Lecture 6: Multivariate generating functions

Robin Pemantle

University of Pennsylvania

pemantle@math.upenn.edu

Minerva Lectures at Columbia University

16 November, 2016
Dinner at the Pemutzle house
Rational series: examples and phenomena
Let 

\[ f(z) = \frac{p(z)}{q(z)} = \sum_{r \geq 0} a_r z^r \]

be a rational function.

Then \( f \) has a partial fraction expansion as a sum of terms of the form \( c(1 - tz)^d \) and therefore there is an exact expression for the coefficient \( a_r \), namely

\[
a_t = \sum_{(t,d,c)} c \binom{n + d}{d} t^r
\]

summed over triples \((t, d, c)\) in the partial fraction expansion. In short, the theory is trivial.
Several variables

Now turn all the indices into multi-indices, which we denote by upper case letters:

\[ F(Z) = \frac{P(Z)}{Q(Z)} = \sum_R a_R Z^R. \]

Here, and throughout, this stands for

\[ F(z_1, \ldots, z_d) = \frac{P(z_1, \ldots, z_d)}{Q(z_1, \ldots, z_d)} = \sum_{r_1, \ldots, r_d} a_{r_1, \ldots, r_d} z_1^{r_1} \cdots z_d^{r_d}. \]

What phenomena are possible for the behavior of the multi-dimensional array \( \{a_R\} \)?
Binomial coefficients

\[ F(x, y) = \frac{1}{1 - xy/2 - y/2} \]

shown: \( a_{i,200} \)
2-D quantum walk

\[ F(x, y) = \frac{1}{1 - (1 - x)y/\sqrt{2} - xy^2} \]

shown: \( a_{i,200} \)
3-D quantum walk

\[ F(x, y) = \det(I - yMU)^{-1}, \]
where \( M \) is a diagonal matrix of monomials and \( U \) is a real orthogonal matrix.

shown: grey scale plot of \( a_{i,j,200} \)
Dimer tiling

\[ F(x, y) = \frac{z/2}{(1 - yz)\left[1 - (x + x^{-1} + y + y^{-1})z + z^2\right]} \]

shown: plot of

\[(r, s) \mapsto \lim_{t \to \infty} a_{rt, st, t}\]
Double-dimer tiling

\[ F = \frac{P}{Q} \text{ where } Q(x, y, z) = 63x^2y^2z^2 - 62(x + y + z)xyz - (x^2y^2 + y^2z^2 + z^2x^2) + 62(xy + yz + zx) + (x^2 + y^2 + z^2) - 63. \]

shown: sample tiling, limiting boundaries
Possible behavior along a ray

Let $R = |R| \cdot \hat{R}$ decompose $R$ into magnitude and direction.

The possible behaviors\(^1\) for $a_R$ with $\hat{R}$ roughly fixed are asymptotically, for some rational $\beta$, and nonnegative integer $\gamma$,

$$a_R = C(\hat{R}) |R|^\beta |\log(R)|^\gamma Z(\hat{R})^{-R}.$$ 

Also possible: a finite sum of such terms.

Such formulae hold piecewise, with $\beta$ and $\gamma$ constant on each piece and $C$ and $Z$ varying analytically within each piece.

Phase boundaries are algebraic curves;
One expects Airy-type behavior at the boundaries.

\(^1\)Precise statement not proved.
Analytic framework
Cauchy integral

Recall the one variable Cauchy integral

\[ a_r = \frac{1}{2\pi i} \int z^{-r} f(z) \frac{dz}{z} \]
Cauchy integral

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\[ a_r = \frac{1}{2\pi i} \int \frac{z^{-r} f(z)}{z} \, dz \]

In \( d \) variables it is nearly the same:

\[ a_R = \frac{1}{(2\pi i)^d} \int_C Z^{-R} \frac{P(Z)}{Q(Z)} \frac{dZ}{Z} \]
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- \( dZ \) is the holomorphic volume form;
- integrand is holomorphic in \( \mathcal{M} := \mathbb{C}^d \setminus \{ Q \prod_{j=1}^d z_j = 0 \} \);
- \( C \) is a chain of integration topologically equivalent to the torus \( \prod_{j=1}^d \gamma_j \) where \( \gamma_j \) is a circle about the origin in the \( j^{th} \) coordinate and the equivalence is in \( H_d(\mathcal{M}) \).
$\mathcal{M}$ is everything other than $V$ and the coordinate axes.
We want an asymptotic formula for $a_r$ as $r \to \infty$ with $r/|r| \to \hat{r}$.

Two levels of accuracy, to be done in two steps:

1. Exponential rate
2. Asymptotic formula

To see how to execute these two steps, recall the univariate case.
Recall the univariate case

Step 1: let $\rho$ be the radius of convergence.

$$\limsup \frac{\log a_n}{n} \leq -\log \rho.$$ 

This completes step one.

Step 2: Use singularity analysis. Find the particular singularity $z_*$ on the radius of convergence and integrate on a contour designed to capture behavior near $z_*$. 
Dominant singularity
Logarithmic coordinates

Instead of a radius of convergence there is a different multi-radius in every direction.

The domain of convergence of a power series or Laurent series is a union of tori

\[ T_X := \{ |z_1| = e^{x_1}, \ldots, |z_d| = e^{x_d} \} \cdot \]

The set of \( X \in \mathbb{R}^d \) for which a series converges is convex.

Map \( \mathbb{C}^d \) to \( \mathbb{R}^d \) via the log-modulus map

\((z_1, \ldots, z_d) \mapsto (\log |z_1|, \ldots, \log |z_d|)\).
Amoebas

The amoeba of $\mathbb{Q}$ is the set $\{\log |Z| : Q(Z) = 0\}$ where the log modulus map is taken coordinatewise.

On each component of the complement of the amoeba there is a convergent Laurent expansion $P/Q = \sum_{R} a_{R} Z^{R}$.

Components of the amoeba complement (white regions) are convex.
Letting $Z = \exp(X + iY)$ and sending $X$ through a component of the complement, to a point on the boundary of the amoeba, the Cauchy integral becomes

$$a_R = (2\pi i)^{-d} \int Z^{-R} F(Z) \, dZ$$

$$= (2\pi)^{-d} e^{-R \cdot X} \int \exp(-iR \cdot Y) f(Y) \, dY$$

where $f(Y) = F(\exp(X + iY))$. 
Choose $X = X_*$ to minimize the magnitude of the integrand.

For fixed $\hat{R}$, this means to make $-\hat{R} \cdot X_*$ as small as possible. This is a convex minimization problem (the Legendre transform).

$B$ is a region of the complement corresponding to an ordinary power series;

$\hat{R}$ is given;

$X_*$ is the minimizing point.
Upper bound

The upper bound is immediate:

$$\limsup \frac{1}{|R|} \log |a_R| \leq -\hat{R} \cdot X_*(R)$$

where $X_*(R)$ is the support point on $\partial B$ normal to $R$. 
Asymptotic evaluation
Contributing points

We turn now to Step 2, namely the asymptotic evaluation of $a_R$. This is the only way to provide a matching lower bound (in fact the limsup and liminf behavior of the coefficients might not be the same).

Recall that every $X$ in the amoeba of $Q$ is $\log |Z|$ for some at least one $Z \in \mathcal{V}$. The Cauchy integral near some of these is what determines the asymptotics in direction $R$.

We next discuss the identification of point(s) $\exp(X_* + iY) \in \mathcal{V}$ that are responsible for the coefficient asymptotics in direction $R$. 
Finally... we get back to hyperbolicity.

Let $T := \mathcal{V} \cap \{\exp(X^* + iY) : Y \in (\mathbb{R}/2\pi \mathbb{Z})^d\}$.

At each $Z \in T$, let $p = p_Z$ be the homogeneous polynomial defined by the leading term of $Q(Z + \cdot)$.

The polynomial $p_Z$ is the **algebraic tangent cone** of $Q$ at $Z$.

**Proposition (Baryshnikov+P 2011)**

1. For every $Z := \exp(X + iY)$ with $X$ on the boundary of the amoeba of $Q$, the polynomial $p_Z$ is hyperbolic.

2. $p_Z$ has a cone of hyperbolicity containing the support cone $B$ to the amoeba complement at $X$. 
Family of cones

This is the key construction for evaluating the Cauchy integral.

Theorem (semi-continuous family of cones)

Let $p$ be any hyperbolic homogeneous polynomial and let $B$ be a cone of hyperbolicity for $p$. There is a family of cones $K(x)$ indexed by the points $x$ at which $p$ vanishes, such that the following hold.

(i) Each $K(x)$ is a cone of hyperbolicity for the tangent cone $p_x$.
(ii) All of the cones $K(x)$ contain $B$.
(iii) $K(x)$ is semi-continuous in $x$, meaning that if $x_n \rightarrow x$, then $K(x) \subseteq \lim \inf K(x_n)$. 

Step 1: By the previous proposition $p_x$ is hyperbolic. Now show that any vector hyperbolic for $p$ is also hyperbolic for $p_x$. This result was originally proved by Atiyah-Bott-Gårding (1970); Borcea (personal communication) gave a short, self-contained proof.

Step 2: Pick $u$ in $B$. Define $K(x)$ to be the cone of hyperbolicity of $p_x$ that contains $u$. This gives $(i)$ and also $(ii)$ because the construction is the same for any $u \in B$.

Step 3: Prove $(iii)$ by showing that these cones obey a condition similar to that of Whitney stratification. This takes a few geometric steps.
Example: orthant

If \( x \) is on a 2-D surface then \( K(x) \) is the halfspace containing \( B \).

If \( x \) is on one of the intersection lines then \( K(x) \) is the quarter-space containing \( B \).

If \( x \) is the origin then then \( K(x) \) is the octant \( B \).
Example: product of two linear polynomials

If \( x = 1 \), the common intersection of the two divisors, then \( K(x) = B \).

If \( x \) is on only one of the lines then \( K(x) \) is a halfspace tangent to one of the two factor amoebas.

Shown: the amoeba for \( p = (3z - x - 2y)(3z - 2x - y) \).

The amoeba of the product is the union of the two factor amoebas.
Counterexample: when \( p \) is not hyperbolic

Along the line \((0, y, 0)\), \( K(y) \) cones of hyperbolicity for \( p_y \) can be chosen but are forced to select the positive or negative \( z \) direction.

One of these violates semi-continuity for \( K(0^+, y, 0) \) while the other choice violates semi-continuity at \( K(0^-, y, 0) \).
Semi-continuous cones give vector field

**Theorem (Morse deformation; BP2011, after ABG1970)**

*If \( \{K(x)\} \) is a semi-continuous family of cones, then a continuous vector field \( \Psi \) may be constructed with \( \Psi(x) \subseteq K(x) \) for each \( x \).*

This allows the chain of integration for the Cauchy integral to be deformed so that the integrand is very small except in a neighborhood of \( Z \).

The deformation is locally projective.
The Cauchy integral and the Riesz kernel

Recall the integral in logarithmic coordinates

$$a_R = (2\pi)^{(1-d)/2} \exp(-R \cdot X) \int \exp(-(iR \cdot Y)f(Y) \, dY.$$ 

Pushing the chain of integration from the imaginary fiber outward in a conical manner produces a homogeneous inverse Fourier transform.

Leading asymptotic behavior only depends on leading behavior of the homogenization $1/p_z$. We recognize the IFT

$$\int_{\gamma} p_z^{-1} \exp(iR \cdot Y)$$

as the Riesz kernel for the homogeneous hyperbolic polynomial $p_z$. 
Inverse Fourier transforms

The estimates needed to establish the existence of the integral rely on the projective deformation.

Relating it to previously computed IFT’s (in, e.g., [ABG1970]) uses the theory of boundaries of holomorphic functions, laid out by Hörmander (1990).

For example, the IFT of a linear function $ax + by + cz$ is a delta function on the ray $\lambda \langle a, b, c \rangle$ in the dual space. (Here the index space $\mathbb{Z}^d$ is the dual space and the real/complex space in which the generating function variables live is the primal space.)
Examples of computed IFT’s

Example (orthant)

The IFT of \(1/(xyz)\) is the constant 1 on the orthant. This corresponds to the generating function

\[
\frac{1}{(1 - x)(1 - y)(1 - z)}.
\]
"Quadratic times linear" describes the homogeneous part of the Aztec Diamond probability generating function

\[
\frac{z/2}{(1 - yz) \left[ 1 - (x + x^{-1} + y + y^{-1})z + z^2 \right]}.
\]
Example: Aztec diamond tilings

IFT is a convolution of a delta function on the ray \((0, \lambda, \lambda)\) with the IFT of a circular quadratic. The quadratic is self-dual, with IFT equal to \(t^2 - r^2 - s^2\).

shown: plot of 
\[(r, s) \mapsto \lim_{t \to \infty} a_{rt, st, t}\]
General case

Inverse Fourier transforms for general hyperbolic homogeneous polynomials have not been effectively computed although quite a number have been worked out, e.g., Atiyah-Bott-Gårding (Acta Math. 1970) “Lacunas for hyperbolic differential operators with constant coefficients, I.”
More to be done

General cubic and quartic integrals are a little trickier to evaluate explicitly than are the quadratics and factored cubics. The so-called fortress tillings are an example of this (work in progress with Y. Baryshnikov).

This cubic arises in analysis of the hexahedron recurrence. Its IFT gives a limit shape theorem (work in progress). At present we can describe the feasible region but not the limit statistics within the region. For a particular parameter value, the central collar becomes a plane and the results for Quadratic times Linear apply.
Analytic Combinatorics in Several Variables

ROBIN PEMANTLE
MARK C. WILSON
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