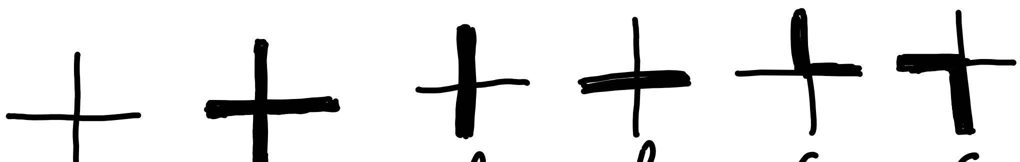
Lecture 3

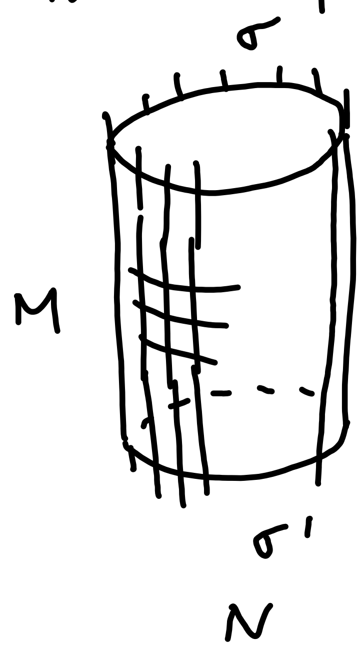
The 6-vertex model on a cylinder

Local states:



a_1 $\quad a_2 \quad b_1 \quad b_2 \quad c_1 \quad c_2$

Partition function on a cylinder:



$$Z_{MN}(\sigma,\sigma') = \sum_{\substack{\text{states}\\ \text{on inner}\\ \text{edges}}} \prod_v w_v(\text{state})$$

$\underset{MN}{Z}(\sigma, \sigma') =$ matrix elements of

$$Z_{MN} : \underset{\substack{\uparrow \\ \text{bottom}}}{(\mathbb{C}^2)^{\otimes N}} \to \underset{\substack{\uparrow \\ \text{top}}}{(\mathbb{C}^2)^{\otimes N}}$$

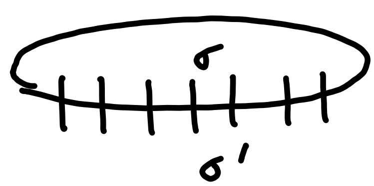
States on vertical edges

$\mathbb{C}^2 =$ basis $e_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $e_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$e_+ \otimes e_+ \otimes e_- \otimes e_- \leftrightsquigarrow$ 

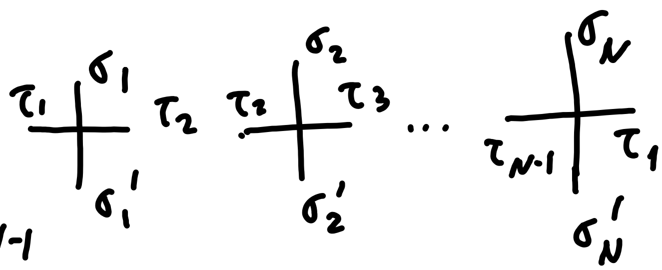
<u>Transfer-matrix</u>

Row-to-raw, periodic boundary cond.

$t(\sigma, \sigma') = Z_{1,N}(\sigma, \sigma')$ 

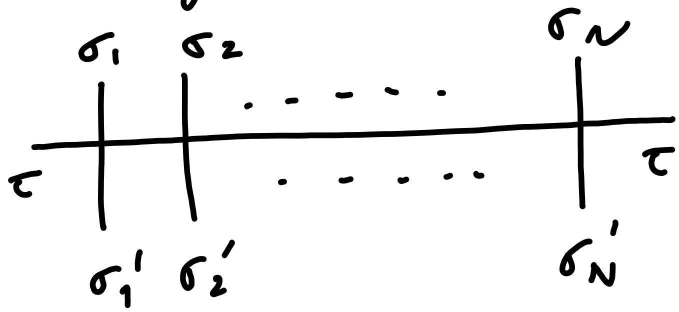
<u>Clear</u>:
$$Z_{M,N} = (Z_{1,N})^M = t^M$$

$$t(\sigma,\sigma') = \sum_{\tau_1 \cdots \tau_{N-1}}$$



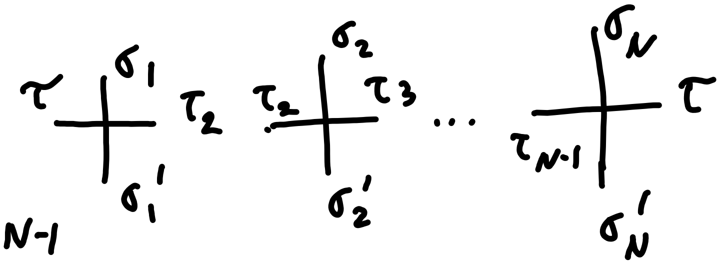
(row-to-row open boundary conditions quantum monodromy matrix)

$$T_{\tau\tau'}(\sigma,\sigma') =$$



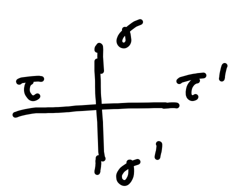
$$=$$

$$= \sum_{\tau_2 \cdots \tau_{N-1}}$$



"Matrix organization" of Boltzmann weights.

$$W^{\tau\sigma}_{\tau'\sigma'} =$$



$$W = \left(\begin{array}{cc|cc} a_1 & 0 & 0 & 0 \\ 0 & b_1 & c_2 & 0 \\ \hline 0 & c_1 & b_2 & 0 \\ 0 & 0 & 0 & a_2 \end{array}\right) \text{ in } \left\{\begin{array}{c} e_+ \otimes e_+ \\ e_+ \otimes e_- \\ e_- \otimes e_+ \\ e_- \otimes e_- \end{array}\right\} \text{ basis}$$

$$W : \mathbb{C}^2 \otimes \mathbb{C}^2 \circlearrowleft$$

Important notations

$$W_{ij} : \mathbb{C}^{2\otimes n} \longrightarrow \mathbb{C}^{2\otimes n}$$

acts as W in $\underset{i}{\mathbb{C}^2} \otimes \underset{j}{\mathbb{C}^2}$ and as 1 on the rest of the factors.

In these notations:

$$T_a = W_{1a} W_{2a} \cdots W_{Na} : \underset{1\,2\,3 \cdots N}{(\mathbb{C}^2)^{\otimes N}} \otimes \underset{a}{\mathbb{C}^2} \circlearrowleft$$

$$t = \mathrm{tr}_a(W_{1a} \cdots W_{Na}) : (\mathbb{C}^2)^{\otimes N} \circlearrowleft$$

$$Z = t^M$$

The goal: describe $N, M \to \infty$ asymptotic using the spectrum of t.

Baxter's parametrization

$$a_1 = a e^{H+V}, \quad a_2 = a e^{-H-V},$$

$$b_1 = b e^{-H+V}, \quad b_2 = b e^{H-V}$$

$$c_1 = c e^{\alpha}, \quad c_2 = c e^{-\alpha}$$

H, V - "magnetic fields"

$$R(a:b:c) = \left[\begin{array}{cc|cc} a & & & \\ & b & c & \\ \hline & c & b & \\ & & & a \end{array}\right] : \mathbb{C}^2 \otimes \mathbb{C}^2 \circlearrowleft$$

<u>Thm</u> (Baxter)

$$R_{12}(a:b:c)\,R_{13}(a':b':c')\,R_{23}(a'':b'':c'') =$$

$$= R_{23}(a'':b'':c'')\,R_{13}(a':b':c')\,R_{12}(a:b:c)$$

iff
$$ac'a'' = bc'b'' + ca'c''$$
$$ab'c'' = ba'c'' + cc'b''$$
$$cb'a'' = ca'b'' + bc'c''$$

2^3 equations

Baxter's parametrization:

$$a = \operatorname{sh}(u+\eta), \quad b = \operatorname{sh} u, \quad c = \operatorname{sh}\eta$$

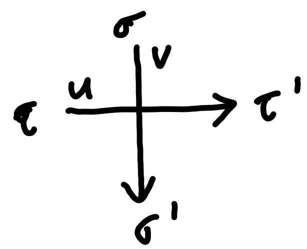
$$\Delta = \frac{a^2+b^2-c^2}{2ab} = \frac{a'^2+b'^2-c'^2}{2a'b'} = \frac{a''^2+b''^2-c''^2}{2a''b''}$$

$$\Delta = 2\operatorname{ch}\eta$$

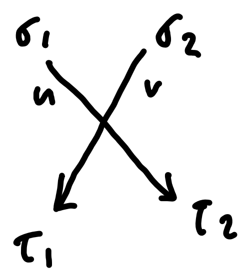
Complex algebraic, real forms later

$$R_{12}(u)\,R_{13}(u+v)\,R_{23}(v) = R_{23}(v)\,R_{13}(u+v)\,R_{12}(u)$$

Yang-Baxter equation



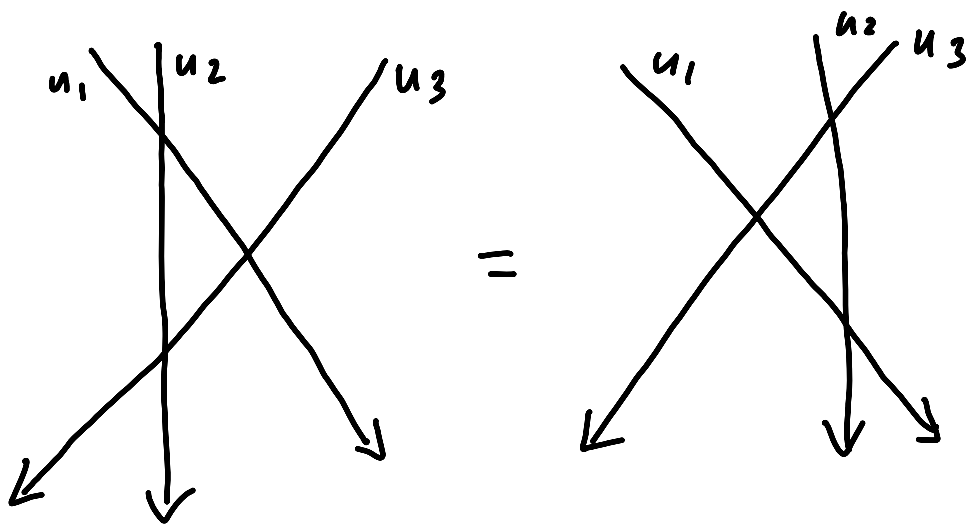
$$\longleftrightarrow \quad R^{\sigma\tau}_{\sigma'\tau'}(u-v)$$



$$\longleftrightarrow \quad (PR)^{\sigma_1\sigma_2}_{\tau_1\tau_2}(u-v) =$$

$$= R^{\sigma_2\sigma_1}_{\tau_1\tau_2}(u-v),$$

Graphical version of the Yang-Baxter equation:



$u = u_1 - u_2, \quad v = u_2 - u_3, \quad u - v = u_1 - u_3$

<u>Reintroduce magnetic fields</u>

Define $$D^a = \begin{pmatrix} e^{a/2} & 0 \\ 0 & e^{-a/2} \end{pmatrix},$$

"Conservation of paths" relation

$$(D^a \otimes D^a) R(u) = R(u) (D^a \otimes D^a)$$

Boltzmann weights of the 6-vertex model

$$W = \left(D^{V/2} \otimes D^{\frac{H+\alpha}{2}}\right) R(u) \left(D^{\frac{V}{2}} \otimes D^{\frac{H-\alpha}{2}}\right)$$

Transfer-matrix:

$$t = \left(D^{V} \otimes \dots \otimes D^{V}\right) \mathrm{tr}_a \left(D_a^{H} R_{1a}(u) D_a^{H} R_{2a}(u)\right.$$

$$\left.\dots D_a^{H} R_{2a}(u)\right) = \left(D^{V}\right)^{\otimes N} t(u),$$

$$t(u) = \mathrm{tr}_a\left(T_a(u)\right), \ (\mathbb{C}^2)^{\otimes N} \circlearrowleft$$

Quantum monodromy matrix:

$$T_a(u) = D_a^{H} R_{1a}(u) D^{H} R_{2a}(u) \dots D^{H} R_{Na}(u)$$

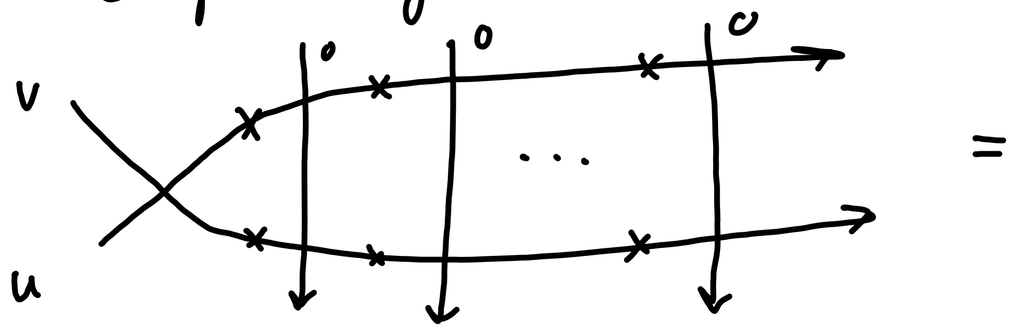
Properties

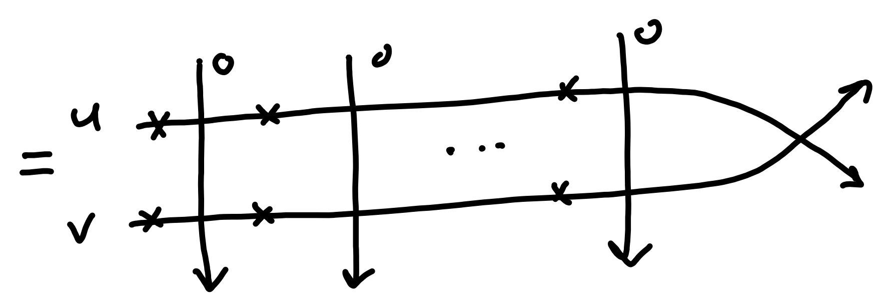
1) $\left[(\mathcal{D}^V)^{\otimes N}, t(u) \right] = 0$

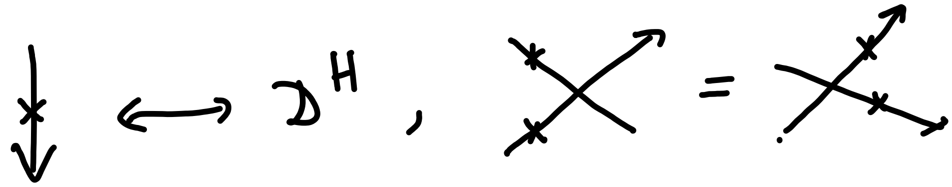
("conservation of paths")

2) $R(u-v)\, T_a(u) T_b(v) = T_b(v) T_a(u) R_{ab}(u-v)$

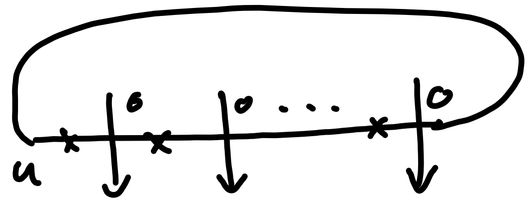
Graphically:







Corollary

$t(u) = \mathrm{tr}_a(T_a(u)) =$ 

form commutative family:

$$t(u)t(v) = \mathrm{tr}_a \mathrm{tr}_b \left(T_a(u) T_b(v) \right)$$

$$= \mathrm{tr}_a \mathrm{tr}_b \left(R_{ab}(u-v)^{-1} T_b(v) T_a(u) R_{ab}(u-v) \right)$$

$$= t(v)t(u),$$

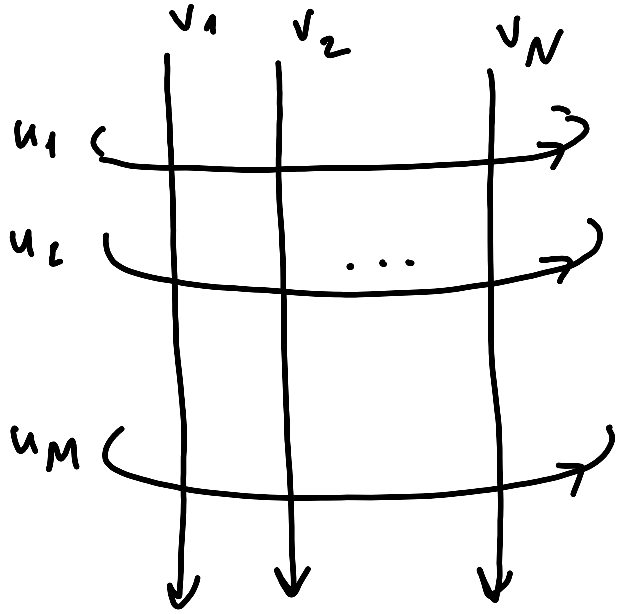
Conclusion:

$$Z_{M,N} = \left((\mathcal{D}^V)^{\otimes N} \right)^M t(u)^M$$

$$[t(u), t(v)] = 0$$

Consequence of YBE and of the commutativity of transfer-matrices:

Inhomogeneous 6-vertex model



$$Z(\{u\}\{v\}) = t(u_1)\cdots t(u_M),$$

$$t(u) = \mathrm{tr}_a\left(D_a^H R_{1a}(u-v_1)\cdots D_a^H R_{Na}(u-v_N)\right)$$

YBE for $R(u) \Rightarrow$:

- $[t(u), t(v)] = 0$

- $Z(\{\ldots u_i \ldots u_j \ldots\}\{v\}) = Z(\{\ldots u_j \ldots u_i\}\{v\})$

- $$\check{R}_{ij}(v_i - v_j)\, Z(\{u\}\{\dots v_i \dots v_j \dots\}) =$$

$$= Z(\{u\}\{\dots v_i \dots v_j \dots\})\, \check{R}_{ij}(v_i - v_j)$$

$$\check{R}(u) = P\, R(u)$$

The spectrum of $t(u)$

$$T(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}$$

$$t(u) = A(u) + D(u)$$

Define

$$\Omega = e_+ \otimes \cdots \otimes e_+$$

$$\Omega(\lambda_1, \dots, \lambda_m) = B(\lambda_1) \cdots B(\lambda_m)\, \Omega$$

<u>Thm</u> (Faddev, Sklyanin, Takhtajan)

$\Omega(\lambda_1, \dots, \lambda_m)$ is an eigenvector of $t(u) = Tr(T(u))$ with the eigenvalue

$$\Lambda(u|\{\lambda\}) = e^{NH} \prod_{k=1}^{N} sh(u - v_k + \eta) \prod_{j=1}^{m} \frac{sh(u - \lambda_j - \eta)}{sh(u - \lambda_j)}$$

$$+ e^{-NH} \prod_{k=1}^{N} sh(u - v_k) \prod_{j=1}^{m} \frac{sh(u - \lambda_j + \eta)}{sh(u - \lambda_j)}$$

if $\{\lambda_j\}_{j=1}^{m}$ satisfy Bethe equations

$$\prod_{k=1}^{N} \frac{sh(\lambda_a - v_k + \eta)}{sh(\lambda_a - v_k)} = e^{-2NH} \prod_{b \neq a} \frac{sh(\lambda_a - \lambda_b + \eta)}{sh(\lambda_a - \lambda_b - \eta)}$$

<u>Remark 1</u> This is a complete system of eigenvectors if $\eta, v_1, \dots, v_N$ are in generic position.

<u>Remark 2</u>. In the tensor product basis eigenvectors for $t(u)$ where first computed by Lieb, Wu and Baxter (inhomogeneus case)

Algebraic $\longleftrightarrow$ Coordinate Bethe ansatz

<u>Real forms</u>

$$\Delta < -1, \quad \Delta = -\operatorname{ch}\eta, \quad \eta > 0$$

$$a : b : c = \operatorname{sh}(\eta - u) : \operatorname{sh} u : \operatorname{sh}\eta, \quad 0 \leq u \leq \eta$$

$$-1 < \Delta \leq 0, \quad \Delta = -\cos\gamma, \quad 0 < \gamma \leq \frac{\pi}{2}$$

$$a : b : c = \sin(u - \gamma) : \sin u : \sin\gamma, \quad \gamma \leq u \leq \frac{\pi}{2}$$

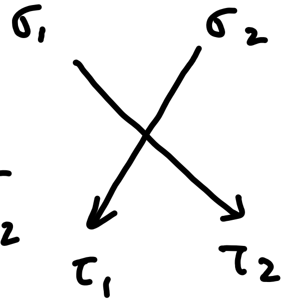
$$0 \leq \Delta < 1, \quad \Delta = \cos\gamma, \quad 0 < \gamma \leq \frac{\pi}{2}$$

$$a : b : c = \sin(\gamma - u) : \sin u : \sin\gamma, \quad 0 \leq u \leq \gamma$$

$$\Delta > 1, \quad \Delta = \operatorname{ch}\eta, \quad \eta > 0$$

$$a : b : c = \operatorname{sh}(u + \eta) : \operatorname{sh} u : \operatorname{sh}\eta$$

Stochastic point

$$\sum_{\tau_1 \tau_2}$$ 

$$= 1$$

$$a_1 = 1\,,\ a_2 = 1\,,\ b_1\,,\ b_2\,,\ 1-b_1\,,\ 1-b_2$$

$$0 \leqslant b_i \leqslant 1$$

Possible only when $\Delta \geqslant 1$

Baxter's parametrization:

$$b_1 = \frac{\operatorname{sh}(u)\, e^{\pm\eta}}{\operatorname{sh}(u+\eta)}\,, \quad b_2 = \frac{\operatorname{sh} u\, e^{\mp\eta}}{\operatorname{sh}(u+\eta)}$$

$+:\qquad b_1 > b_2$

$-:\qquad b_1 < b_2$

$$H = -V = \pm\eta/2\,, \qquad (+)$$

Transfer-matrix:

$$t(u) = \mathrm{tr}_a\Big(D_a^{1/2} R_{1a}(u)\, D_a^{1/2} R_{2a}(u) \cdots$$
$$\cdots D_a^{1/2} R_{Na}(u)\Big)$$

<u>Thm</u> (Borodin, Corwin, Gorin ($N=\infty$); Sridhar, R. ($N<\infty$))

$\exists\ \mathcal{H}^{(m)} \subset (\mathbb{C}^2)^{\otimes N}$ weight subspace

then

$$M^{(m)} = \frac{1}{1+b_1^{m} b_2^{N-m}}\, t^{(m)}(u) \quad \text{is Markov matrix}$$

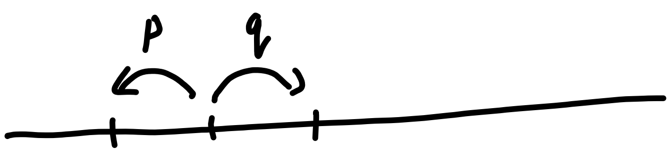
and

$$[t(u), t(v)] = 0$$

The relation to ASEP (on S^1)

$$H_{ASEP} = \sum_{i=1}^{N} H_{i,i+1}(p,q)$$

$$H(p,q) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -q & p & 0 \\ 0 & q & -p & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} : (\mathbb{C}^2)^{\otimes 2} \circlearrowleft$$



$$\frac{dP}{dt} = H_{ASEP} P$$

<u>Proposition</u>

$$H_{ASEP} = t'(0)t(0)^{-1} - \mathrm{cth}(\eta)\cdot I$$

$$p = \frac{e^{\eta}}{\mathrm{sh}\,\eta}, \quad q = \frac{e^{-\eta}}{\mathrm{sh}\,\eta}, \quad p:q \text{ is fixed by } \eta$$

TASEP: $\eta \to +\infty$

$p \to 1 \,, \quad q \to 0$

The ground state as $N \to \infty$

$\Delta > 1, \quad \Delta = \operatorname{ch}\eta, \quad \eta > 0$

$a : b : c = \operatorname{sh}(u+\eta) : \operatorname{sh}(u) : \operatorname{sh}(\eta)$

Bethe equations:

$$\left(\frac{\operatorname{sh}(i\alpha_a + \eta/2)}{\operatorname{sh}(i\alpha_a - \eta/2)}\right)^N = e^{-2NH} \prod_{b \neq a} \frac{\operatorname{sh}(i\alpha_a - i\alpha_b + \eta)}{\operatorname{sh}(i\alpha_a - i\alpha_b - \eta)}$$

$$e^{ip(\alpha)} = \frac{\operatorname{sh}(i\alpha + \frac{\eta}{2})}{\operatorname{sh}(i\alpha - \frac{\eta}{2})}, \quad e^{i\Theta(\alpha)} = \frac{\operatorname{sh}(i\alpha + \eta)}{\operatorname{sh}(i\alpha - \eta)}$$

$$p(\alpha_j) = 2iH + \frac{2\pi I_j}{N} + \frac{1}{N}\sum_{k \neq j}^{m} \Theta(\alpha_j - \alpha_k)$$

Conjecture 1. The ground state max. eigenvalue correspond to

$$I_j = \frac{n+1-2j}{2}$$

Conjecture 2. As $N \to \infty$, $n \to \infty$, $\rho = \frac{n}{N}$ is fixed, solutions $\{\alpha_a\}_{a=1}^{n}$ converge to $\{\alpha(t)\}_{t\in[-\rho,\rho]}$ (distributionally)

$$\sum_{a=1}^{n} \alpha_a \delta\left(t - \frac{n-1-2j}{2N}\right) \to \alpha(t)$$

and

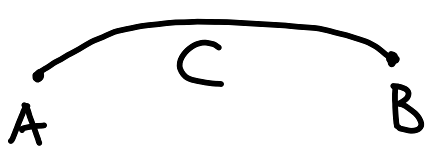
$$2\pi t = p(\alpha(t)) - 2Hi - \int_{-\rho/2}^{\rho/2} \Theta\left(\alpha(t) - \alpha(s)\right) ds,$$

$$\alpha : [-\rho/2, \rho/2] \to \mathbb{C}, \quad C = \mathrm{Im}(\alpha)$$

$$t : C \to [-\tfrac{\rho}{2}, \rho/2], \quad t = \alpha^{-1}$$

Density $\rho(\alpha) = \frac{\partial t}{\partial \alpha}$

$$\begin{cases} 2\pi\rho(\alpha) = p'(\alpha) - \int_A^B \Theta'(\alpha-\beta)\rho(\beta)\,d\beta \\ \\ \rho(\beta)\,d\beta\Big|_C = \text{real} \end{cases}$$



<u>Corollary</u>

$$\lambda^{max}(u) = \exp\Big(N\mathcal{H}_u(\rho, H)\big(1 + o(1)\big)\Big)$$

$$\mathcal{H}_u(H,\rho) = \max_{\pm}\Big(\pm H + \ln \operatorname{sh}\big(u + \tfrac{\eta}{2} \pm \tfrac{\eta}{2}\big) + \int_{-\rho/2}^{\rho/2} \ln\Big(\frac{\operatorname{sh}(u - i\alpha(t) + \frac{\eta}{2} \mp \eta)}{\operatorname{sh}(u - i\alpha(t) + \frac{\eta}{2})}\Big)\,dt\Big)$$