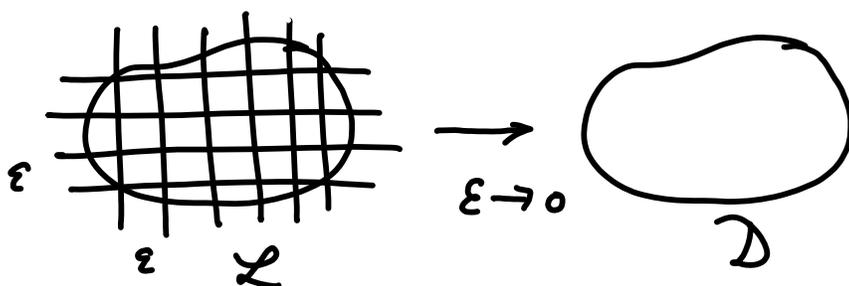


Lect 3

Lect 1-2: Height functions for dimer models on bipartite plane graph develop limit shapes



$w(\varepsilon)$ are periodic on \mathcal{L} with the fundamental domain \mathcal{L}_0 , then

$$\varepsilon h \longrightarrow \varphi_0$$

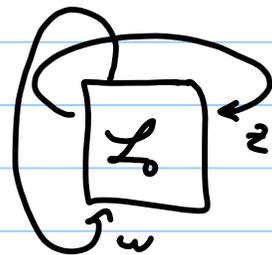
h_0 is the minimizer of the large deviation rate functional

$$S[\varphi] = \iint_{\mathcal{D}} \sigma(\vec{\nabla} \varphi) dx dy$$

$$\sigma(s, t) = \max_{H, V} (Hs + Vt - f(H, V))$$

$$f(H, V) = \frac{1}{(2\pi i)^2} \int_{|z|=e^H} \int_{|w|=e^V} P(z, w) \frac{dz}{z} \frac{dw}{w}$$

$$P(z, w) = \det(A_K^{\varphi_0})$$



Euler Lagrange equations

(on regions where φ_0 is smooth)

$$\partial_x h = \frac{1}{\pi} \arg w, \quad \partial_y h = -\frac{1}{\pi} \arg z$$

$$\left\{ \begin{array}{l} \frac{\partial_x z}{z} + \frac{\partial_y w}{w} = 0 \end{array} \right.$$

KO, 2007

$$\left\{ \begin{array}{l} P(z, w) = 0 \end{array} \right.$$

algebraic geometric solutions

Complex Burgers type equation.

Fluctuations around the limit shape

$$\mathbb{E} \left[(\varepsilon h(f_1) - h_0(x_1)) \dots (\varepsilon h(f_1) - h_0(x_n)) \right] \rightarrow$$

$C_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} ?$

A) $x_i \neq x_j$

Convergence to Gaussian free field

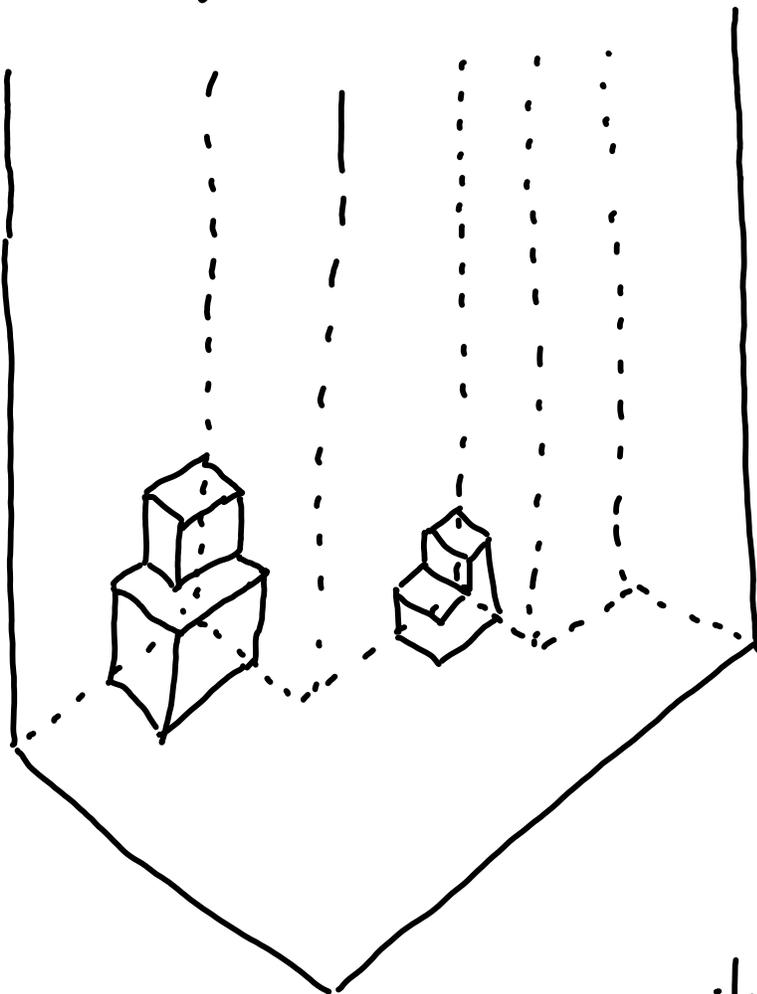
Kenyon 2007, Borodin Ferrari 2008,

... Petrov 2015, Bufetov Knizel 2016

B) When $x_1 = \dots = x_n$ converges to
a determinantal processes at a
smaller scale.

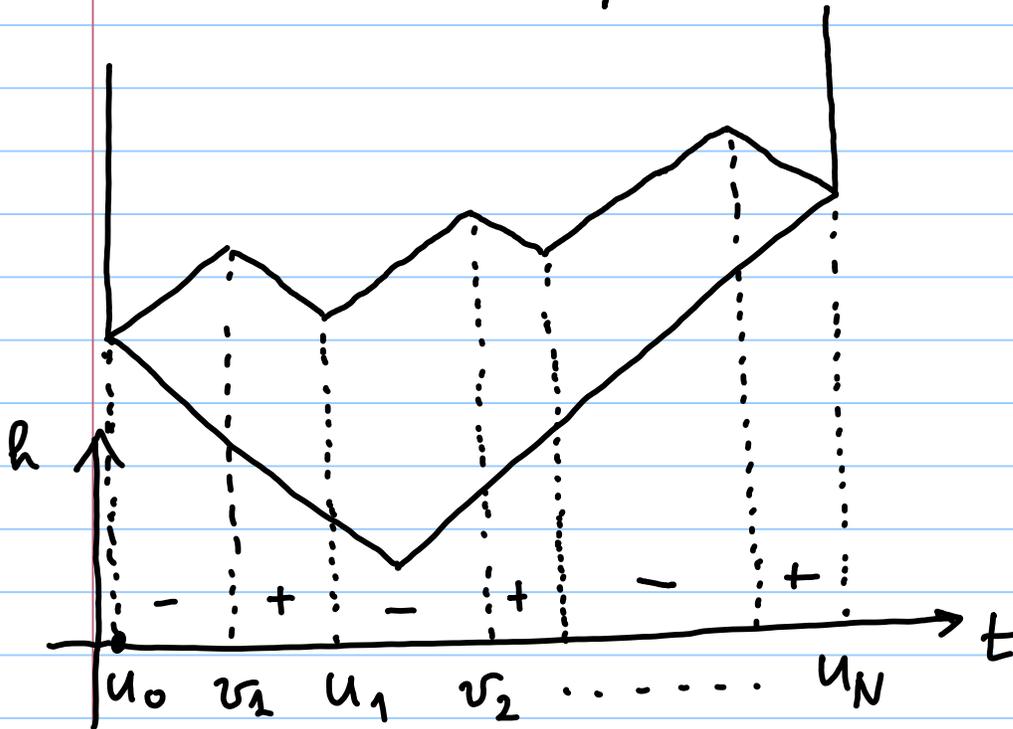
Special case
(Schur process)

Dimers on a semiinfinite region
in a hexagonal lattice

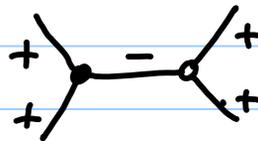


Dimers \leftrightarrow rhombi tilings \leftrightarrow piles of cubes
3D partitions

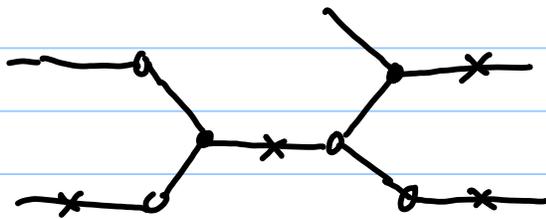
$$\text{Prob}(\pi) \propto q^{|\pi|}$$



Kasteleyn orientation



$$K = A^{-1}$$



$$K((t_1, h_1)(t_2, h_2)) - K((t_1, h_1)(t_2, h_2)) +$$

$$+ q^{t_1 - \frac{1}{2}} K((t_1, h_1)(t_2, h_2)) = \delta_{t_1, t_2} \delta_{h_1, h_2}$$

$$t_1 \in D_+$$

$$K((t_1, h_1)(t_2, h_2)) - K((t_1, h_1)(t_2, h_2)) +$$

$$+ q^{-t_1 + \frac{1}{2}} K((t_1, h_1)(t_2, h_2)) = \delta_{t_1, t_2} \delta_{h_1, h_2}$$

$$t_1 \in D_-$$

$$K((t_1, h_1)(t_2, h_2)) = \left(\frac{1}{2\pi i}\right)^2 \int_{C_z} \int_{C_w}$$

$$\frac{\psi_-(z, t_1) \psi_+(w, t_2)}{\psi_+(z, t_1) \psi_-(z, t_2)} z^{-h_1 - \frac{t_1}{2} - u_0 - \frac{1}{2}}$$

$$w^{h_2 + \frac{t_2}{2} + u_0 - \frac{1}{2}} dz dw$$

$$\Psi_+(z, t) = \prod_{m=t}^{u_i} (1 - z q^m) \prod_{m=v_{i+1}}^{u_{i+1}} (1 - z q^m) \dots \left. \vphantom{\prod_{m=t}^{u_i}} \right\} t \in \mathcal{D}_+$$

$$\Psi_-(z, t) = \prod_{m=u_{i-1}}^{v_i} (1 - z q^m) \prod_{m=u_{i-2}}^{v_{i-1}} (1 - z q^m) \dots$$

$$\Psi_+(z, t) = \prod_{m=v_{i+1}}^{u_{i+1}} (1 - z q^m) \prod_{m=v_{i+2}}^{u_{i+2}} (1 - z q^m) \dots \left. \vphantom{\prod_{m=v_{i+1}}^{u_{i+1}}} \right\} t \in \mathcal{D}_-$$

$$\Psi_-(z, t) = \prod_{m=u_i}^t (1 - z q^m) \prod_{m=u_{i-1}}^{v_{i-1}} (1 - z q^m) \dots$$

Correlation functions

$$E(\sigma_{\vec{x}_1} \dots \sigma_{\vec{x}_n}) = \det(K_{\vec{x}_1, \vec{x}_2})$$

↑

probability that dimers are sitting
on edges $\vec{x}_1, \dots, \vec{x}_n$

$$\mathbb{E}(\sigma_{\vec{x}_1}) = K_{\vec{x}, \vec{x}}$$

density of horizontal dimers

$$\mathbb{E}(\sigma_{\vec{x}_1} \sigma_{\vec{x}_2}) = K_{\vec{x}_1 \vec{x}_1} K_{\vec{x}_2 \vec{x}_2} -$$

$$- K_{\vec{x}_1 \vec{x}_2} K_{\vec{x}_2 \vec{x}_1},$$

$$\rightarrow \mathbb{E}((\sigma_{\vec{x}_1} - \mathbb{E}(\sigma_{\vec{x}_1}))(\sigma_{\vec{x}_2} - \mathbb{E}(\sigma_{\vec{x}_2}))) =$$

$$= - K_{\vec{x}_1 \vec{x}_2} K_{\vec{x}_2 \vec{x}_1}$$

Scaling limit: $q = e^{-\varepsilon}$, $\varepsilon \rightarrow 0$

$$\varepsilon u_i = U_i, \quad \varepsilon v_i = V_i$$

To find the limit shape and correlation functions we should find asymptotic of $K_{\vec{x}, \vec{y}}$ as $\varepsilon \rightarrow 0$

1) Assume $\varepsilon t_i \rightarrow \tau_i$, $\varepsilon h_i \rightarrow \chi_i$

$$K((t_1, h_1)(t_2, h_2)) \rightarrow \frac{S'(z, \tau_1, \chi_1) - S'(w, \tau_2, \chi_2)}{\varepsilon}$$

$$\rightarrow \left(\frac{1}{2\pi i}\right)^2 \int_{C_z} \int_{C_w} e^{\dots}$$

$$\cdot \sqrt{\frac{f(z)(we^{-\tau_2} - 1)}{f(w)(ze^{-\tau_1} - 1)}} \frac{\sqrt{zw}}{z-w} \frac{dz}{z} \frac{dw}{w}$$

$$S'(z, \tau, \chi) = -\left(\chi + \frac{\tau}{2} - u_0\right) \ln z +$$

$$+ \sum_{i=0}^N \text{Li}_2(ze^{-u_i}) - \sum_{i=1}^N \text{Li}_2(ze^{-v_i}) - \text{Li}_2(ze^{-\tau})$$

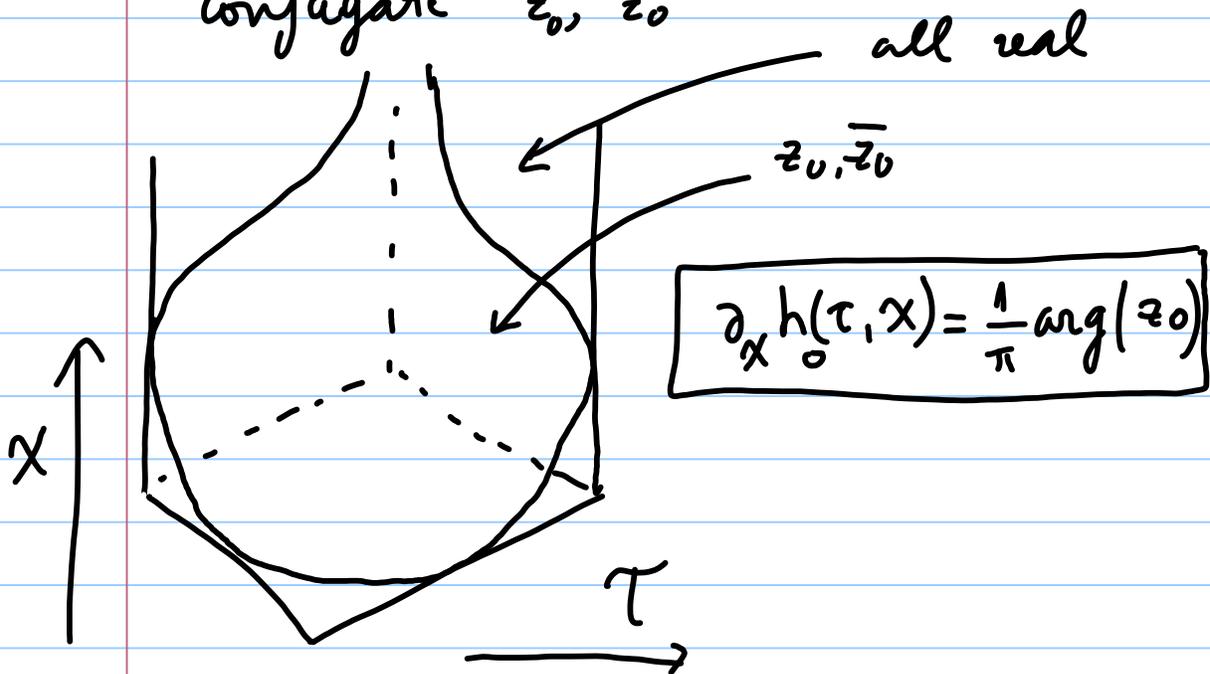
$$\text{Li}_2(z) = \int_0^z \frac{\ln(1-t)}{t} dt$$

$$f(z) = \frac{\prod_{i=0}^N (1 - z e^{-u_i})}{\prod_{i=1}^N (1 - z e^{-v_i})},$$

Proposition 1) Critical points of $S(z)$:

$$e^{x + \tau/2} = \frac{f(z)}{1 - z e^{-\tau}},$$

2) Either N real solutions, or
 $N-2$ real and two complex
conjugate z_0, \bar{z}_0



$$E(\sigma_{(t,h)}) = K((t,h), (t,h)) = \varepsilon \partial_{\alpha} h_0(\tau, \chi)$$

$$\partial_{\alpha} h_0(\tau, \chi) = \frac{1}{\pi} \arg(z_0)$$

Steepest descent: $\varepsilon t_i \rightarrow \tau_i, \varepsilon h_i \rightarrow \chi_i$

$$K((t_1, h_1), (t_2, h_2)) = - \frac{\varepsilon e^{\chi_{1,2} + \frac{\tau_{1,2}}{2}}}{2\pi}$$

$$\left(\frac{e^{\frac{S_1(z_1) - S_2(w_2)}{\varepsilon}}}{(z_1 - w_2) \sqrt{-w_2 S_2''(w_2)} \sqrt{z_1 S_1''(z_1)}} - \frac{e^{\frac{S_1(z_1) - S_2(\bar{w}_2)}{\varepsilon}}}{(z_1 - \bar{w}_2) \sqrt{-\bar{w}_2 S_2''(\bar{w}_2)} \sqrt{z_1 S_1''(z_1)}} + \text{c.c.} \right) \cdot (1 + o(1))$$

$$\begin{aligned}
& \mathbb{E} \left((\sigma_{\bar{x}_1} - \mathbb{E}(\sigma_{\bar{x}_1})) (\sigma_{\bar{x}_2} - \mathbb{E}(\sigma_{\bar{x}_2})) \right) = \\
& = -K_{12} K_{21} = \\
& = -\frac{\varepsilon^2}{(2\pi)^2} \left[\frac{\frac{\partial z_1}{\partial x_1} \frac{\partial w_2}{\partial x_2}}{(z_1 - w_2)^2} - \frac{\frac{\partial z_1}{\partial x_1} \frac{\partial \bar{w}_2}{\partial x_2}}{(z_1 - \bar{w}_2)^2} \right. \\
& \quad \left. + \text{c.c.} \right] (1 + o(1))
\end{aligned}$$

$$\frac{1}{\varepsilon} (\sigma_{\bar{x}} - \mathbb{E}(\sigma_{\bar{x}})) \rightarrow \partial_x \varphi(\tau, x)$$

Gaussian free field ↗

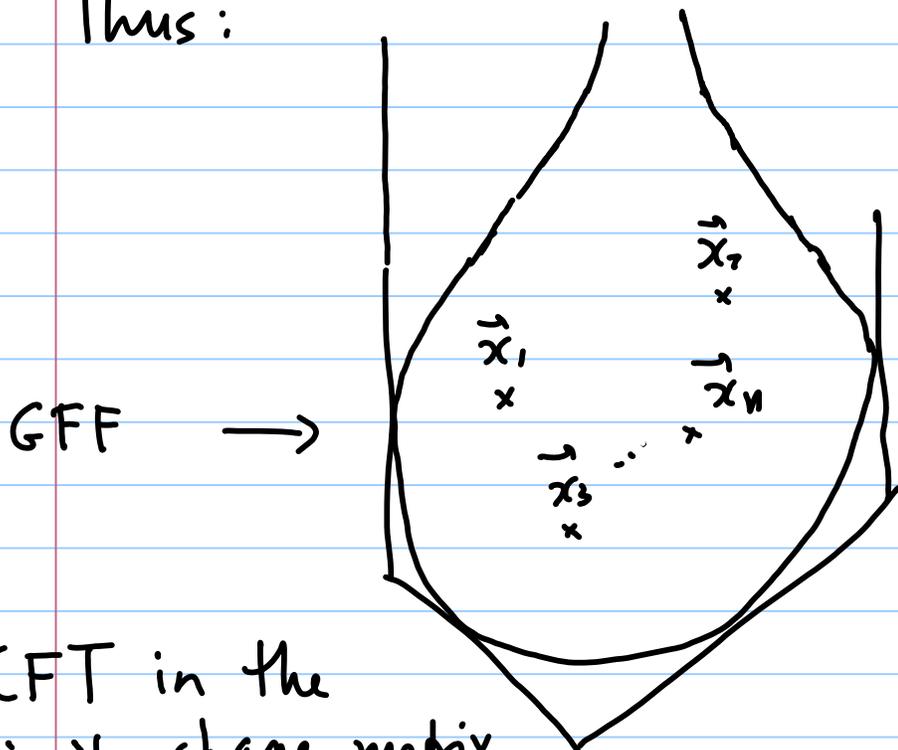
$$\mathbb{E}(\varphi(\tau_1, x_1) \varphi(\tau_2, x_2)) = G(z_1, w_2)$$

GFF

$$G(z, w) = -\frac{1}{4\pi} \ln \left| \frac{z - w}{z - \bar{w}} \right|$$

$$\frac{1}{\varepsilon^n} \prod_i^n (\sigma_{\vec{x}_i} - \mathbb{E}(\sigma_{\vec{x}_i})) \rightarrow \prod_{i=1}^n \partial_{x_i} \mathcal{P}(\tau_i, x_i)$$

Thus:



(FT in the
limit shape matrix

Fermions:

$$\sigma_{\vec{x}} = \psi_{\vec{x}} \psi_{\vec{x}}^*$$

$$\frac{1}{\sqrt{\varepsilon}} \psi_{\vec{x}} = \left(\psi(z_0(\tau, x)) e^{\frac{i}{\varepsilon} \text{Im}(S(z_0))} + \psi(\bar{z}_0(\tau, x)) e^{-\frac{i}{\varepsilon} \text{Im}(S(z_0))} \right) (1 + o(1))$$

$$\frac{1}{\sqrt{\epsilon}} \psi_{\vec{x}}^* = \left(\psi(z_0(\tau, \chi))^* e^{-\frac{i}{\epsilon} \text{Im}(S(z_0))} + \psi(\bar{z}_0(\tau, \chi))^* e^{\frac{i}{\epsilon} \text{Im}(S(z_0))} \right) (1 + o(1))$$

$$\mathbb{E} \left(\psi(z) \psi^*(w) \right) = \frac{1}{z - w},$$

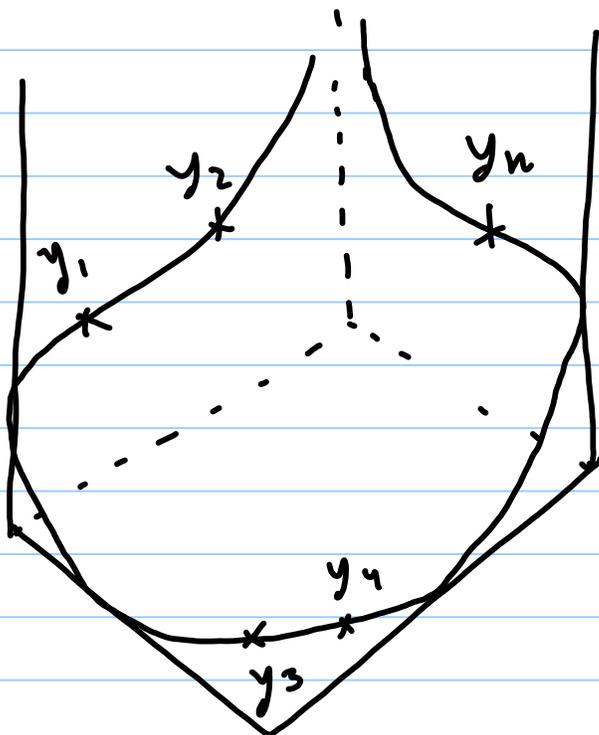
$$\epsilon \sigma_{\vec{x}} = \partial_{\chi} \varphi_0(\tau, \chi) + : \tilde{\psi}(z, \bar{z}) \tilde{\psi}^*(z, \bar{z}) :$$

Fermi - Bose correspondence in 2d.

2) Similarly

$$\vec{y}_1, \dots, \vec{y}_n \in \partial(\text{Limit shape})$$

Determinantal Airy process

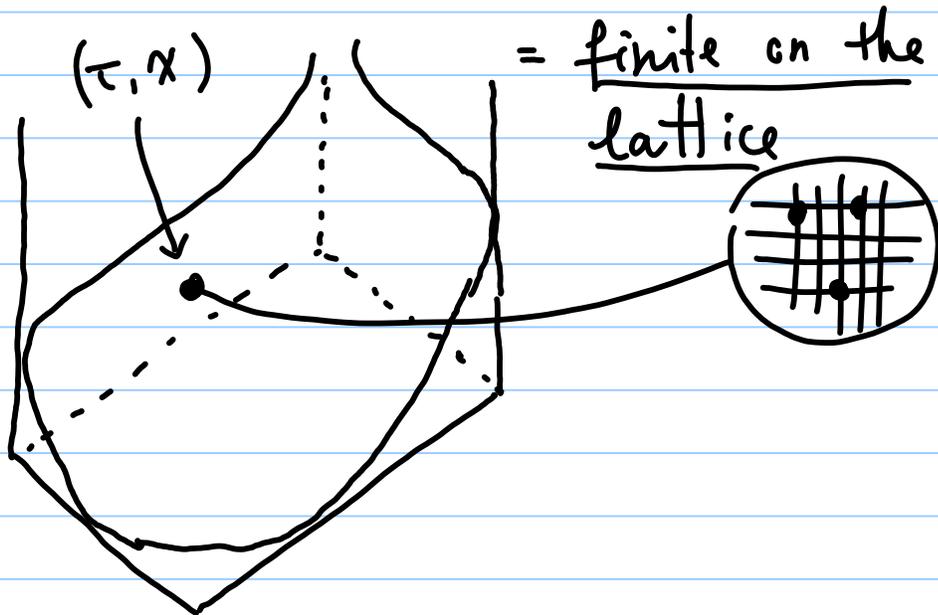


Work in progress

(with D. Keating, A. Sridhar)

3) Microscopic scale

a) $\varepsilon \vec{x}_i \rightarrow (\tau, \chi)$, $\vec{x}_i - \vec{x}_j =$



$$(s, t) = (\partial_\chi h_0(\tau, \chi), \partial_\tau h_0(\tau, \chi))$$

$$\mathbb{E}(\sigma_{\vec{x}_1} \cdots \sigma_{\vec{x}_K}) \rightarrow$$

$$\rightarrow G^{(s, t)}(n_1 m_1; \cdots; n_K m_K)$$

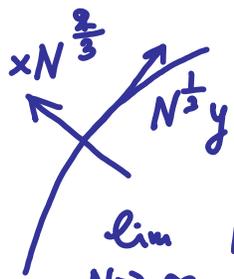
$$G^{(s,t)}(n_1, m_1; \dots; n_k, m_k) =$$

$$= \det \left(K^{(s,t)}(n_i - n_j, m_i - m_j) \right)_{i,j=1}^k$$

$$K^{(s,t)}(\Delta n, \Delta m) = \int_{\bar{z}_0}^{z_0} (1 - e^{\tau w}) w^{-\Delta m - 1} dw$$

KOS 2006, OR 2003, ...

(b) Near the interface

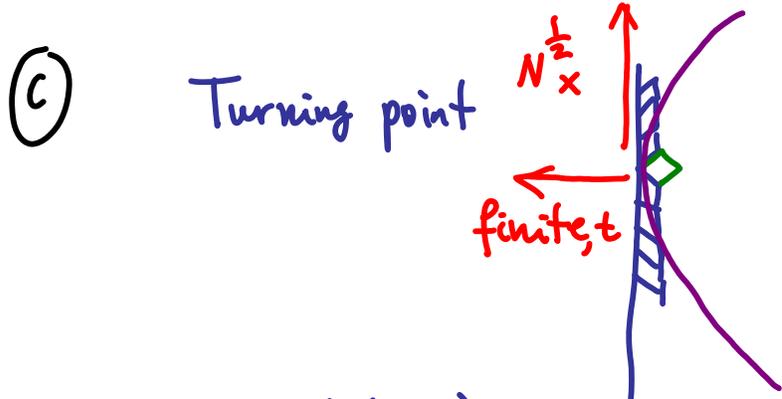


$$\lim_{N \rightarrow \infty} N^{\#} (B^{-1})_{(N^{2/3}x_1, N^{1/2}y_1)(N^{2/3}x_2, N^{1/2}y_2)}$$

= Airy process

Johansson; Okounkov, N.R.

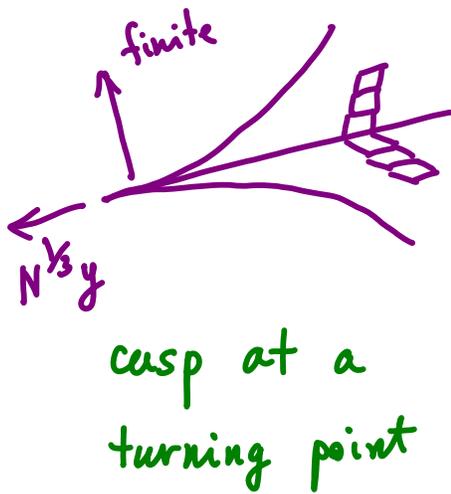
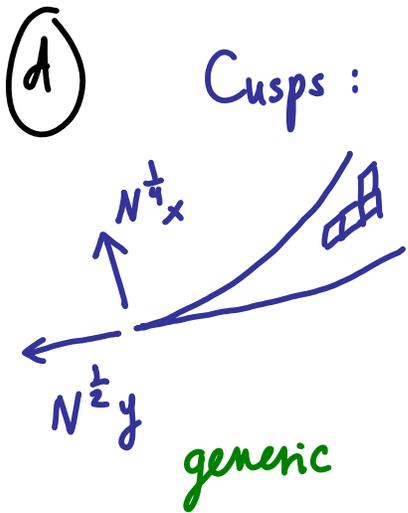
$$N \sim \frac{1}{\varepsilon}$$



$$\lim_{N \rightarrow \infty} N^* (B^{-1}) (N^{\frac{1}{2}}_{x_1}, t_1) (N^{\frac{1}{2}}_{x_2}, t_2) =$$

(Okounkov, N.R.)

GUE
distributed
principal
minors



Pearse process

