Lect 5

Recall:

$$R(u) = \begin{bmatrix} \operatorname{sh}(u+\eta) & 0 & 0 & 0 \\ 0 & \operatorname{sh} u & \operatorname{sh}\eta & 0 \\ 0 & \operatorname{sh}\eta & \operatorname{sh} u & 0 \\ 0 & 0 & 0 & \operatorname{sh}(u+\eta) \end{bmatrix}$$

satisfies the Yang-Baxter equation

$$t(u) = \operatorname{tr}_a \left(D_a^H R_{1a}(u) D_a^H R_{2a}(u) \cdots \right.$$
$$\left. \cdots D_a^H R_{Na}(u) \right)$$

Eigenvalues (corresponding to Bethe vectors):

$$\Lambda(u|\{\alpha\}) = e^{NH} \operatorname{sh}(u+\eta)^N \prod_{j=1}^{m} \frac{\operatorname{sh}(u - i\alpha_j - \frac{\eta}{2})}{\operatorname{sh}(u - i\alpha_j + \frac{\eta}{2})}$$

$$+ e^{-NH} \prod_{k=1}^{N} \operatorname{sh}(u - v_k) \prod_{j=1}^{m} \frac{\operatorname{sh}(u - i\alpha_j + \frac{3\eta}{2})}{\operatorname{sh}(u - i\alpha_j + \frac{\eta}{2})},$$

Here $\{\alpha_j\}_{j=1}^{m}$ satisfy Bethe equations

$$\left(\frac{sh(i\alpha_a+\frac{\eta}{2})}{sh(i\alpha_a-\frac{\eta}{2})}\right)^N = e^{-2NH}\prod_{b\neq a}\frac{sh(i\alpha_a-i\alpha_b+\eta)}{sh(i\alpha_a-i\alpha_b-\eta)}$$

$$e^{ip(\alpha)} = \frac{sh(i\alpha+\frac{\eta}{2})}{sh(i\alpha-\frac{\eta}{2})}, \quad e^{i\Theta(\alpha)} = \frac{sh(i\alpha+\eta)}{sh(i\alpha-\eta)}$$

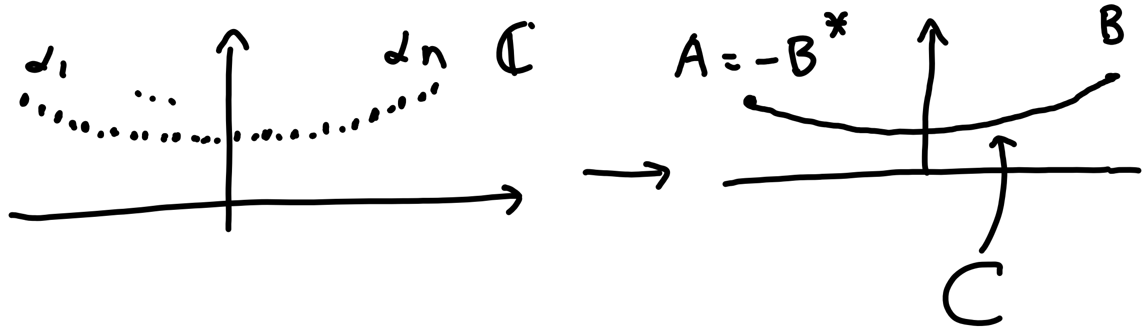
$$p(\alpha_j) = 2iH + \frac{2\pi I_j}{N} + \frac{1}{N}\sum_{k\neq j}^{m}\Theta(\alpha_j-\alpha_k)$$

<u>Conjecture 1.</u> The ground state max. eigenvalue correspond to

$$I_j = \frac{n+1-2j}{2}$$ (true when $\Delta=0$)

Conjecture 2. As $N \to \infty$, $n \to \infty$, $\rho = \frac{n}{N}$ is fixed, solutions $\{\alpha_a\}_{a=1}^{n}$ converge to $\{\alpha(t)\}_{t \in [-\rho, \rho]}$ (distributionally) (true when $\Delta = 0$)

$$\sum_{a=1}^{n} \alpha_a \delta\left(t - \frac{n-1-2j}{2N}\right) \to \alpha(t)$$



$$2\pi t = p(\alpha(t)) - 2Hi \, \fint_{-\rho/2}^{\rho/2} \Theta(\alpha(t) - \alpha(s)) \, ds,$$

$\alpha : [-\rho/2, \rho/2] \to \mathbb{C}$, $C = \mathrm{Im}(\alpha)$

$t : C \to [-\frac{\rho}{2}, \rho/2]$, $t = \alpha^{-1}$

Density $\rho(\alpha) = \frac{\partial t}{\partial \alpha}$

$$\begin{cases} 2\pi\rho(\alpha) = p'(\alpha) - \int_A^B \Theta'(\alpha-\beta)\rho(\beta)d\beta \\ \rho(\beta)d\beta\Big|_C = \text{real} \end{cases}$$

$\rho(\alpha) = \rho(\alpha; \rho, H)$, B, A = fncns in ρ, H

From here and from the formula for eigenvalues of $t(u)$ we obtain:

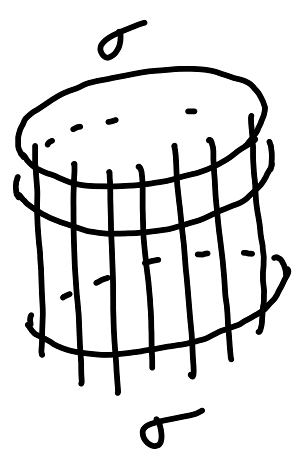
$$\lambda^{max}(u) = \exp\left(N \mathcal{H}_u(H, \rho)(1 + o(1))\right)$$

$$\mathcal{H}_u(H, \rho) = \max_{\pm}\left(\pm H + \ln \operatorname{sh}\left(u + \frac{\eta}{2} \pm \frac{\eta}{2}\right) + \right.$$

$$\left. + \int_{-\rho/2}^{\rho/2} \ln\left(\frac{\operatorname{sh}(u - i\alpha(t) + \frac{\eta}{2} \mp \eta)}{\operatorname{sh}(u - i\alpha(t) + \frac{\eta}{2})}\right) dt\right)$$

Corollary of the ground state conjecture:
The ground state vector (eigenvector with λ_{max}) does not depend on u.

The partition function for the torus

a) $Z_{NM}^{torus} = \sum_{\{\sigma\}}$  $= tr(t^M)$

as $M \to \infty$ $$Z_{NM}^{torus} = d\,\lambda_{max}^{M}\left(1 + O(e^{-\alpha M})\right)$$

- For the 6-vertex with generic H, V
  $d = 1$
- passing to $N \to \infty$ we obtain:

$$Z^{torus}_{M,N} = tr\left(t(u)^M (D^V)^{\otimes M}\right)$$

$$\simeq \int_0^1 e^{NM\mathcal{H}(H,\rho)+\rho VNM} d\rho$$

$$\simeq \exp\left(-NM f(H,V)\right)$$

$$f_u(H,V) = \max_\rho \left(\mathcal{H}_u(H,\rho) - \rho V\right)$$

We assumed $M \gg N$

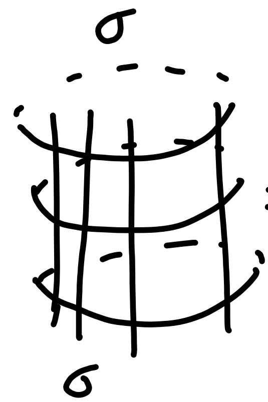
<u>Conjecture</u> (supported by the wealth of evidence)

$$f_u(H,V) = \lim_{\substack{N,M\to\infty \\ N:M \text{ finite}}} \frac{1}{NM} \ln\left(Z^{torus}_{NM}\right)$$

(away from resonant values, see for example Kenyon, Wilson, 2001)

b) similarly fixing the number of paths passing through the boundary define

$$Z^{\text{torus}}_{N,M}(n) = \sum_{\substack{\{\sigma\} \\ \text{with } n \text{ paths}}}$$

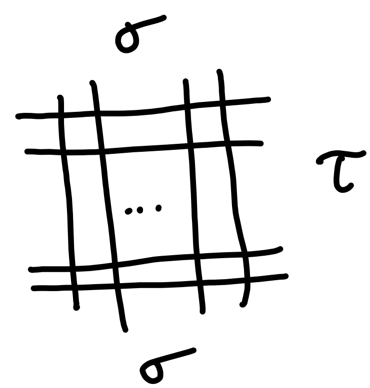


$$= \mathrm{tr}_{\mathcal{H}^{(n)}}\, t(u)^M$$

$$\mathcal{H}(\rho, H) = \lim_{\substack{N,M,n\to\infty \\ n/N=\rho,\ N/M \text{ finite}}} \frac{1}{NM} \ln Z^{\text{torus}}_{N,M}(n)$$

c) fixing the number of paths through each side of the fundamental domain of the torus define the partition function

$$Z^{\text{torus}}_{N,M}(n,m) = \sum_{\substack{\tau,\sigma \\ \#(\tau)=m,\ \#(\sigma)=n}}$$



$$\sigma(s,t) = \lim \frac{1}{NM} \ln Z^{\text{torus}}_{NM}(n,m)$$

$$s = \frac{n}{N},\ t = \frac{m}{N},\ N \to \infty$$

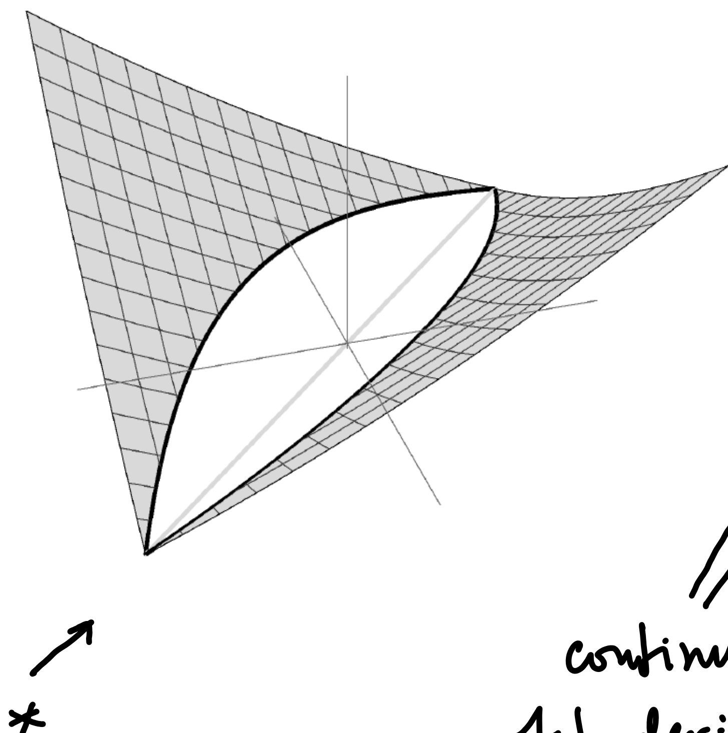
This function is the Legendre transform of $\mathcal{H}(H,t)$ in t:

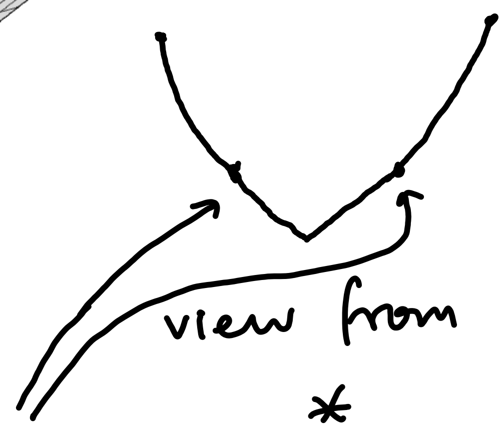
$$\sigma(s,t) = \max_{H} \left(sH - \mathcal{H}(H,t) \right)$$

$$= \max_{H,V} \left(sH + tV - f(H,V) \right)$$

The graph of the function $\sigma(s,t)$ has the following shape ($\Delta > 1$)

$0 \le s, t \le 1$

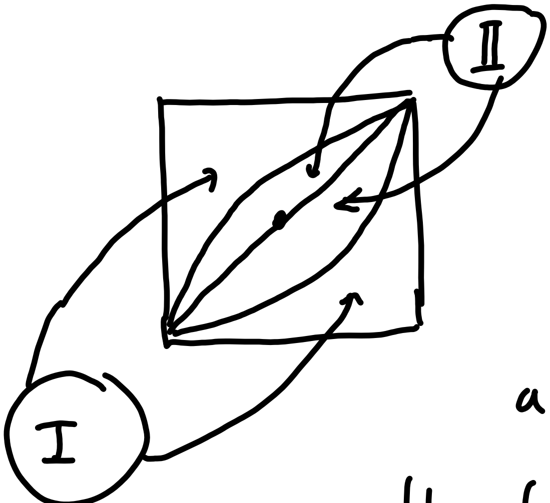




Critical curves:

$$s = \frac{t \pm \operatorname{th}(u+\eta)}{1 \pm t \operatorname{th}(u+\eta)},$$

$$\mathrm{Hess}(\sigma) = \det(\partial_i \partial_j \sigma) > 0, \quad \text{in } \mathrm{I}$$



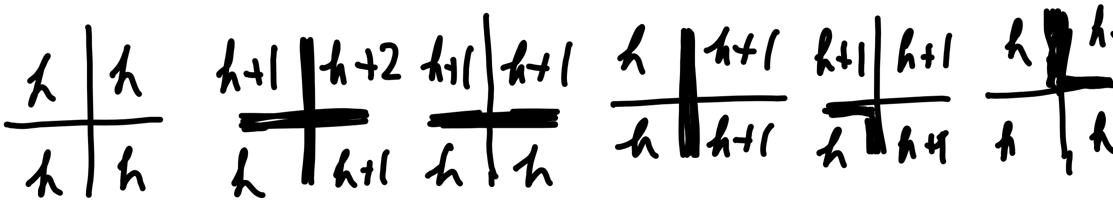
$$\mathrm{Hess}(\sigma) = 0 \quad \text{in } \mathrm{II}$$

$\mathrm{Hess}(\sigma)$ is a continuous function,

$$\mathrm{Hess}(\sigma)\Big|_{\mathrm{I}\cap\mathrm{II}} = 0$$

(Bukman, Shore, 1995)

Height function for the 6-vertex



$\Delta h_+ = \Delta h_- = n$

and is the same along each horizontal slice.

Given h_+, h_- define $\mathcal{H}_{h_+ h_-}$ to be the space of all possible height functions with these boundary conditions.

Thermodynamic limit:

$$N = \frac{L}{\varepsilon}, \quad M = \frac{T}{\varepsilon}, \quad \varepsilon \to 0$$

Normalized height functions:

$\varepsilon h(n,m) = \varphi(\varepsilon n, \varepsilon m)$ as $\varepsilon \to 0$ $\varphi(x,y)$:

$$\begin{aligned} |\varphi(x,y) - \varphi(x',y)| &< |x-x'| \\ |\varphi(x,y) - \varphi(x,y')| &< |y-y'| \end{aligned} \qquad (*)$$

Stabilization of boundary conditions

$$\varepsilon h(n,0) \to \varphi(x,0) = \varphi_-(x)$$

$$\varepsilon h(n,M) \to \varphi(x,T) = \varphi_+(x)$$

Continuous counterpart of the space of height functions: $H_{\varphi_+\varphi_-}(L,T)$ which satisfy $(*)$ and have boundary values φ_+, φ_-.

<u>Conjecture</u>:

(P. Zinn-Justin; Palamarchuk-R., Sridhar-R.)

The asymptotic of $Z_{NM}(\sigma,\sigma')$ in the thermodynamic limit:

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^2} \ln Z_{NM}(\sigma,\sigma') = S_u[\varphi_0]$$

assuming stabilization $h_+ \to \varphi_+$, $h_- \to \varphi_-$ where φ_0 is the minimizer of

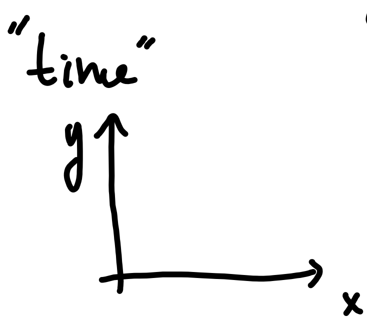
$$S_u[\varphi] = \int_0^T \int_0^L \sigma_u(\partial_x \varphi, \partial_y \varphi)\, dx\, dy$$

on $\mathcal{H}_{\varphi_+, \varphi_-}$

Hamiltonian version

of the variational problem.

Consider $S_u[\varphi]$ as the action functional of a Lagrangian field theory in a 2-dimensional space-time



"space"

Its first order version involves two fields: $\pi(x,y)$ and $\varphi(x,y)$

The first order action functional

$$S_u[\pi, \varphi] = \int_0^T \int_0^L \left(\pi \partial_y \varphi - H_u(\pi, \partial_x \varphi)\right) dx\, dy$$

- Euler-Lagrange equations

$$(*) \quad \begin{cases} \partial_y \varphi = \partial_1 H_u(\pi, \partial_x \varphi) \\ \partial_y \pi = \partial_x \partial_2 H_u(\pi, \partial_x \varphi) \end{cases}$$

Describe the Hamiltonian flow on T^*(boundary height functions) with the natural symplectic structure

$$\{\pi(x), \varphi(y)\} = \delta(x-y), \quad \{\pi(x), \pi(y)\} = 0$$

$$\{\varphi(x), \varphi(y)\} = 0$$

Here $\pi(x), \varphi(x)$ are distributional "coordinate function" ...

The flow (*) is generated by the Hamiltonian

$$H_u = \int_0^L \mathcal{H}(\pi, \partial_x \varphi)\,dx$$

- Equations (*) imply the Euler-Lagrange equations for $S_u[\varphi]$:

$$\partial_x \partial_1 \sigma(\partial_x \varphi, \partial_y \varphi) + \partial_y \partial_2 \sigma(\partial_x \varphi, \partial_y \varphi) = 0$$

- If π_0 is the critical value of $S_u[\pi, \varphi]$ with fixed φ,

$$\tilde{S}_u[\pi, \varphi] = \tilde{S}_u[\varphi]$$

This is easy to check using the Legendre transform.

It turns out that (*) have ∞ many conservation laws. This follows from the following

<u>Thm</u> $\{H_u, H_v\} = 0$

if $\partial_1^2 H_u \partial_2^2 H_v - \partial_2^2 H_u \partial_1^2 H_v = 0$ (**)

<u>Proof</u>. $\{H_u, H_v\} =$

$$= \int_0^L \left(A \partial_x p + B \partial_x^2 \varphi\right) dx,$$

$$A = \partial_1^2 H_u \partial_2 H_v - \partial_2 H_u \partial_1^2 H_v$$

$$B = \partial_1 \partial_2 H_u \partial_2 H_v - \partial_1 \partial_2 H_v \partial_2 H_u$$

The integrant is a total derivative iff

$$\partial_2 A - \partial_1 B = 0$$

This is equivalent to (**) ∎

Now let us show that $\partial_1^2 \mathcal{H}_u / \partial_2^2 \mathcal{H}_u$

is u-independent

<u>Lemma</u>

(Kim, Noh, 1995)

$$\frac{\partial^2 \mathcal{H}_u(H,\rho)}{\partial H^2} = -\frac{4D_-^2}{\pi} \operatorname{Im}\left(\frac{\xi_u}{\rho(B)}\right)$$

$$\frac{\partial^2 \mathcal{H}_u(H,\rho)}{\partial \rho^2} = \frac{\pi}{D_-^2} \operatorname{Im}\left(\frac{\xi_u}{\rho(B)}\right)$$

The complex-valued function ξ_u is defined as

$$\xi_u = 2\pi i \left(f_u'(B) + \int_C f_u'(\alpha) R(\alpha, B) d\alpha \right)$$

$$f_u(\alpha) = \ln\left(\frac{\operatorname{sh}(u - i\alpha + \frac{\eta}{2} \mp \eta)}{\operatorname{sh}(u - i\alpha + \frac{\eta}{2})} \right)$$

Here R is the resolvent of an integral operator

$$\left(1 + \frac{1}{2\pi} K\right)(1 + R) = 1$$

$$(K g)(\alpha) = \int_C \theta'(\alpha - \beta) g(\beta) d\beta$$

The function $\theta(\alpha)$ is defined earlier and

$$D_- = \frac{1}{2\pi}\left(1 + F(B, B) - F(B, A)\right)$$

where

$$F(\alpha, \gamma) + \frac{1}{2\pi} \int_C \theta'(\alpha - \beta) F(\beta, \gamma) = \frac{1}{2\pi} \theta(\alpha - \gamma)$$

Corollary $$\frac{\partial_1^2 \mathcal{H}_u}{\partial_2^2 \mathcal{H}_u} = -\frac{4}{\pi^2} \Delta_-^4$$

and therefore is u- independent

Corollary For the 6 vertex model

$$\{H_u, H_v\} = 0 \quad \text{(Sridhar, R., 2015)}$$

This is "expected" if one thinks about the thermodynamic limit as a semiclassical limit.

(Poisson commutativity of Hamiltonians)

$$\Updownarrow$$

(Commutativity of transfer-matrices)