Cauchy integral

Recall the one variable Cauchy integral

$$a_r = \frac{1}{2\pi i} \int z^{-r} f(z) \frac{dz}{z}$$

In d variables it is nearly the same:

$$a_R = \frac{1}{(2\pi i)^d} \int_{\mathcal{C}} Z^{-R} \, \frac{P(Z)}{Q(Z)} \, \frac{dZ}{Z}$$

- dZ is the holomorphic volume form;
- integrand is holomorphic in $\mathcal{M} := \mathbb{C}^d \setminus \{Q \prod_{j=1}^d z_j = 0\}$;
- $\mathcal{C}$ is a chain of integration topologically equivalent to the torus $\prod_{j=1}^d \gamma_j$ where γ_j is a circle about the origin in the j^{th} coordinate and the equivalence is in $H_d(\mathcal{M})$.

Imaginary fiber through a point on the boundary

Letting $\mathbf{Z} = \exp(\mathbf{X} + \mathbf{i}\mathbf{Y})$ and sending $\mathbf{X}$ through a component of the complement, to a point on the boundary of the amoeba, the Cauchy integral becomes

$$\begin{aligned} \mathbf{a_R} &= (2\pi\mathbf{i})^{-\mathbf{d}} \int \mathbf{Z}^{-\mathbf{R}} \mathbf{F}(\mathbf{Z})\, \mathbf{dZ} \\ &= (2\pi)^{-\mathbf{d}} \mathbf{e}^{-\mathbf{R}\cdot\mathbf{X}} \int \exp(-\mathbf{iR} \cdot \mathbf{Y})\, \mathbf{f}(\mathbf{Y})\, \mathbf{dY} \end{aligned}$$

where $\mathbf{f}(\mathbf{Y}) = \mathbf{F}(\exp(\mathbf{X} + \mathbf{iY}))$.

Family of cones

This is the key construction for evaluating the Cauchy integral.

Theorem (semi-continuous family of cones)

Let p be any hyperbolic homogeneous polynomial and let B be a cone of hyperbolicity for p. There is a family of cones $K(x)$ indexed by the points x at which p vanishes, such that the following hold.

(i) *Each $K(x)$ is a cone of hyperbolicity for the tangent cone p_x.*

(ii) *All of the cones $K(x)$ contain B.*

(iii) *$K(x)$ is* **semi-continuous** *in x, meaning that if $x_n \to x$, then $K(x) \subseteq \liminf K(x_n)$.*

Theorem (Riesz kernel supported on the dual cone)

(i) *For each cone K of hyperbolicity of p there is a soluton E_x to $D_p E_x = \delta_x$ in $\mathbb{R}^d$ supported on the dual cone K^*.*

(ii) *This solution is called the Riesz kernel and is defined by*

$$E_x(r) := (2\pi)^{-d} \int_{\mathbb{R}^d} q(x+iy)^{-1} \exp[r \cdot (x+iy)]\, dy\,.$$

(iii) *The boundary value problem $D_p f = 0$ on the halfspace $x \cdot r > 0$ with boundary values g and normal derivatives vanishing to order $\deg(p) - 1$ is given by $\int E_x(r) g(x) dx$.*

We will discuss the proof next lecture when extracting coefficients of rational multivariate generating functions.

References I

M. Atiyah, R. Bott, and L. Gårding.
Lacunas for hyperbolic differential operators with constant coefficients, I.
Acta Mathematica, 124:109–189, 1970.

Y. Baryshnikov and R. Pemantle.
Asymptotics of multivariate sequences, part III: quadratic points.
Adv. Math., 228:3127–3206, 2011.

Lars Hörmander.
An Introduction to Complex Analysis in Several Variables.
North-Holland Publishing Co., Amsterdam, third edition, 1990.

R. Pemantle and M.C. Wilson.
Asymptotics of multivariate sequences. I. Smooth points of the singular variety.
J. Combin. Theory Ser. A, 97(1):129–161, 2002.

R. Pemantle and M.C. Wilson.
Asymptotics of multivariate sequences, II. Multiple points of the singular variety.
Combin. Probab. Comput., 13:735–761, 2004.

References II

R. Pemantle and M.C. Wilson.
Twenty combinatorial examples of asymptotics derived from multivariate generating functions.
SIAM Review, 50:199–272, 2008.

R. Pemantle and M.C. Wilson.
Asymptotic expansions of oscillatory integrals with complex phase.
In *Algorithmic Probability and Combinatorics*, volume 520, pages 221–240. American Mathematical Society, 2010.