A Singular Journey In Optimisation problems
Involving Index Processes

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The Magic world of optimisation

• At the end of 80’s, Ioannis introduces me at new (for me) optimization problem :
  – Singular control problem
  – Finite fuel
  – Multi armed Bandit problem

• All had in common the same type of methodology :
  – their are convex problems with respect to some (eventually artificial parameter)
  – the derivatives of the value function with respect to this parameter is easy to compute
  – Come back to the primitive problem by simple integration give new and useful representation
Reply to Remarks at NICOLE (dated 30 Apr 93) on our CONTINUOUS-TIME DYNAMIC ALLOCATION Paper

Have changed everything accordingly.

\[ \{ w \mid \sigma(t, w) > \theta_2 \} = [0, M(t, \theta_2)], \] same for \( \theta_1 \).

It seems to me that, for this to work, we need to take \( M(t, \cdot) \) right-continuous, as in the picture (looked at, of course, from the other side of the paper!).

This \( M(t, \cdot) \) is indeed characterized by

\[ M(t, \theta) = \sup \{ w \mid \sigma(t, w) > \theta \}, \quad \inf \{ w \mid \sigma(t, w) < \theta \} \]

and I am making this correction.
On note \( H(t) = \inf \{ m; \sigma_1(m) + \sigma_2(m) \leq t \} \)
\[ \sigma_1(m) + \sigma_2(m) \leq t \]

\[ T_1(t) = \sigma_2(H(t)) \quad T_2(t) = \sigma_2(H(t)) \]
\[ T_1(t) + T_2(t) = t \]

Cela est faux, on doit faire attention aux paliers de \( H(t) \).

On a identiquement

\[ H^\ast(T_1(t)) = H_1 \circ \sigma_2(H(t)) \]
\[ = H(t) = H^2(T_2(t)) \]
\[ H(t) = \varphi^2(W^2_{T_3(t)}) \]

en un point de croissance du \( T_2(t) : (\varphi^2)^{-1}(H(t)) = W^1_{T_3(t)} \)

Par suite

\[ H(t) = \sup \left( \varphi^2(W^2_{T_1(t)}), \varphi^2(W^2_{T_2(t)}) \right) \]

On traduit de cette manière que le stratègique \( (T_1(t), T_2(t)) \) suit l'indice.

Par suite

\[ H(t) - \varphi^2(W^2_{T_1(t)}) > 0 \quad \text{et} \quad H(t) - \varphi^2(W^2_{T_2(t)}) > 0 \]

Par conséquent

\[ S^+(t) = \varphi^{-1}_1(H(t)) - W^2_{T_1(t)} \geq 0 \]
\[ S^-(t) = \varphi^{-1}_2(H(t)) - W^2_{T_2(t)} \geq 0 \]

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Introduction to Bandit Problem

What is a Multi-Armed bandit problem?

- There are $d$-independent projects (investigations, arms) among which effort to be allocated.
- By engaging one project, a stochastic reward is accrued, influencing the time-allocation strategy

⇒ Trade-off between exploration (trying out each arm to find the best one) and exploitation (playing the arm believed to give the best payoff)

- Discrete-time version is well-understood for a long time (Gittins (74-79), Whittle (1980))
- Continuous-time version received also a lot of attention (Karatzas (84), Mandelbaum (87), Menaldi-Robin (90), Tsitsiklis (86), NEK-Karatzas (93,95,97)}
Renewed interest in Economy

- RD problems (Weitzman &... (1979, 81))
- Strategic experimentation with learning on the quality of some project (Poisson uncertainty) (Keller, Rady, Cripps (2005))
- Learning in matching markets such as labor and consumer good markets: Jovanovic (1979) applies a bandit problem to a competitive labor markets.
- Strategic Trading and Learning about Liquidity (Hong & Rady (2000))

Principle of the solution (Gittins, Whittle)

⇒ To associate to each projet some rate of performance (Gittins index)
⇒ To maximize Gittins indices over all projects and at any time engaged a project with maximal current Gittins index
⇒ The essential idea is that the evolution of each arm does not depends on the running time of the other arms.
General Framework

Several projects \((i = 1, \ldots, d)\) are competing for the attention of a single investigator

- \(T_i(t)\) is the total time allocated to project \(i\) during the time \(t\), with

\[
\sum_{i=1}^{d} T_i(t) = (\leq) t
\]

- By engaging project \(i\) at time \(t\), the investigator accrues a certain reward \(h_i(T_i(t))\) per unit time,
  - discounted at the rate \(\alpha > 0\) and multiplied by the intensity \(i(t) = dT_i(t)/dt\) with which the project is engaged.
  - \(h_i(t)\) is a progressive process adapted to the filtration \(\mathcal{F}_i\), independent of the other.

\[\Rightarrow\] The objective is to allocate sequentially the time between these projects optimally

\[
\Phi := \sup_{(T_i)} \mathbb{E} \left[ \sum_{i=1}^{d} \int_{0}^{\infty} e^{-\alpha t} h_i(T_i(t)) dT_i(t) \right].
\]
**Decreasing Rewards**

Pathwise solution without probability

**Deterministic case and concave analysis** (modified pay-off with $\alpha = 0$, and finite horizon $T$)

- Let $(h_i)$ be the family of right-continuous decreasing positive pay-offs, with $h_i(0) > 0$ ($h_i(t) = 0$ for $t \geq \zeta$) and $H_i(t)$ the primitive of $h_i$ with $H_i(0) = 0$, assumed to be constant after some date $\zeta$.

- $H_i$ is a concave increasing function, with convex decreasing Fenchel conjuguate $G_i(m) = \sup_{t \leq T} \{H_i(t) - tm\}$ with derivative $G_i'(m) = \sigma_i(m)$.

- The criterium is now

$$\phi_T := \sup_{(T_i)} \sum_{i=1}^d \int_0^T h_i(T_i(t))dT_i(t) = \sup J_T(\mathcal{T})$$

over all strategies: $\mathcal{T} = (T_i)$ with $\sum_{i=1}^d T_i(t) = t$. 

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Criterium Transformation

\[ J_T(T) := \sum_{i=1}^{d} \int_{0}^{T} h_i(T_i(t))dT_i(t) = \sum_{i=1}^{d} H_i(T_i(T)) \]

Proof

\begin{itemize}
  \item \( h_i(T_i(t)) = \int_{0}^{\infty} 1_{\{m<h_i(T_i(t))\}} dm = \int_{0}^{\infty} 1_{\{T_i(t)<\sigma_i(m)\}} dm \)
  \item \( \sum_{i=1}^{d} 1_{\{T_i(t)<\sigma_i(m)\}} dT_i(t) = \sum_{i=1}^{d} d(T_i(t) \wedge \sigma_i'(m)) \)
\end{itemize}

\[ J_T(T) = \int_{0}^{\infty} dm \int_{0}^{T} d(T_i(t) \wedge \sigma_i(m)) = \int_{0}^{\infty} dm T_i(T) \wedge \sigma_i(m) \]

Remark: Assume that the reward functions \( h_i \) are not decreasing. The same properties hold true by using the concave envelope of \( \int_{0}^{t} h_i(s)ds \), defined through its conjugate \( G_i(m) = \sup_t \{ \int_{0}^{t} (h_i(s) - m)ds \} \).
Max-convolution problem

New formulations

- The bandit problem becomes

\[ \Phi_T := \sup \left\{ \sum_{i=1}^{d} H_i(T_i(T)) \middle| T_i \text{ increasing, and } \sum_{i=1}^{d} T_i(t) = t, \forall t \leq T \right\} \]

- The Max-Convolution problem with value function \( V(t) \) is:

\[ V(t) := \sup \left\{ \sum_{i=1}^{d} H_i(\theta_i(t)) \middle| \sum_{i=1}^{d} \theta_i(t) = t, \right\} \]

- Showing that the problems are equivalent is obtained by constructing a monotone optimal solution for the Max-convolution problem.
Optimal Time Allocation in Max-Convolution Pb

- **Main property** The conjugate $U(m)$ of the Max-Convolute $V(t)$ is the sum of the conjugate functions $U(m) = \sum_{i=1}^{d} G_i(m)$, with derivative $\tau(m) = \sum_{i=1}^{d} \sigma_i(m)$.

- $V(\tau(m)) = \tau(m)m - U(m) = \sum_{i=1}^{d} (m\sigma_i(m) - G_i(m)) = \sum_{i=1}^{d} H_i(\sigma_i(m))$

**Optimal time allocation**

- Let $V'(t) = M_t$ be the decreasing derivative of $V$, also the inverse of $\tau(m)$, and called the **Gittins Index** of the problem.

- The optimal time allocation is the increasing process $\theta_i^*(t) = \sigma_i(V'(t))$

- The optimal allocation is of **Index type**, i.e. maximizing the index $V'(t) = \sup_i h_i(\theta_i^*(t)) = \sup_i h_i(\sigma_i(V'(t)))$.

In the case of strictly decreasing continuous pay-offs, all projects may be engaged at the same time.
The Stochastic Decreasing case

Pathwise static problem

- Assume the decreasing pay-off as \( h_i(t, \omega) = \inf_{0 \leq u \leq t} k_i(u, \omega) \) where \( k_i(t) \) is \( \mathcal{F}_i(t) \)-adapted.
- The inverse process of \( h_i(t) \) is given by the stopping time
  \[ \sigma_i(m) = \sup \{ t \mid h_i(t) \leq m \} \]
- The strategic allocation \( T_i(t) \) is an \( \mathcal{F}_i(t) \)-adapted non decreasing cadlag process.
- All the previous results hold true, but the optimality is more difficult to establish, because the \( \mathcal{F}_i(t) \)-mesurability constraint.
- We have to use multi-parameter stochastic calculus, as Mandelbaum (92), Nek.Karatzas(93-97)

Today, we are concerned by the one-dimensional problem, which consists in replacing any adapted and positive process \( h_i \) by a decreasing process

\[ M_i(t) = \sup_{s < t} M_i(s) \] where \( M_i \) is called the Index process.
Max-Plus decomposition
Different Type of Max-Plus decomposition

- In our context, the problem is to find an adapted Index process $M(t)$

$$V_t = \mathbb{E}[\int_t^\infty e^{-\alpha s} h(s) ds | \mathcal{F}_t] = \mathbb{E}[\int_t^\infty e^{-\alpha s} \sup_{t<u<s} M(u) ds | \mathcal{F}_t] = \mathbb{E}[\int_t^\infty e^{-\alpha s} M_{t,s} ds | \mathcal{F}_t]$$

- More generally, in a Markov framework (Foellmer - Nek (05), (Foellmer, Riedel), the problem is to represent any fonction $u(x)$ as

$$u(x) = \mathbb{E}_x[\int_0^{\zeta} \sup_{0<u<t} f(X_t) dB_t], \quad B \text{ additive fonctional}$$

- In Bank-Nek (04), Bank-Riedel (01) the problem motivated by consumption problem is to solve for "any " adapted process $X$

$$X_t = \mathbb{E}[\int_t^\infty G(s, \sup_{t<u<s} L_s) ds | \mathcal{F}_t], \quad G(s, l) \text{ decreasing in } l$$
The class of supermartingale decomposition II

- Nek-Meziou (2002, 2005) for general process

- Foellmer Knispel (2006)

Max-plus algebra Calculus

It is an idempotent **semiring** :

\[ \Rightarrow \bigoplus = \text{max} \text{ is a commutative, associative and idempotent operation : } a \bigoplus a = a, \]

the **zero** = \( \epsilon \), is given by \( \epsilon = -\infty \),

\[ \Rightarrow \bigotimes \text{ is an associative } \textbf{product} \text{ distributive over addition, with a unit element } \]

\( e = 0 \). \( \epsilon \) is absorbing for \( \bigotimes : \epsilon \bigotimes a = a \bigotimes \epsilon = \epsilon, \forall a. \)

\[ \Rightarrow \mathbb{R}_{\text{max}} \text{ can be equipped with the natural order relation :} \]

\[ a \succeq b \iff a = a \oplus b. \]

\[ \Rightarrow \textbf{Linear Equation.} \text{ The set of solutions } x \text{ of } z \oplus x = m \text{ is empty if } m \leq z. \text{ If not, the set has a greatest element } x = m. \]
Max-Plus Supermartingale Decomposition

Let $Z$ be a càdlàg supermartingale in the class ($\mathcal{D}$) defined on $[0, \zeta]$.

- There exists $L = (L_t)_{t \leq \zeta}$ adapted, with upper-right continuous paths with running supremum $L_{t,s}^* = \sup_{t \leq u \leq s} L_u$, s.t.

\[
Z_t = \mathbb{E}\left[\left( \sup_{t \leq u \leq \zeta} L_u \right) \vee Z_\zeta | \mathcal{F}_t\right] = \mathbb{E}\left[\left( L_{t,\zeta}^* \oplus Z_\zeta\right) | \mathcal{F}_t\right] = \mathbb{E}\left[\int_t^\zeta L_u \oplus Z_\zeta | \mathcal{F}_t\right]
\]

- Let $M^\oplus$ be the martingale: $M_t^\oplus := \mathbb{E}\left[\left( L_{0,\zeta}^* \oplus Z_\zeta\right) | \mathcal{F}_t\right]$. Then,

\[
M_t^\oplus \geq \max (Z_t, L_{0,t}^*) = Z_t \oplus L_{0,t}^* \quad \leq t \leq \zeta
\]

and the equality holds at times when $L^*$ increases or at maturity $\zeta$:

\[
M_S^\oplus = \max (Z_S, L_{0,S}^*) = Z_S \oplus L_{0,S}^* \quad \text{for all stopping times } S \in \mathcal{A}_{L^*} \cup \{\zeta\}.
\]
Uniqueness in the Max-Plus decomposition

Let $Z \in \mathcal{D}$ be a cadlag supermartingale and assume that

- there exist two increasing adapted processes $\Lambda^1_t$ and $\Lambda^2_t$ ($\Lambda^i_{-0} = -\infty$) and two u.i. martingales $M^1$ and $M^2$ such that $M^i_\zeta = \Lambda^i_\zeta \lor Z_\zeta$ and $M^i_0 = Z_0$
- $\Lambda^i$ only increases at times when the martingale $M^i$ hits the supermartingale $Z$,
  \(\text{(flat-off condition)}\)
  \[
  \int_{[0, \zeta]} (M^i_t - Z^i_t) d\Lambda^i_t = 0
  \]
- \((M^i, \Lambda^i)\) are two (max-+) decompositions of $Z$ ($\oplus = \lor = \max$)

\[
M^1_t \geq Z_t \oplus \Lambda^1_t, \quad M^2_t \geq Z_t \oplus \Lambda^2_t.
\]

$\Rightarrow$ $M^1$ and $M^2$ are indistinguishable processes.

$\Rightarrow$ Given such a martingale $M^\oplus$, the set $\mathcal{K}$ of $\Lambda$ satisfying the above conditions has a maximal element $\Lambda^{\max}$ which is also in $\mathcal{K}$.

If $Z$ is bounded by below, $\Lambda^{\max}$ is also bounded by below with the same constant.
Sketch of the proof when $Z$ and $\Lambda$ are bounded by below

Recall the assumption $\int_0^\zeta (M^i_s - Z_s) d\Lambda^i_s = 0$ with $\Lambda^i_\zeta \geq Z_\zeta$

Then, for any regular convex function ($C^2$ with linear growth) $g$, $g(0) = 0$.

\[
g(M^1_\zeta - M^2_\zeta) \leq g'(M^1_\zeta - M^2_\zeta)(M^1_\zeta - M^2_\zeta) = g'(\Lambda^1_\zeta - \Lambda^2_\zeta)(M^1_\zeta - M^2_\zeta)
\]

\[
\mathbb{E}[g(M^1_\zeta - M^2_\zeta)] \leq 
\]

\[
\mathbb{E}[g'(\Lambda^1_0 - \Lambda^2_0)(M^1_\zeta - M^2_\zeta)] + \mathbb{E}[(M^1_\zeta - M^2_\zeta) \int_0^\zeta g''(\Lambda^1_t - \Lambda^2_t)(d\Lambda^1_t - \Lambda^2_t)]
\]

\[
= \mathbb{E}[- \int_0^\zeta (M^1_t - M^2_t)g''(\Lambda^1_t - \Lambda^2_t)(d\Lambda^1_t - \Lambda^2_t)]
\]

\[
= \mathbb{E}[- \int_0^\zeta (Z_t - M^2_t)g''(\Lambda^1_t - \Lambda^2_t)d\Lambda^1_t - \int_0^\zeta (M^1_t - Z_t)g''(\Lambda^1_t - \Lambda^2_t)d\Lambda^2_t] \leq 0
\]

by the flat condition and the convexity of $g$.

In particular, $\mathbb{E}[g(M^1_\zeta - M^2_\zeta)] = 0$ for $g(x) = x^+$
Introduction

DLT have studied American Call options with infinite horizon on discrete time supermartingale, sum of iid r.v. with negative expectation. They gave a large place to the running supremum of these variables.

- \( Z \) is a supermartingale on \([0, \zeta]\) and \( \mathbb{E}[|Z^*_{0,\zeta}|] < +\infty \), \( \mathbb{E}[|Z^*_t,\zeta|] < +\infty \)
- Assume \( Z \) to be a conditional expectation of some running supremum process \( L^*_{s,t} = \sup_{s \leq u \leq t} L_u \), such that \( \mathbb{E}[|L^*_{0,\zeta}|] < +\infty \) and \( Z_t = \mathbb{E}[L^*_{t,\zeta}|F_t] \)

American Call options

Let \( C_t(Z, m) \) be the American Call option with strike \( m \),

\[
C_t(Z, m) = \text{ess sup}_{t \leq S \leq \zeta} \mathbb{E}[(Z_S - m)^+|F_t].
\]

Then

\[
C_t(Z, m) = \mathbb{E}[(L^*_{t,\zeta} \vee Z_\zeta - m)^+|F_t]
\]

and the stopping time \( D_t(m) = \inf\{s \in [t, \zeta]; L_s \geq m\} \) is optimal.
Proof

\[ \mathbb{E}[(L_{t,\zeta}^* - m)^+ | \mathcal{F}_t] \] is a supermartingale dominating \( \mathbb{E}[L_{t,\zeta}^* | \mathcal{F}_t] - m = Z_t - m \), and so \( C_t(Z, m) \)

\[ \mathbb{E}[(L_{t,\zeta}^* - m)^+ | \mathcal{F}_t] \] is a supermartingale dominating \( \mathbb{E}[L_{t,\zeta}^* | \mathcal{F}_t] - m = Z_t - m \), and so \( C_t(Z, m) \)

\[ \Rightarrow \] Conversely, since on \( \{ \theta = D_t(m) < \infty \} \), \( L_{\theta,\zeta}^* \geq m \), at time \( \theta = D_t(m) \), we can omit the sign +, and replace \( (L_{\theta,\zeta}^* - m) \) by its conditional expectation \( Z_{D_t(m)} - m \), still nonnegative.

Main question :

To find numerical method to calculate a Max-Plus Index

- Directly by using AY-martingale (elementary)
- By characterization through optimization problems (Gittins, Karatzas, Foellmer)
Closed Formulae
based on Azéma-Yor martingales
Azéma-Yor Martingales (1979)

**Definition** Let $X$ be a càdlàg local semimartingale with $X_0 = a$ and $X^*_t = \sup_{0 \leq s \leq t} X_s$ its running supremum assumed to be nonnegative. Then for any finite variation function $u$, with locally integrable right-hand derivative $u'$, the process $M^u(X)$

$$M^u_t(X) = u(X^*_t) + u'(X^*_t)(X_t - X^*_t)$$

is a local martingale, called the **Azéma-Yor** martingale associated with $(u, X)$.

**Main properties**

$\Rightarrow$ $M^u_t(X) = M^u_0(X) + \int_0^t u'(X^*_s) \, dX_s,$ \hspace{1cm} (1)

$\Rightarrow$ If $u'$ is only defined on $[a, b)$, $M^u(X)$ may be defined up to the exit time $\zeta$ of $[a, b)$ by $X$.

$\Rightarrow$ Assume $u'$ to be non negative. Then the running supremum of $M^u(X)$ is given by $u(N^*_t)$
Bachelier equation

First introduced by Bachelier in 1906.

**Def** : Let \( \phi : [a^*, \infty) \) be a locally bounded away from 0 function and \( X \) a local martingale with continuous running supremum. The Bachelier equation is

\[
dY_t = \phi(Y^*_t) dX_t
\]

**Example** Let \( u \) be an increasing function, \( v \) the inverse function of \( u \), and \( \phi = u' \circ v = 1/v' \). Then \( M^u(X) \) the AY-martingale associated with \( u \) is a solution of the Bachelier equation.
Bachelier equation, (suite)

Th : Let \( \phi : [a^*, \infty) \rightarrow (0, \infty) \) be a Borel function locally bounded away from zero, and \((X_t : t \geq 0), X_0 = a\), a càdlàg semimartingale as before.

- Define \( v(y) = a + \int_{a^*}^{y} \frac{ds}{\phi(s)} \) and \( u(x) = v^{-1}(x) \). So \( u'(x) = (v^{-1})'(x) = \phi \circ v(x) \).

\[ \Rightarrow \] Then the Bachelier equation

\[ dY_t = \phi(Y_t^*) \, dX_t, \quad Y_0 = a^* \]

has a strong, pathwise unique, solution defined up to its explosion time \( \zeta_Y = T_{V(\infty)} \).

- The solution is given by \( Y_t = M_t^u(X) \), \( t < T_{V(\infty)} \).

For any process \( X \) as before, and any increasing function \( u \) function (with locally bounded derivative) with inverse function \( v \), we have

\[ X_t = M_t^u(M^v(X)) \]
Maximum distribution

Well-known result.

**Th :** Let \((N_t), N_0 = 1\) be a non-negative local martingale with a continuous running supremum and with \(N_t \to 0\) a.s. Then \(1/N_*\) has a uniform distribution on \([0, 1]\).

**Proof :** Let \(u(x) = (K - x)^+\) the “Put “function. Then, \(M^U(N)\) is bounded and u.i. martingale, such that

\[
\mathbb{E}\left( (K - N_*^+) + 1_{\{K > N_*\}^+} \right) = K \mathbb{P}(K \geq N_*^+) = K - 1
\]

Moreover if \(b \geq 1\) is a constant such that for \(\zeta = T_b, N_\zeta \in \{0, b\}\), then

\[
\mathbb{P}(N_*^* = b) = 1/b
\]

and conditionally to \(\{N_*^* < b\}^\prime\), \(1/N_*^*\) is uniformly distributed on 
\([1/b, 1]\).
Surmartingale decomposition and running supremum

- Let $N$ be a local martingale with continuous running supremum, and going to 0 at $\beta$

- Let $u$ be a increasing convex function, such that $\mathbb{E}(|u(N)|^\infty) < \infty$

⇒ The supermartingale $u(N_t)$ is the conditional expectation of the running supremum between $t$ and $\infty$ of $L_t = v(N_t)$ where $v(x) = u(x) - x u'(x)$ is an non decreasing function, that is

$$Z_t = u(N_t) = \mathbb{E}(\sup_{t,\infty} v(N_u)|\mathcal{F}_t)$$

- More generally, $g$ is a continuous increasing function on $\mathbb{R}^+$ whose increasing concave envelope $u$ is finite.

- Galtchouk, Mirochnitchenko Result (1994) : The process $Z_t = u(N_t)$ is the Snell envelope of $Y = g(N)$. 

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Concave envelop of $u \lor m$
Max-Plus decomposition of Supermartingales with Independent Increments

Continuous case Let $N$ be a geometric Brownian motion with return $= 0$ and volatility to be specified. Let $Z$ be a supermartingale defined on $[0, \infty]$ such that

- a geometric Brownian motion with negative drift,
  \[
  \frac{dZ_t}{Z_t} = -rdt + \sigma dW_t, \quad Z_0 = z > 0.
  \]
- Setting $\gamma = 1 + \frac{2r}{\sigma^2}$, $N_t = Z_t^\gamma$ is a local martingale, with volatility $\gamma \sigma$
- $Z_t = u(N_t)$ where $u$ is the increasing concave function $u(x) = x^{1/\gamma}$.

- $v(x) = u(x) - xu'(x) = \frac{\gamma-1}{\gamma} x^{1/\gamma} = \frac{\gamma-1}{\gamma} z$,
- Let $Z$ be a Brownian motion with negative drift $-(r + \frac{1}{2} \sigma^2) \geq 0$
  \[
  dZ_t = -(r + \frac{1}{2} \sigma^2) dt + \sigma dW_t, \quad Z_0 = z.
  \]
  Then $Z_t = \frac{1}{\gamma} \ln(N_t)$, $v(z) = z - \frac{1}{\gamma}$ and the Call American boundary is $y^*(m) = m + \frac{1}{\gamma}$.
- the exponentional of a Lévy process with jumps

Assume $Z$ to be a supermartingale with a continuous and integrable supremum.
Then the same result holds with a modified coefficient $\gamma_{Levy}$, such that $Z_t^{\gamma_{Levy}}$ defines a local martingale that goes to 0 at $\infty$.

- **Finite horizon** $T$ without Azéma-Yor martingale
  
  Same kind of solution: we have to find a function $b(.)$ such that at any time $t$

  $$Z_t = \mathbb{E}\left[ \sup_{t \leq u \leq T} b(T - u)Z_u | \mathcal{F}_t \right]$$

  Can we find a direct and efficient method to calculate the boundary $b(T - t)$?

**References**: Previous papers are relative to processes with independent increments (E. Mordecki (2001) - S. Asmussen, F. Avram and M. Pistorius (2004) - L. Alili and A. E. Kiprianou (2005)).
Universal Index and Pricing Rule

**Framework**: Let \( Z = u(N) \) be an increasing concave function of the cadlag local martingale \( N \) going to 0 at infinity, with continuous running supremum. Assume \( \mathbb{E}[|Z_{0,\infty}|] < +\infty \).

- Let \( \varphi \) be the increasing convex, inverse function of \( u \), such that \( \varphi(Z) = N \) is a local martingale and \( \psi(z) = v \circ \varphi(z) = z - \frac{\phi(z)}{\phi'(z)} \). Then

\[
Z_t = \mathbb{E}[\psi(Z_{t,\infty})|\mathcal{F}_t], \quad C_t^Z(m) = \mathbb{E}[(\psi(Z_{t,\infty}) - m)^+|\mathcal{F}_t]
\]

\[
y^*(m) = \psi^{-1}(m) = m + \frac{\varphi(y^*(m))}{\varphi'(y^*(m))}
\]

\[
C_t^Z(m) = \begin{cases} 
(Z_t - m) & \text{if } Z_t \geq y^*(m) \\
\frac{y^*(m) - m}{\varphi'(y^*(m))} \varphi(Z_t) & \text{if } Z_t \leq y^*(m)
\end{cases}
\]
Optimality of Azema-Yor martingale
Martingale optimization problem

The optimization problem

Let $Y_t = g(N_t)$ be a floor process and $Z^Y_t = u(N_t)$ the Snell envelope of $Y$ where $u$ is the concave envelope of $g$.

The following problem is motivated by portfolio insurance:

$\mathcal{M}(x) = \{(M_t)_{t \geq 0} \mid \text{u.i. martingale } |M_0 = x \text{ and } M_t \geq g(N_t) \quad \forall t \in [0, \zeta]\}$

- We aim at finding a martingale $(M^*_t)$ in $\mathcal{M}(x)$ such that for all martingales $(M_t)$ in $\mathcal{M}(x)$, and for any utility function, (concave, increasing) $V$ such that the following quantities have sense

  $\mathbb{E}(V(M^*_\zeta)) \geq \mathbb{E}(V(M_\zeta))$

- The initial value of any martingale dominating $Y$ must be at least equal to the one of the Snell envelope $Z^Y$, that is $u(N_0)$. 
The $u$-Azéma-Yor martingale is optimal

The martingale $M_t^{AY} = u(N_t^*) + u'(N_t^*)(N_t - N_t^*)$ martingale is optimal for the concave order of the terminal value. In particular, $dM_t^{Y,\oplus} = u'(N_t^*)dN_t$ is less variable than the martingale of the Doob Meyer Decomposition $dM_t^{DM} = u'(N_t)dN_t$.

**Sketch of proof**: Let $M$ be in $\mathcal{M}_t^Y(Z_0^Y)$. Since $M$ dominates $Z_t^Y$, the American Call option $C_t(M, m)$ also dominates $C_t(Z_t^Y, m)$. By convexity,

$$C_t(M, m) = \mathbb{E}[(M_\zeta - m)^+ | \mathcal{F}_S] \geq \mathbb{E}[(L_{S,\zeta}^Y \vee Y_\zeta - m)^+ | \mathcal{F}_S] \quad \forall S \in \mathcal{T}.$$ 

More generally, this inequality holds true for any convex function $g$, and

$$\mathbb{E}[g(M_\zeta)] \geq \mathbb{E}[g(L_{0,\zeta}^Y \vee Y_\zeta)] = \mathbb{E}[g(M_\zeta^{Y,\oplus})]$$

**Initial condition** $x \geq Z_0^Y$ Same result by using $L_t^{Y,*}S, \zeta \vee m$ in place of $L_{S,\zeta}^Y$. 

Skew Brownian Motion
Strategic Process


Framework : Two arms Brownian Bandit

- With two independent Brownian motions \((W^1, W^2)\) we associate two pay-off strictly increasing functions \(\eta^i(W^i_t)\) and their Gittins Index \(\nu^i(W^i)\)
  - \(\nu^i\) is also positive strictly increasing, with inverse function \(\mu^i\) (assumed to have the same domain.)

- Let \(\sigma^i(m) = \inf\{t; \nu^i(W^i_t) \leq m\}\), \(\gamma^i(\alpha) = \inf\{t; W^i_t \leq \alpha\}\)\((\alpha \leq 0)\), \(\sigma^i(m) = \gamma^i(\mu^i(m))\).

- The minimum rewards : \(\underline{W}^i(t) = \inf_{u \leq t} W^i(u)\) is the inverse function of \(\gamma^i(\alpha)\), and is flat on the excursions of the reflected Brownian motion \(R(W^i)(t) = W^i(t) - \underline{W}^i(t)\)

- \(\underline{M}^i_t = \inf_{u \leq t} \nu^i(W^i_u) = \nu^i(\underline{W}^i(t))\) is the inverse function of \(\sigma^i\)
A reflected Brownian motion

Optimal strategies

• Let $M$ the continuous inverse of $\sigma^1(m) + \sigma^2(m)$, $T^i(t) = \sigma^i(M(t))$ when $t$ is a decreasing time of $M$, and $M(t) = M^i(T^i(t))$, $i = 1, 2$ at any time.

• Let $S^i = R(W^i)(T_i) = W^i(T_i) - \overline{W^i}(T_i) = \mu^i(M_i(T_i)) - \mu^i(M) \geq 0$. Then $S^i(t) > 0$ if $T^i(t)$ belongs to an excursion of $R(W^i)$, and does not belongs to the support of $M$.

$\Rightarrow$ Lemma Let $\nu$ the inverse fonction of $\mu^1 + \mu^2$ and $\mu$ the inverse function of $\nu$.

Put $S(t) = S^1(t) + S^2(t) = W_t + L^W_t$. Then $W(t) = W^1(T_1(t)) + W^2(T_2(t))$ is a $G = \mathcal{F}^1 \lor \mathcal{F}^2$-Brownian motion, and $L^W = -\sum_{i=1}^2 \mu^i(M) = -\mu(M)$.

- By the previous remarks, $L^W$ only increases when $S(t) = 0$ and $S(t) = S^1(t) + S^2(t)$ is a reflected Brownian motion;

- by uniqueness of the Skohorod problem $\mu(M)(t) = \inf_{u \leq t} W^1(T_1(t)) + W^2(T_2(t))$.

By classical result, the distribution of $L^W_t$ is well-known.
Skew Brownian motion

Pathwise construction

- Let $X = S^1 - S^2 = B + V$ where $B = W^1(T_1) - W^2(T_2)$ is a Brownian motion, and $V = \mu^1(M) - \mu^2(M) = \phi(L^W)$ where

$$\phi(l) = (\mu^1 - \mu^2)(\nu)(-l) = (\mu^1 - \mu^2)((\mu^1 + \mu^2)^{-1})(-l).$$

- Because $S^1_t S^2_t = 0$, $S^1 = X^+$ and $S^2 = X^-$, and $|X| = S$ is a reflected Brownian motion, with local time $L^X = L^W$. Then, $X$ is solution of the following problem, involving the local time $L^X$, where the function $\phi \in C^1$ and $|\phi| \leq 1$

$$X_t = \phi(L^X)_t + B_t$$

- **Examples**:
  - $\nu^1 = \nu^2$, $\phi \equiv 0$, and $X$ is a Brownian motion
  - $\nu^1(x) = \nu^2(\alpha x)$, $\alpha \in (0, 1]$, then $\phi(l) = \beta l$ with $\beta = \frac{1-\alpha}{1+\alpha}$, and $X$ is the Skew Brownian motion (Harrison Kreps(1981), Walsh(78))
Multidimensional case

Assume a bandit problems with $d$ projects

- By the same way, we still have that $S^i(t) > 0$ only outside of the open support of $M$, and $S(t) = \sum_{i=0}^{d} S^i(t)$ is a reflected Brownian motion, with intrinsic local time $-\mu(M)$

- How describe the muti-dimensional process $S$ which are reflected independent Brownian motions with different scales of times
To finish...

• In 1993, my daughter Imen (6 years) asks me: 
  *but Mom, why do you argue with Ioannis always bandit problems with multiple guns, you are not police?*
  She was really surprised.

• Explanation: in french the word bandit is the same, but the word arm means weapon

Thank you Ioannis for these moments

so stimulating and friendly

Happy Birthday

Next Year in Paris