K(1+|x|), \end{aligned} \tag{2}$$

This implies for any $(t, x) \in [0, T] \times \mathbb{R}^d$ and $\alpha \in \mathcal{A}$, (1) admits a unique strong solution $X_\cdot^{t,x,\alpha}$.

Assumptions on f, g, and c

f and g are rewards, c is the discount rate $\Rightarrow$ assume $f, g, c \geq 0$.

In addition, Assume:

- $f : [0, T] \times \mathbb{R}^d \times M \mapsto \mathbb{R}$ is Borel measurable, and $f(t, x, u)$ continuous in (x, u), and continuous in x uniformly in $u \in M$.
- $g : \mathbb{R}^d \mapsto \mathbb{R}$ is continuous,
- $c : [0, T] \times \mathbb{R}^d \mapsto \mathbb{R}$ is continuous and bounded above by some real number $\bar{c} > 0$.
- f and g satisfy a polynomial growth condition

$$|f(t, x, u)| + |g(x)| \leq K(1 + |x|^{\bar{p}}) \text{ for some } \bar{p} \geq 1. \qquad (3)$$

Reduction to the Mayer form

- Set $F(x,y,z) := z + yg(x)$. Observe that

$$V(t,x) = \sup_{\alpha \in \mathcal{A}_t} \inf_{\tau \in \mathcal{T}^t_{t,T}} \mathbb{E}\left[Z^{t,x,1,0,\alpha}_\tau + Y^{t,x,1,\alpha}_\tau g(X^{t,x,\alpha}_\tau)\right] = \sup_{\alpha \in \mathcal{A}_t} \inf_{\tau \in \mathcal{T}^t_{t,T}} \mathbb{E}\left[F(\mathbf{X}^{t,x,1,0,\alpha}_\tau)\right], \quad (4)$$

  where $\mathbf{X}^{t,x,y,z,\alpha}_\tau := (X^{t,x,\alpha}_\tau, Y^{t,x,y,\alpha}_\tau, Z^{t,x,y,z,\alpha}_\tau)$.
- More generally, for any $(x,y,z) \in \mathcal{S} := \mathbb{R}^d \times \mathbb{R}^2_+$, define

$$\bar{V}(t,x,y,z) := \sup_{\alpha \in \mathcal{A}_t} \inf_{\tau \in \mathcal{T}^t_{t,T}} \mathbb{E}\left[F(\mathbf{X}^{t,x,y,z,\alpha}_\tau)\right].$$

  Let $J(t,\mathbf{x};\alpha,\tau) := \mathbb{E}[F(\mathbf{X}^{t,\mathbf{x},\alpha}_\tau)]$. We can write V as

$$V(t,x) = \sup_{\alpha \in \mathcal{A}_t} \inf_{\tau \in \mathcal{T}^t_{t,T}} J(t,(x,1,0);\alpha,\tau).$$

Conditional expectation

Lemma

Fix $(t, \mathbf{x}) \in [0, T] \times \mathcal{S}$ and $\alpha \in \mathcal{A}$. For any $\theta \in \mathcal{T}_{t,T}$ and $\tau \in \mathcal{T}_{\theta,T}$,

$$\mathbb{E}[F(\mathbf{X}_{\tau}^{t,\mathbf{x},\alpha}) \mid \mathcal{F}_{\theta}](\omega) = J\left(\theta(\omega), \mathbf{X}_{\theta}^{t,\mathbf{x},\alpha}(\omega); \alpha^{\theta,\omega}, \tau^{\theta,\omega}\right) \ \mathbb{P}\text{-a.s.}$$

$$\left(= \mathbb{E}\left[F\left(\mathbf{X}_{\tau^{\theta,\omega}}^{\theta(\omega), \mathbf{X}_{\theta}^{t,\mathbf{x},\alpha}(\omega), \alpha^{\theta,\omega}}\right)\right]\right)$$

Outline

For $(t, x, p, A) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{M}^d$, define

$$H^a(t, x, p, A) := -b(t, x, a) - \frac{1}{2} Tr[\sigma\sigma'(t, x, a)A] - f(t, x, a),$$

and set

$$H(t, x, p, A) := \inf_{a \in M} H^a(t, x, p, A).$$

Subsolution Property of U^*

Proposition 4.2

The function U^* is a viscosity subsolution on $[0, T) \times \mathbb{R}^d$ to the obstacle problem of an HJB equation

$$\max \left\{ c(t,x)w - \frac{\partial w}{\partial t} + H_*(t,x,D_x w, D_x^2 w), w - g(x) \right\} \leq 0.$$

Proof: Assume the contrary, i.e. $\exists\ h \in C^{1,2}([0, T) \times \mathbb{R}^d)$ and $(t_0, x_0) \in [0, T) \times \mathbb{R}^d$ s.t.

$$0 = (U^* - h)(t_0, x_0) > (U^* - h)(t,x),\ \forall\,(t,x) \in [0, T) \times \mathbb{R}^d \setminus (t_0, x_0),$$

and

$$\max \left\{ c(t_0,x_0)h - \frac{\partial h}{\partial t} + H_*(t_0,x_0,D_x h, D_x^2 h), h - g(x_0) \right\} (t_0,x_0) > 0. \tag{5}$$

Proof (continued)

Since by definition $U \leq g$, the USC of g implies $h(t_0, x_0) = U^*(t_0, x_0) \leq g(x_0)$. Then, we see from (5) that

$$c(t_0, x_0)h(t_0, x_0) - \frac{\partial h}{\partial t}(t_0, x_0) + H_*(\cdot, D_x h, D_x^2 h)(t_0, x_0) > 0.$$

Define the function $\tilde{h}(t, x) := h(t, x) + \varepsilon(|t - t_0|^2 + |x - x_0|)^4$. Note that $(\tilde{h}, \partial_t \tilde{h}, D_x \tilde{h}, D_x^2 \tilde{h})(t_0, x_0) = (h, \partial_t h, D_x h, D_x^2 h)(t_0, x_0)$. Then, by LSC of H_*, $\exists\ r > 0$, $\varepsilon > 0$ such that $t_0 + r < T$ and

$$c(t, x)\tilde{h}(t, x) - \frac{\partial \tilde{h}}{\partial t}(t, x) + H^a(\cdot, D_x \tilde{h}, D_x^2 \tilde{h})(t, x) > 0, \qquad (6)$$

for all $a \in M$ and $(t, x) \in \overline{B_r(t_0, x_0)}$.

PROOF (CONTINUED)

Define $\eta > 0$ by $\eta e^{\bar{c}T} := \min_{\partial B_r(t_0,x_0)}(\tilde{h} - h) > 0$.
Take $(\hat{t}, \hat{x}) \in B_r(t_0, x_0)$ s.t. $|(U - \tilde{h})(\hat{t}, \hat{x})| < \eta/2$. For $\alpha \in \mathcal{A}_{\hat{t}}$, set

$$\theta^\alpha := \inf \left\{ s \geq \hat{t} \;\middle|\; (s, X_s^{\hat{t},\hat{x},\alpha}) \notin B_r(t_0, x_0) \right\} \in \mathcal{T}_{\hat{t},T}^{\hat{t}}.$$

Applying the product rule to $Y_s^{\hat{t},\hat{x},1,\alpha}\tilde{h}(s, X_s^{\hat{t},\hat{x},\alpha})$, we get

$$\tilde{h}(\hat{t}, \hat{x}) = \mathbb{E}\Bigg[Y_{\theta^\alpha}^{\hat{t},\hat{x},1,\alpha}\tilde{h}(\theta^\alpha, X_{\theta^\alpha}^{\hat{t},\hat{x},\alpha})$$

$$+ \int_{\hat{t}}^{\theta^\alpha} Y_s^{\hat{t},\hat{x},1,\alpha} \left(c\tilde{h} - \frac{\partial \tilde{h}}{\partial t} + H^\alpha(\cdot, D_x\tilde{h}, D_x^2\tilde{h}) + f \right)(s, X_s^{\hat{t},\hat{x},\alpha})ds \Bigg]$$

$$> \mathbb{E}\left[Y_{\theta^\alpha}^{\hat{t},\hat{x},1,\alpha} h(\theta^\alpha, X_{\theta^\alpha}^{\hat{t},\hat{x},\alpha}) + \int_{\hat{t}}^{\theta^\alpha} Y_s^{\hat{t},\hat{x},1,\alpha} f(s, X_s^{\hat{t},\hat{x},\alpha}, \alpha_s)ds \right] + \eta$$

PROOF (CONTINUED)

By our choice of $(\hat{t}, \hat{x})$, $U(\hat{t}, \hat{x}) + \eta/2 > \tilde{h}(\hat{t}, \hat{x})$. Thus,

$$U(\hat{t}, \hat{x}) > \mathbb{E}\left[Y_{\theta^\alpha}^{\hat{t},\hat{x},1,\alpha} h(\theta^\alpha, X_{\theta^\alpha}^{\hat{t},\hat{x},\alpha}) + \int_{\hat{t}}^{\theta^\alpha} Y_s^{\hat{t},\hat{x},1,\alpha} f(s, X_s^{\hat{t},\hat{x},\alpha}, \alpha_s) ds \right] + \frac{\eta}{2},$$

for any $\alpha \in \mathcal{A}_{\hat{t}}$.

How to get a contradiction to this??

PROOF (CONTINUED)

By the definition of U,

$$U(\hat{t},\hat{x}) \leq \sup_{\alpha\in\mathcal{A}_{\hat{t}}} \mathbb{E}\left[F\left(\mathbf{X}_{\tau^*}^{\hat{t},\hat{x},1,0,\alpha}\right)\right]$$
$$\leq \mathbb{E}\left[F\left(\mathbf{X}_{\tau^*}^{\hat{t},\hat{x},1,0,\hat{\alpha}}\right)\right] + \frac{\eta}{4}, \text{ for some } \hat{\alpha}\in\mathcal{A}_{\hat{t}}.$$
$$\leq \mathbb{E}\left[Y_{\theta^{\hat{\alpha}}}^{\hat{t},\hat{x},1,\hat{\alpha}} h(\theta, X_{\theta^{\hat{\alpha}}}^{\hat{t},\hat{x},\hat{\alpha}}) + Z_{\theta^{\hat{\alpha}}}^{\hat{t},\hat{x},1,0,\hat{\alpha}}\right] + \frac{\eta}{4} + \frac{\eta}{4},$$

The blue part is the weak DPP we want to prove!

Weak DPP I

Proposition

Fix $(t, \mathbf{x}) \in [0, T] \times \mathcal{S}$ and $\varepsilon > 0$. For any $\alpha \in \mathcal{A}_t$, $\theta \in \mathcal{T}_{t,T}^t$, and $\varphi \in LSC([0, T] \times \mathbb{R}^d)$ with $\varphi \geq U$, there exists $\tau^*(\alpha, \theta) \in \mathcal{T}_{t,T}^t$ such that

$$\mathbb{E}[F(\mathbf{X}_{\tau^*}^{t,\mathbf{x},\alpha})] \leq \mathbb{E}[Y_\theta^{t,x,y,\alpha}\varphi(\theta, X_\theta^{t,x,\alpha}) + Z_\theta^{t,x,y,z,\alpha}] + 4\varepsilon.$$

Continuity of an Optimal Control Problem

Lemma 4.3

Fix $t \in [0, T]$. For any $\tau \in \mathcal{T}^t_{t,T}$, the function $L^\tau : [0, t] \times \mathcal{S}$ defined by

$$L^\tau(s, \mathbf{x}) := \sup_{\alpha \in \mathcal{A}_s} J(s, \mathbf{x}; \alpha, \tau)$$

is continuous.

Idea of Proof: Generalize the arguments in Krylov[1980].

Proof of Weak DPP I

Step 1: **Separate $[0, T] \times \mathcal{S}$ into small pieces.** By Lindelöf covering thm, take $\{(t_i, x_i)\}_{i\in\mathbb{N}}$ s.t. $\bigcup_{i\in\mathbb{N}} B(t_i, x_i; r^{(t_i,x_i)}) = (0, T] \times \mathcal{S}$. Take a disjoint subcovering $\{A_i\}_{i\in\mathbb{N}}$ of the space $(0, T] \times \mathcal{S}$ s.t. $(t_i, x_i) \in A_i$.

Step 2: **Construct desired stopping time $\tau^{(t_i,x_i)}$ in each A_i.** For each (t_i, x_i), by def. of $\bar{U}$, $\exists\ \tau^{(t_i,x_i)} \in \mathcal{T}^{t_i}_{t_i,T}$ s.t.

$$\sup_{\alpha\in\mathcal{A}_{t_i}} J(t_i, x_i; \alpha, \tau^{(t_i,x_i)}) \leq \bar{U}(t_i, x_i) + \varepsilon. \qquad (7)$$

Set $\bar{\varphi}(t, x, y, z) := y\varphi(t, x) + z$. For any $(t', x') \in A_i$,

$$L^{\tau^{(t_i,x_i)}}(t', x') \underset{\text{usc}}{\leq} L^{\tau^{(t_i,x_i)}}(t_i, x_i) + \varepsilon \leq \bar{U}(t_i, x_i) + 2\varepsilon$$
$$\leq \bar{\varphi}(t_i, x_i) + 2\varepsilon \underset{\text{lsc}}{\leq} \bar{\varphi}(t', x') + 3\varepsilon.$$

Proof of the Weak DPP I (continued)

Step 3: Construct desired stopping time τ on the whole space $[0, T] \times \mathcal{S}$. For any $n \in \mathbb{N}$, set $B^n := \cup_{0 \le i \le n} A_i$ and define

$$\tau^n := T 1_{(B^n)^c}(\theta, \mathbf{X}_\theta^{t,\mathbf{x},\alpha}) + \sum_{i=0}^{n} \tau^{(t_i,x_i)} 1_{A_i}(\theta, \mathbf{X}_\theta^{t,\mathbf{x},\alpha}) \in \mathcal{T}_{t,T}^t.$$

Step 4: Estimations.

$$\mathbb{E}[F(\mathbf{X}_{\tau^n}^{t,\mathbf{x},\alpha})] = \mathbb{E}\left[F(\mathbf{X}_{\tau^n}^{t,\mathbf{x},\alpha}) 1_{B^n}(\theta, \mathbf{X}_\theta^{t,\mathbf{x},\alpha})\right] + \mathbb{E}\left[F(\mathbf{X}_{\tau^n}^{t,\mathbf{x},\alpha}) 1_{(B^n)^c}(\theta, \mathbf{X}_\theta^{t,\mathbf{x},\alpha})\right]$$

Proof of Weak DPP I (continued)

By Lemma 2.4 and Properties 1 & 2,

$$\begin{aligned}
&\mathbb{E}[F(\mathbf{X}^{t,\mathbf{x},\alpha}_{\tau^n}) \mid F_\theta](\omega)\, 1_{B^n}(\theta(\omega), \mathbf{X}^{t,\mathbf{x},\alpha}_\theta(\omega)) \\
&= \sum_{i=0}^{n} J\left(\theta(\omega), \mathbf{X}^{t,\mathbf{x},\alpha}_\theta(\omega); \alpha^{\theta,\omega}, \tau^{(t_i,x_i)}\right) 1_{A_i}(\theta(\omega), \mathbf{X}^{t,\mathbf{x},\alpha}_\theta(\omega)) \\
&\leq \sum_{i=0}^{n} L^{\tau^{(t_i,x_i)}}\left(\theta(\omega), \mathbf{X}^{t,\mathbf{x},\alpha}_\theta(\omega)\right) 1_{A_i}(\theta(\omega), \mathbf{X}^{t,\mathbf{x},\alpha}_\theta(\omega)) \\
&\leq \left[\bar{\varphi}\left(\theta(\omega), \mathbf{X}^{t,\mathbf{x},\alpha}_\theta(\omega)\right) + 3\varepsilon\right] 1_{B^n}(\theta(\omega), \mathbf{X}^{t,\mathbf{x},\alpha}_\theta(\omega)).
\end{aligned}$$

Thus,

$$\begin{aligned}
&\mathbb{E}\left[F(\mathbf{X}^{t,\mathbf{x},\alpha}_{\tau^n}) 1_{B^n}(\theta, \mathbf{X}^{t,\mathbf{x},\alpha}_\theta)\right] = \mathbb{E}\left[\mathbb{E}[F(\mathbf{X}^{t,\mathbf{x},\alpha}_{\tau^{\varepsilon,n}}) \mid \mathcal{F}_\theta] 1_{B^n}(\theta, \mathbf{X}^{t,\mathbf{x},\alpha}_\theta)\right] \\
&\leq \mathbb{E}[\bar{\varphi}(\theta, \mathbf{X}^{t,\mathbf{x},\alpha}_\theta) 1_{B^n}(\theta, \mathbf{X}^{t,\mathbf{x},\alpha}_\theta)] + 3\varepsilon \leq \mathbb{E}[\bar{\varphi}(\theta, \mathbf{X}^{t,\mathbf{x},\alpha}_\theta)] + 3\varepsilon.
\end{aligned}$$

Proof of Weak DPP I(continued)

Step 5: Conclusion.

$$\mathbb{E}[F(\mathbf{X}_{\tau^n}^{t,\mathbf{x},\alpha})] \leq \mathbb{E}[\bar{\varphi}(\theta, \mathbf{X}_\theta^{t,\mathbf{x},\alpha})] + 3\varepsilon + \mathbb{E}[F(\mathbf{X}_T^{t,\mathbf{x},\alpha}) 1_{(A^n)^c}(\theta, \mathbf{X}_\theta^{t,\mathbf{x},\alpha})].$$

Now, take $n^* \in \mathbb{N}$ large enough s.t.

$$\begin{aligned}\mathbb{E}[F(\mathbf{X}_{\tau^{n^*}}^{t,\mathbf{x},\alpha})] &\leq \mathbb{E}[\bar{\varphi}(\theta, \mathbf{X}_\theta^{t,\mathbf{x},\alpha})] + 4\varepsilon \\ &= \mathbb{E}[Y_\theta^{t,x,y,\alpha} \varphi(\theta, X_\theta^{t,x,\alpha}) + Z_\theta^{t,x,y,z,\alpha}] + 4\varepsilon.\end{aligned}$$

Outline

Supersolution Property of V_*

Proposition

The function V_* is a viscosity supersolution on $[0, T) \times \mathbb{R}^d$ to the obstacle problem of an HJB equation

$$\max\left\{c(t,x)w - \frac{\partial w}{\partial t} + H(t,x,D_x w, D_x^2 w),\ w - g(x)\right\} \geq 0. \tag{8}$$

Weak DPP II

Proposition

Fix $(t, \mathbf{x}) \in [0, T] \times \mathcal{S}$ and $\varepsilon > 0$. Take arbitrary $\alpha \in \mathcal{A}_t$, $\theta \in \mathcal{T}_{t,T}^t$ and $\varphi \in USC([0, T] \times \mathbb{R}^d)$ with $\varphi \leq V$. We have the following:

(I) $\mathbb{E}[\bar{\varphi}^+(\theta, \mathbf{X}_\theta^{t,\mathbf{x},\alpha})] < \infty$;

(II) If, moreover, $\mathbb{E}[\bar{\varphi}^-(\theta, \mathbf{X}_\theta^{t,\mathbf{x},\alpha})] < \infty$, then there exists $\alpha^* \in \mathcal{A}_t$ with $\alpha_s^* = \alpha_s$ for $s \in [t, \theta]$ such that

$$\mathbb{E}[F(\mathbf{X}_\tau^{t,\mathbf{x},\alpha^*})] \geq \mathbb{E}[Y_{\tau\wedge\theta}^{t,x,y,\alpha}\varphi(\tau \wedge \theta, X_{\tau\wedge\theta}^{t,x,\alpha}) + Z_{\tau\wedge\theta}^{t,x,y,z,\alpha}] - 4\varepsilon, \tag{9}$$

for any $\tau \in \mathcal{T}_{t,T}^t$.

Continuity of an Optimal Stopping Problem

Lemma

Fix $t \in [0, T]$. Then for any $\alpha \in \mathcal{A}_t$, the function

$$G^\alpha(s, \mathbf{x}) := \inf_{\tau \in \mathcal{T}^s_{s,T}} J(s, \mathbf{x}; \alpha, \tau)$$

is continuous on $[0, t] \times \mathcal{S}$.

Idea: Express optimal stopping problem as a solution to RBSDE and then use continuity results for RBSDE.

Outline

To state an appropriate comparison result, we assume

A. for any $t, s \in [0, T]$, $x, y \in \mathbb{R}^d$, and $u \in M$,

$$|b(t,x,u)-b(s,y,u)|+|\sigma(t,x,u)-\sigma(s,y,u)| \leq K(|t-s|+|x-y|).$$

B. $f(t,x,u)$ is uniformly continuous in (t,x), uniformly in $u \in M$.

The conditions **A** and **B**, together with the linear growth condition on b and σ, imply that the function H is continuous, and thus $H = H_*$.

Comparison Result

Proposition

Assume **A** and **B**. Let u (resp. v) be an USC viscosity subsolution (resp. a LSC viscosity supersolution) with polynomial growth condition to (8), such that $u(T, x) \leq v(T, x)$ for all $x \in \mathbb{R}^d$. Then $u \leq v$ on $[0, T) \times \mathbb{R}^d$.

$U^*(T,\cdot) = V_*(T,\cdot)$

Lemma

For all $x \in \mathbb{R}^d$, $V_*(T,x) \geq g(x)$.

Main Result

Theorem

Assume **A** and **B**. Then $U^* = V_*$ on $[0, T] \times \mathbb{R}^d$. In particular, $U = V$ on $[0, T] \times \mathbb{R}^d$, i.e. the game has a value, which is the unique viscosity solution to (8) with terminal condition $U(T, x) = g(x)$ for $x \in \mathbb{R}^d$.

Thank you very much for your attention!
Happy Birthday Yannis!

- E. Bayraktar and V.R. Young, *Proving Regularity of the Minimal Probability of Ruin via a Game of Stopping and Control*, (2010) To appear in Finance and Stochastics. Available at http://arxiv.org/abs/0704.2244.
- B. Bouchard, and N. Touzi, *Weak Dynamic Programming Principle for Viscosity Solutions*, SIAM Journal on Control and Optimization, 49 No.3 (2011), pp. 948–962.
- N. El Karoui, C. Kapoudjian, E. Pardoux, S. Peng, and M.C. Quenez *Reflected Solutions of Backward SDE's, and Related Obstacle Problems for PDE's*, the Annals of Probability, 25 No.2 (1997), pp. 702–737.
- I. Karatzas and S.G. Kou, *Hedging American Contingent Claims with Constrained Portfolios*, Finance & Stochastics, 2 (1998), pp. 215–258.

- I. Karatzas and W.D. Sudderth, *The Controller-and-stopper Game for a Linear Diffusion*, The Annals of Probability, 29 No.3 (2001), pp. 1111–1127.
- I. Karatzas and H. Wang, *A Barrier Option of American Type*, Applied Mathematics and Optimization, 42 (2000), pp. 259–280.
- I. Karatzas and I.-M. Zamfirescu, *Game Approach to the Optimal Stopping Problem*, Stochastics, 8 (2005), pp. 401–435.
- I. Karatzas and I.-M. Zamfirescu, *Martingale Approach to Stochastic Differential Games of Control and Stopping*, The Annals of Probability, 36 No.4 (2008), pp. 1495–1527.
- N.V. Krylov, *Controlled Diffusion Processes*, Springer-Verlag, New York (1980).

H. Pham, *Continuous-time stochastic control and optimization with financial applications*, Springer-Verlag, Berlin (2009).