Lecture 5: Hyperbolic polynomials

Robin Pemantle

University of Pennsylvania

pemantle@math.upenn.edu

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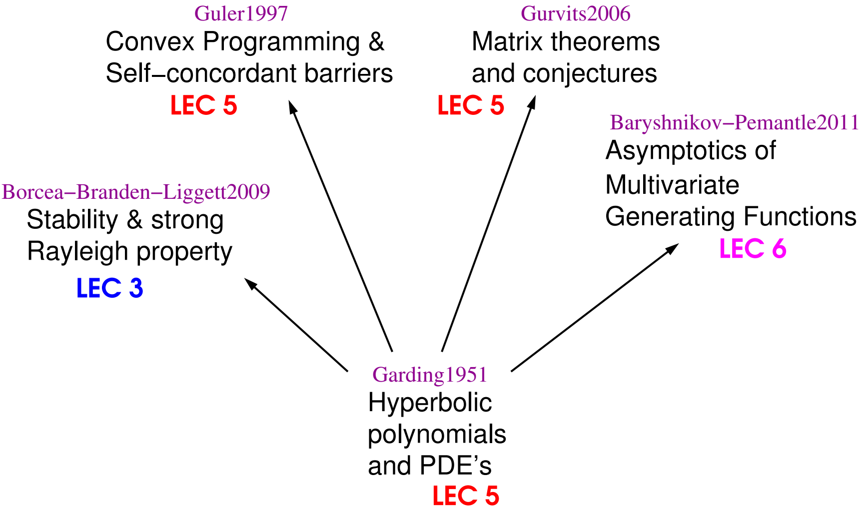
Ubiquity of zeros of polynomials

"The one contribution of mine that I hope will be remembered has consisted in pointing out that all sorts of problems of combinatorics can be viewed as problems of the location of the zeros of certain polynomials..."

– Gian-Carlo Rota (1985)

- PDE's (hyperbolicity)
- Number theory (zeta function)
- Statistical physics and probability (Lee-Yang theory)


Guler1997
Convex Programming &
Self–concordant barriers
LEC 5
Gurvits2006
Matrix theorems
and conjectures
LEC 5
Baryshnikov–Pemantle2011
Asymptotics of
Multivariate
Generating Functions
LEC 6
Borcea–Branden–Liggett2009
Stability & strong
Rayleigh property
LEC 3
Garding1951
Hyperbolic
polynomials
and PDE's
LEC 5

- Basics of the theory
- PDE's
- Other uses: convex programming and matrix theorems
- Lecture 6: Applications to multivariate generating functions

Hyperbolicity: equivalences and examples

Definitions

Definition (stability)

A polynomial q is said to be **stable** if $\mathbf{q}(\mathbf{z}) \neq 0$ whenever each coordinate $\mathbf{z_j}$ is in the strict upper half plane.

Definition (hyperbolicity)

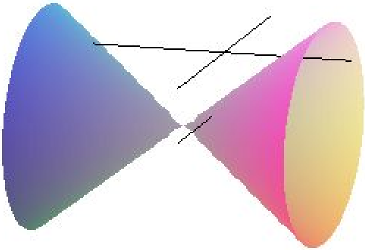
A homogeneous polynomial $\mathbf{p}$ of degree $\mathbf{m}$ is said to be **hyperbolic** in direction $\mathbf{x} \in \mathbb{R}^{\mathbf{d}}$ if $\mathbf{p}(\mathbf{x} + \mathbf{iy}) \neq 0$ for all $\mathbf{y} \in \mathbb{R}^{\mathbf{d}}$.

Proposition (equivalences)

- *Hyperbolicity in direction* $\mathbf{x}$ *is equivalent to the univariate polynomial* $\mathbf{p}(\mathbf{y} + \mathbf{tx})$ *having only real roots for all* $\mathbf{y} \in \mathbb{R}^{\mathbf{d}}$.
- *A real homogeneous polynomial is stable if and only if it is hyperbolic in every direction in the positive orthant.*

Example: Lorentzian quadratic

Let $\mathbf{p}$ be the Lorentzian quadratic $\mathbf{t}^2 - \mathbf{x}_2^2 - \cdots - \mathbf{x}_\mathbf{d}^2$, where we have renamed $\mathbf{x}_1$ as "$\mathbf{t}$" because of its interpretation as the time axis in spacetime; then $\mathbf{p}$ is hyperbolic in every timelike direction, that is, for each direction $\mathbf{x}$ with $\mathbf{p}(\mathbf{x}) > 0$.



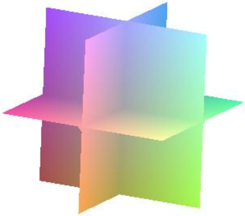
The time axis is left-right

Example: coordinate planes

The coordinate function $\mathbf{x_j}$ is hyperolic in direction $\mathbf{y}$ if and only if $\mathbf{y_j} \neq 0$ (this is true for any linear polynomial).

It is obvious from the definition that the product of polynomials hyperbolic in direction $\mathbf{y}$ is again hyperbolic in direction $\mathbf{y}$.

It follows that $\prod_{\mathbf{j}=1}^{\mathbf{d}} \mathbf{x_j}$ is hyperbolic in every direction not contained in a coordinate plane, that is, in every open orthant.



Cones of hyperbolicity

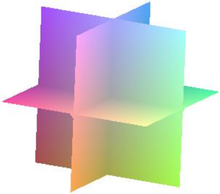
The real variety $\mathcal{V} := \{\mathbf{p} = 0\} \subseteq \mathbb{R}^{\mathbf{d}}$ plays a special role in hyperbolicity theory.

Proposition (cones of hyperbolicity)

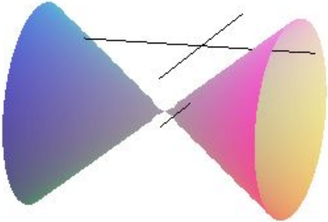
Let ξ be a direction of hperbolicity for $\mathbf{p}$ and let $\mathbf{K}(\mathbf{p}, \xi)$ denote the connectied component of the set $\mathbb{R}^{\mathbf{d}} \setminus \mathcal{V}$ that contains ξ.

- $\mathbf{p}$ *is hyperbolic in direction* $\mathbf{x}$ *for every* $\mathbf{x} \in \mathbf{K}(\mathbf{p}, \xi)$.
- *The set* $\mathbf{K}(\mathbf{p}, \xi)$ *is an open convex cone; we call this a* **cone of hyperbolicity** *for* $\mathbf{p}$.

Examples, continued



Every component of $\mathbb{R}^3 \setminus \mathcal{V}$ is a cone of hyperbolicity for $\mathbf{p} := \mathbf{xyz}$

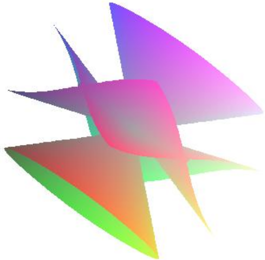


The forward and backward light cones are cones of hypebolicity for $\mathbf{p} := \mathbf{x}^2 - \mathbf{y}^2 - \mathbf{z}^2$

Example: Fortress polynomial

$$\mathbf{w}^4 - \mathbf{u}^2\mathbf{w}^2 - \mathbf{v}^2\mathbf{w}^2 + \frac{9}{25}\mathbf{u}^2\mathbf{v}^2$$

is the projective localization of the denominator (cleaned up a bit) of the so-called Fortress generating polynomial. It follows from [BP11, Proposition 2.12] that this polynomial is hyperbolic in certain directions, e.g., forward and backward cones (pictured pointing NE and SW).



Hyperbolicity and PDE's

These results are not needed for applications to other areas but they serve to explain where the consructions originate.

Stable evolution of PDE's

Let $\mathbf{p}$ be a polynomial in $\mathbf{d}$ variables and denote by $\mathbf{D_p}$ the operator $\mathbf{p}(\partial/\partial\mathbf{x})$ obtained by replacing each $\mathbf{x_i}$ by $\partial/\partial\mathbf{x_i}$.

Let $\mathbf{r}$ be a vector in $\mathbb{R}^{\mathbf{d}}$, let $\mathbf{H_r}$ be the hyperplane orthogonal to $\mathbf{r}$, and consider the equation

$$\mathbf{D_p}(\mathbf{f}) = 0 \tag{1}$$

in the halfspace $\{\mathbf{r} \cdot \mathbf{x} \geq 0\}$ with boundary conditions specified on $\mathbf{H_r}$ (typically, $\mathbf{f}$ and its first $\mathbf{d} - 1$ normal derivatives).

We say that (1) evolves **stably** in direction $\mathbf{r}$ if convergence of the boundary conditions to 0 implies convergence[1] of the solution to 0.

[1] Uniform convergence of the function and its derivatives on compact sets.

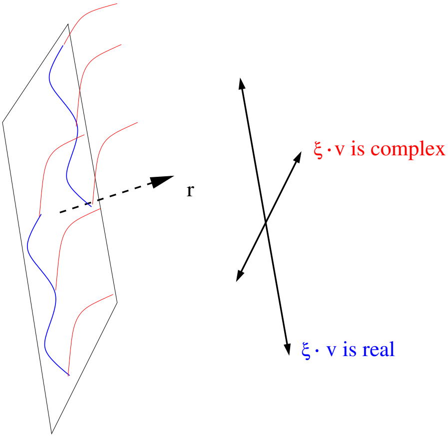
Gårding's Theorem

Theorem ([Går51, Theorem III])

The equation $D_p f = 0$ evolves stably in direction r if and only if p is hyperbolic in direction r.

Let us see why this should be true.

We begin with the observation that if $\xi \in C^d$ is any vector with $p(\xi) = 0$ then $f_\xi(x) := \exp(i\xi \cdot x)$ is a solution to $D_p f = 0$. (Our solutions are allowed to be complex but live on $\mathbb{R}^d$.)



Exponential growth in direction v when $\xi \cdot v$ is non-real;

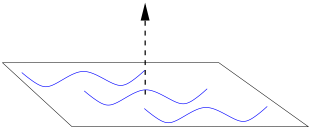
Bounded growth when $\xi \cdot v$ is real.

Stability implies hyperbolicity:

Assume WLOG that $r = (0, \ldots, 0, 1)$.

Suppose p is not hyperbolic. Unraveling the definition, p has at least one root $\xi = (a_1, \ldots, a_{d-1}, a_d \pm bi)$, with $\{a_i\}$ and b real and nonzero. In other words, $v \mapsto \xi \cdot v$ is real when restricted to $r^\perp$ and non-real on $\langle r \rangle$.

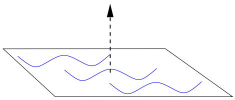
Therefore, f_ξ grows exponentially in direction r but is bounded on $r^\perp$.



The same is true for $f_{\lambda\xi}$. Sending $\lambda \to \infty$, we may take initial conditions going to zero such that $f_{\lambda\xi}(r) = 1$ for all λ.

Hyperbolicity implies stable evolution

SKETCH: Suppose q is hyperbolic and WLOG $r = e_d$. For every real $r' = (r_1, \ldots, r_{d-1})$ in frequency space there are d real values of r_d such that $q(r) := q(r', r_d) = 0$. For each such r, the function f_r is a solution to $D_q(f) = 0$, traveling unitarily.



d bounded solutions

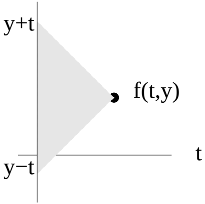
These d solutions are a unitary basis for the space $V_{r'}$ that they span, of solutions to (1) that restrict on H_ξ to $e^{ir' \cdot x}$, and the same is true at any later time. Difficult argument: d independent solutions in the boundary plane, for each frequency, is enough to give boundary values up to $d-1$ derivatives. $\square$

Light cone

Hyperbolicity can also be used to establish finite propagation speed. Contrast to parabolic equations such as the heat equation, where $f(x,t)$ depends on $f(y,0)$ for all y.

Example (2-D wave equation)

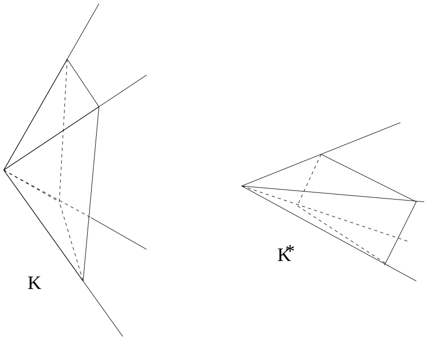
Let f solve $f_{tt} - f_{yy} = 0$ with boundary conditions $f(0,y) = g(y)$ and $f_t(0,y) = h(y)$. Then an explicit formula for f in the right half plane is given by



$$f(t,y) = \frac{1}{2}\left[g(y+t) + g(y-t) + \int_{y-t}^{y+t} h(u)\,du\right].$$

Dual cone

The more general result is that the solution from δ-function initial conditions propagates on the **dual cone** (Paley-Wiener Theorem) and that the solution in general is given by the **Riesz kernel**. Let $K \subseteq \mathbb{R}^d$ be a cone and let K^* denote the dual cone, that is the set of all y such that $x \cdot y \geq 0$ for all $x \in K$.



Theorem (Riesz kernel supported on the dual cone)

(i) *For each cone K of hyperbolicity of p there is a soluton E_x to $D_p E_x = \delta_x$ in $\mathbb{R}^d$ supported on the dual cone K^*.*

(ii) *This solution is called the Riesz kernel and is defined by*

$$E_x(r) := (2\pi)^{-d} \int_{\mathbb{R}^d} q(x+iy)^{-1} \exp[r \cdot (x+iy)]\, dy\,.$$

(iii) *The boundary value problem $D_p f = 0$ on the halfspace $x \cdot r > 0$ with boundary values g and normal derivatives vanishing to order $\deg(p) - 1$ is given by $\int E_x(r) g(x) dx$.*

We will discuss the proof next lecture when extracting coefficients of rational multivariate generating functions.

Convex programming and self-concordant barriers

Interior point method

Convex programming is the algorithmic location of the minimum of a convex function f on a convex set $K \in \mathbb{R}^d$.

Without loss of generality, $f(x) = c^T x$ is linear (intersect K with the region above the graph of f) and the minimum on K occurs on ∂K.

The **interior point** method is to let ϕ be a **barrier function** going to infinity at ∂K and to define a family of interior minima

$$x_*(t) := \mathrm{argmin}_x \; c^T x + t^{-1}\phi(x)$$

converging to the true minimum x_* as $t \to \infty$. This is useful because from $x_*(t)$, it is often possible to approximate $x_*(t + \Delta t)$ very quickly.

Inequalities that make this work well

Suppose $\phi \in C^3$ is defined on the interior of a convex cone K, is homogeneous of degree $-m$, and goes to infinity at ∂K.

If $F := \log \phi$ satisfies

$$|D^3 F(x)[h, h, h]| \leq 2 \left(D^2 F(x)[x, x] \right)^{3/2}$$

then ϕ serves as a barrier function. If, in addition,

$$|DF(x)[h]|^2 \leq m D^2 F(x)[h, h]|$$

then the interior method will converge well.

Hyperbolicity and self-concordant barriers

Theorem (Nesterov and Todd 1997; Guler 1997)

If p is a homogeneous polynomial vanishing on the boundary of a convex cone K, then $\phi(x) = p(x)^{-\alpha}$ satisfies the above inequalities, bounding second directional derivative from below by constant multiples of the 2/3-power of the third derivative and the square first derivative, all in the same direction.

Hyperbolicity is universal for homogeneous cones

This establishes a long-step interior point algorithm on the cone of hyperbolicity of any hyperbolic polynomial. It turns out this is more general than one might think. Say a cone is **homogeneous** if the linear maps preserving it as a set act transitively on its interior.

Proposition

All homogenous convex cones are cones of hyperbolicity for some polynomial.

Matrix theorems and conjectures

van der Waerden conjecture

Theorem (van der Waerden conjecture)

The minimum permanent of an $n \times n$ doubly stochastic matrix is $n!/n^n$, uniquely obtained when all entries are $1/n$.

This was proved by Falikman (value of the minimum) and Egorychev (uniqueness) in 1981.

Gurvits (2008) found a much simpler proof using stability. Let $\mathcal{C}_n$ be the set of homogeneous polynomials of degree n in n variables with nonnegative coefficients.

Permanent as a stable polynomial in $\mathcal{C}_n$

If A is an $n \times n$ matrix then its permanent may be represented as a mixed partial derivative of a homogeneous stable polynoimal.

Define

$$p_A(x) := \prod_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_j \, .$$

Proposition (Gurvits 2008)

The polynomial p is stable and

$$\mathrm{per}\,(A) = \frac{\partial^n}{\partial x_1 \cdots \partial x_n} p_A(0, \ldots, 0) \, .$$

Gurvits' argument

Define $\mathrm{Cap}\,(p) := \inf \dfrac{p(x_1, \ldots, x_n)}{\prod_{i=1}^{n} x_i}$.

The following two results, also from Gurvits (2008), immediately imply the van der Waerden conjecture.

Proposition

If A is doubly stochastic then $\mathrm{Cap}\,(p_A) = 1$.

Theorem (Gurvits' inequality)

If $p \in \mathcal{C}_n$ is stable then

$$\frac{\partial^n}{\partial x_1 \cdots \partial x_n} p(0, \ldots, 0) \geq \mathrm{Cap}\,(p) \frac{n!}{n^n} .$$

More mileage

Note: both propositions are immediate and the theorem is a short induction.

Further mileage may be obtained from this. The theorem also implies the Schrijver-Valiant conjecture (1980, proved by Schrijver in 1998), concerning the minimum permanent of $n \times n$ nonnegative integer matrices whose row and column sums are k.

Monotone Column Permanent (MCP) Conjecture

Say that the $n \times n$ matrix A is a **monotone column matrix** if its entries are real and weakly decreasing down each column, that is, $a_{i,j} \geq a_{i+1,j}$ for $1 \leq i \leq n-1$ and $1 \leq j \leq n$. Let J_n denote the $n \times n$ matrix of all ones. It was conjectured (Haglund, Ono and Wagner 1999) that whenever A is a monotone column matrix, the univariate polynomial $\mathrm{Per}\,(zJ_n + A)$ has only real roots. A proof was given there for the case where A is a zero-one matrix.

MCP theorem

Theorem (Brändén, Haglund, Ono and Wagner 2009)

Let Z_n be the diagonal matrix whose entries are the n indeterminates $z_1, \ldots, z_n$. Let A by an $n \times n$ monotone column matrix. Then $\mathrm{Per}\,(J_n Z_n + A)$ is a stable polynomial in the variables $z_1, \ldots, z_n$. Specializing to $z_j \equiv z$ for all j preserves stability, hence the original conjecture follows.

The proof is via a lemma making use of the multivariate Pólya-Schur theorem to reduce to the special case, already proved, where all entries are 0 or 1.

End of Lecture 5

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