Generalized Malliavin Calculus and Stochatstic PDEs

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Abstract

The origins of Malliavin Calculus can be traced to Malliavin's work on hypoellipticity of PDEs. However, the Malliavin Divergence Operator (MDO) has some other precursors, in particular, Skorokhod integral, Wick product, creation operator in Quantum Physics, etc. Moreover, both Skorokhod integral and MDO are related to "stochastic quantization", a methodology developed in Quantum Physics to mitigate divergence of certain potentials (see e.g. B. Simon, "The $P(\varphi)_2$ Euclidean Quantum Field Theory", Princeton University Press, 1974). Roughly speaking, this methodology is based on the replacement of standard products $u \cdot v$ of functionals on Wiener space by the Wick product $u \diamond v$. The relations between Skorokhod (Itô) integral, Wick product, and MDO could be illustrated by the following formula

$$\int_0^T \delta_{\dot{W}}\left(f\left(t\right)\right) dt = \int_0^T f\left(t\right) dW\left(t\right) = \int_0^T f\left(t\right) \diamond \dot{W}\left(t\right) dt$$

Currently, the predominant *driving random source* in Malliavin calculus is an isonormal Gaussian process (white noise) $\dot{W}$ on a separable Hilbert space. In this lectures we will discuss two main topics:

(a) Extension of Malliavin calculus to the driving random source given by a nonlinear functional $v := v\left(\dot{W}\right)$ of white noise. More specifically, we will discuss the main operators of Malliavin calculus: Malliavin derivative $\mathbf{D}_v\left(f\right)$; divergence operator $\boldsymbol{\delta}_v\left(f\right)$, and Ornstein-Uhlenbeck operator $\mathcal{L}_v\left(f\right)$ with respect to a generalized random field v.

(b) Applications of the extended Malliavin calculus to Elliptic and Parabolic stochastic PDEs.

A simple but representative example of stochastic PDEs to be discussed in the lectures is given by the following equation:

$$-\left(a\left(x\right) u_x(x)\right)_x = f(x),\ x \in (0,1)\,,\ u\left(0\right) = u\left(1\right) = 0, \tag{0.1}$$

with $a(x) = \bar{a}(x) + \epsilon(x)$, where $\bar{a}(x)$ is the mean and $\epsilon(x) = \sum_{k\geq 1} \sigma_k(x)\xi_k$ is a Gaussian noise term, is a typical example of stochastic PDEs we will discuss. Recently, this equation was investigated in the context of uncertainty quantification for mathematical and computational models. Problem (0.1) is ill posed because the diffusion coefficient a may take negative values. However, a natural stochastic quantization reduces (0.1) to equation

$$\begin{gathered} -\left(\bar{a}\left(x\right) v_x(x)\right)_x + \left(\delta_{\epsilon(x)}\left(v_x\left(x\right)\right)\right)_x = f(x), \\ x \in (0,1)\,,\ v\left(0\right) = v\left(1\right) = 0, \end{gathered} \tag{0.2}$$

where '$\boldsymbol{\delta}_{\epsilon(x)}$' stands for Malliavin divergence operator (Skorokhod integral) with respect to Gaussian noise $\epsilon(x)$. In contrast to (0.1), equation (0.2) is well posed and uniquely solvable.

Familiarity of attendees with Malliavin calculus will not be assumed.

Course Description

Lecture 1. Stochastic Quantization and Navier-Stokes Equation.

This lecture will consists ow two parts: (a) Stochastic quantization of Navier-Stokes equation; (b) Introduction to Wiener Chaos. Part (a) will be a seminar type review of stochastic quantization of Navier Stokes equation. Part (b) will be the first part of a detailed review of Wiener Chaos.

Lecture 2. Introduction to Malliavin calculus.

We will first review the construction of the standard Malliavin calculus in non-adapted and adapted settings. Then we will discuss several simple examples of SPDEs driven by strictly spatial white noise. Given a typical SPDE, we will derive its *propagator,* a system of deterministic PDEs for the coefficients of the Wiener chaos expansion of the solution. By analyzing the propagator, we will discover that a solution of a typical SPDE driven by spatial white noise has infinite variance.

Lecture 3. Generalized Malliavin calculus

To deal with random fields with infinite second moments, we will develop generalized Malliavin calculus. This extension of Malliavin calculus is based on the procedure known in Quantum Physics as "second quantization". In mathematical terminology, the second quantization corresponds to a special rescaling/weighting of the Wiener Chaos expansions of the generalized random fields v and f in the main operators of Malliavin calculus: Malliavin derivative $\mathbf{D}_v\,(f)$; divergence operator $\boldsymbol{\delta}_v\,(f)$, and Ornstein-Uhlenbeck operator $\mathcal{L}_v\,(f)$. The rest of the lecture will be dedicated to exploration of continuity of the main operators of Malliavin Calculus.

Lecture 4. Bi-linear stochastic PDEs driven by stationary noise

This lecture deals with bilinear stochastic parabolic and elliptic PDEs driven by purely spatial white noise. We will discuss solvability of such equations in weighted Wiener chaos spaces and study the long-time behavior of the solutions of evolution equations.

We will also discuss analytical and numerical issues related to elliptic equations with random coefficients which are generally nonlinear functions of white noise. The existence and uniqueness of the solutions will be established under rather weak assumptions, the main of which requires only that the expectation of the highest order (differential) operator is a non-degenerate elliptic operator. It will be shown that the deterministic coefficients of the Wiener Chaos expansion of the solution solve a lower-triangular system of linear elliptic equations (the propagator). This structure of the propagator insures linear complexity of the related numerical algorithms.