KPZ Equation Limit of Interacting Particle Systems

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joint work with Ivan Corwin and with Amir Dembo

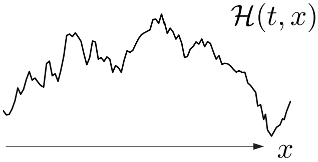
The KPZ equation

The KPZ (Kardar-Parisi-Zhang) equation

$$\partial_t \mathcal{H} = \tfrac{1}{2}\partial_{xx}\mathcal{H} - \tfrac{1}{2}(\partial_x \mathcal{H})^2 + \xi$$

is a paradigmatic equation describing random surface growth with smoothing effect and slope dependent growth.

- $\partial_{xx}\mathcal{H}$: smoothing;
- $(\partial_x \mathcal{H})^2$: slope depend. growth;
- ξ : spacetime white noise,
  $\mathbf{E}(\xi(t,x)\xi(t,x')) = \delta(t-t')\delta(x-x')$



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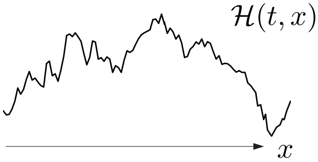

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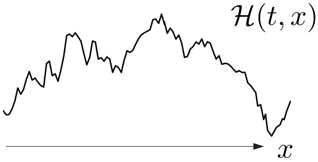

- Exhibit non-Gaussian distribution.
- Is an integrable model, with asymptotic ($t \to \infty$) distribution related to RMT.

A Discrete Example

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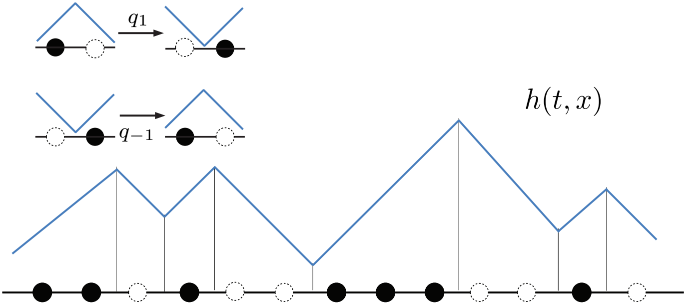

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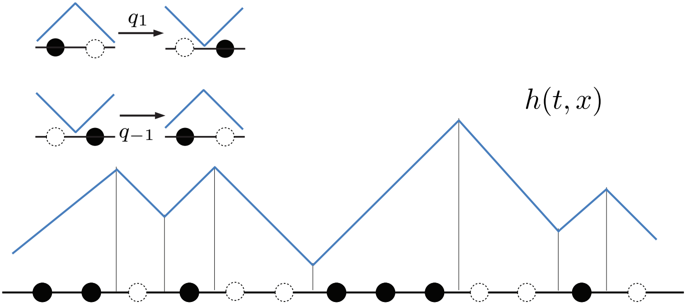

- Under the *weak asymmetry* scaling: $q_{-1} - q_1 := \varepsilon \to 0$,

$$b\varepsilon^1 (h(t\varepsilon^{-4}, x\varepsilon^{-2}) - at\varepsilon^{-3}) \Longrightarrow \mathcal{H}(t,x).$$

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- Under $\mathcal{H}_\varepsilon(t,x) := \varepsilon^b \mathcal{H}(\varepsilon^{-z}t, \varepsilon^{-a}x)$, the KPZ equation scales as

$$\partial_t \mathcal{H}_\varepsilon = \varepsilon^{2a-z}\tfrac{1}{2}\partial_{xx}\mathcal{H} - \varepsilon^{2a-z-b}\tfrac{1}{2}(\partial_x\mathcal{H})^2 + \varepsilon^{b-\frac{z}{2}+\frac{a}{2}}\xi.$$

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The weak asymmetry causes $\frac{1}{2} \mapsto \frac{\varepsilon}{2}$, so $(z,a,b) = (4,2,1)$ preserves the KPZ equation.

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- Such a convergence is conjectured to hold *universally*
  a) for models in the KPZ universality class: e.g. directed polymer, last passage percolation, q-TASEP.
  b) regardless of details of microscopic interaction,

  under scalings that preserve the KPZ equation.

The Hopf-Cole (HC) transformation

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- $\mathcal{Z}(t,x) := e^{-\mathcal{H}(t,x)}$ formally converts the KPZ equation

$$\partial_t \mathcal{H} = \tfrac{1}{2}\partial_{xx}\mathcal{H} - \tfrac{1}{2}(\partial_x \mathcal{H})^2 + \xi$$

  into the Stochastic Heat Equation (SHE)

$$\partial_t \mathcal{Z} = \tfrac{1}{2}\partial_{xx}\mathcal{Z} + \mathcal{Z}\xi.$$

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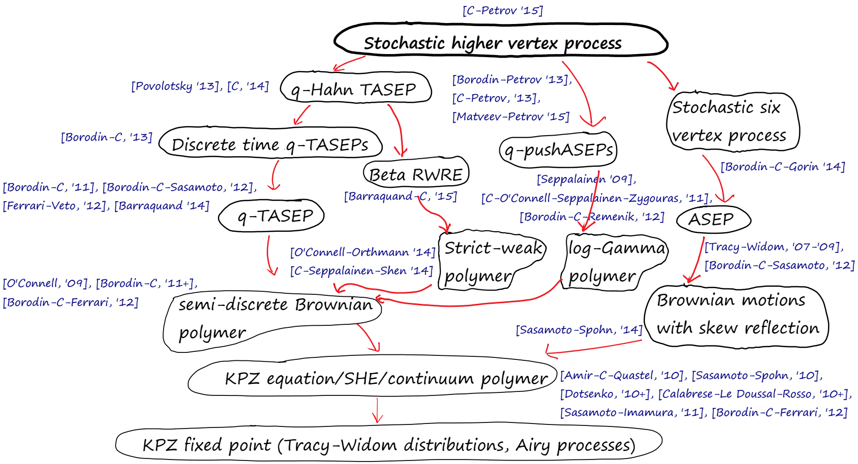
- A: for *all* integrable models.

Higher Spin Exclusion Processes (Courtesy of Corwin)

Degenerations to known integrable stochastic systems in KPZ class



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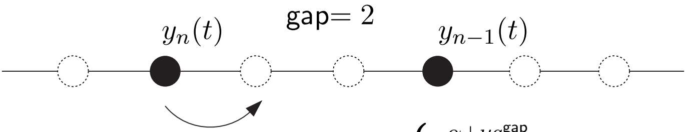
- Consider only $J = 1$ for simplicity.



with probability $\begin{cases} \frac{\alpha+\nu q^{\text{gap}}}{1+\alpha}, & \text{if } y_{n-1} \text{ moved };\\ \frac{1+\alpha(1-q^{\text{gap}})}{1+\alpha}, & \text{otherwise }. \end{cases}$

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$$\lambda := \tfrac{1+\alpha f}{1+\alpha q f} > 0,\ \mu := \tfrac{a-a'}{b-b'} > 0;$$

$$a := \tfrac{\alpha f}{1+\alpha f},\ a' := \tfrac{\alpha q f}{1+\alpha q f},\ b := \tfrac{f}{1-f},\ b' := \tfrac{\nu f}{1-\nu f},\ f := \tfrac{1-\rho}{1-\nu\rho}.$$

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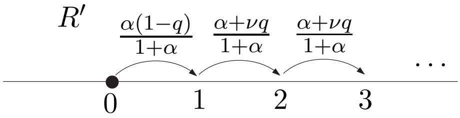
- Let $p(y,x) := \mathbf{P}(R = x - y)$ and $L(y,x) := p(y,x) - \mathbf{1}_{y=x+\mu}$ be the semigroup and pseudo generator of the the following random walk.



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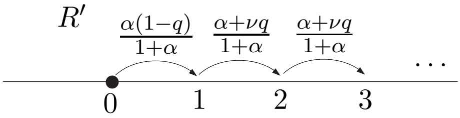

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$\sum_{n=0}^{\infty} \lambda \rho^n \, \mathbf{P}(R' = n) = 1.$

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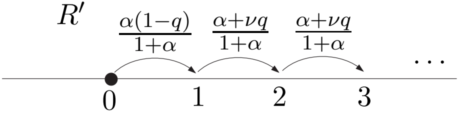

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$\sum_{n=0}^{\infty} \lambda \rho^n \, \mathbf{P}(R' = n) = 1. \qquad \mathbf{E}(R) = 0.$

The HC transformation for HSEP

Proposition (Corwin and Tsai, 2015)

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\begin{aligned}
&Z(t+1,x) - Z(t,x+\mu) \\
&\quad = \big[LZ(t,\bullet)\big](x) + Z(t,x+\mu)W(t,x+\mu),
\end{aligned}
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where $W(t,y)$ *is an (explicit)* $\mathscr{F}$*-martingale,*

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where $W(t,y)$ is an (explicit) $\mathscr{F}$-martingale, with

$$\begin{aligned} &Z(t,y)Z(t,y')\,\mathbf{E}(W(t,y)W(t,y)|\mathscr{F}(t)) \\ &\quad = \big(\tfrac{\nu+\alpha}{1+\alpha}\big)^{|y-y'|}\Theta_1(t,y_1\wedge y_2)\Theta_2(t,y_1\wedge y_2), \end{aligned}$$

$\Theta_1(t,y) := q\lambda Z(t,y) - [pZ(t,\bullet)](t)$ *and*
$\Theta_2(t,y) := -\lambda Z(t,y) + [pZ(t,\bullet)](t)$.

KPZ equation limit for HSEP, scaling

Scaling

- fix $(\rho, \alpha, \nu) \in (0,1) \times [0,1)^2$, and let $q = q_\varepsilon := e^{-\varepsilon}$.

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Here $r_* := (b - b')^{-1} > 0$, $\frac{\mu_\varepsilon}{\varepsilon} = \mu + O(\varepsilon)$, and
$t^*_\varepsilon := \varepsilon^{-1}(a^2 - {a'_\varepsilon}^2 - (a - a'_\varepsilon)(b + b'))^{-1} = t^* + O(\varepsilon)$.

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Assume the initial condition $Z_\varepsilon(0,\bullet)$ satisfies the following moment conditions: for all $n\in\mathbb{N}$ and $v\in(0,\frac{1}{2})$, there exists $C=C(v,n)<\infty$ such that

$$\mathbf{E}(Z_\varepsilon(0,x)^n) \le C_n e^{C_n|x|},$$
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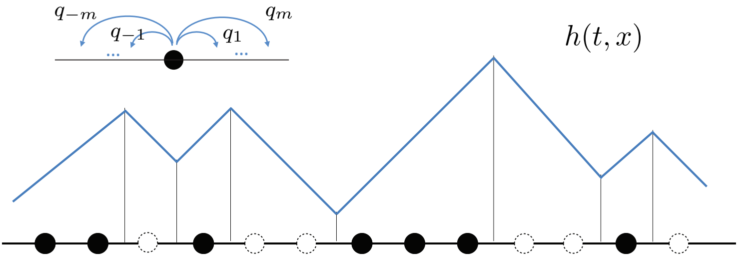
Theorem (Corwin and Tsai, 2015)

Let $\mathcal{Z}(t,x)$ be the $C(\mathbb{R}_+,\mathbb{R})$-valued solution of the SHE starting from $\mathcal{Z}_{ic}(x) \in C(\mathbb{R})$. If $Z_\varepsilon(0,\bullet) \Rightarrow \mathcal{Z}_{ic}(\bullet)$, then

$$Z_\varepsilon(\bullet,\bullet) \Rightarrow \mathcal{Z}(\bullet,\bullet).$$

Non-nearest Neighbor Exclusion Processes



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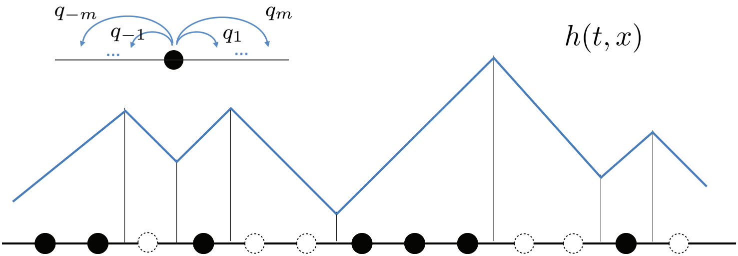

- Height function $h(t,x)$ defined for $x \in \mathbb{Z}$;
- Occupation variable $\eta(y)$ defined for $y \in \mathbb{Z} + \frac{1}{2}$, as

$$\eta(y) := \begin{cases} 1 & \text{, if occupied,} \\ -1, & \text{if empty.} \end{cases}$$

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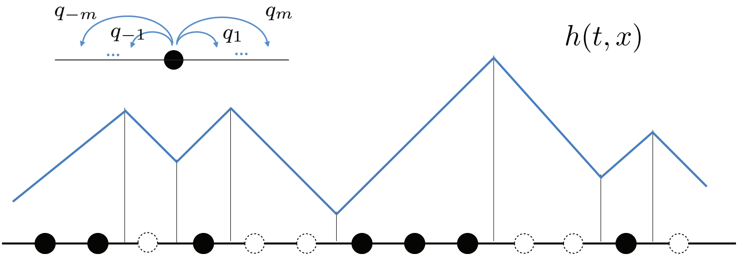

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- Weak asymmetry:

$$q_i := \tfrac{1}{2}(\kappa_i - \varepsilon^1 \gamma_i), \;\; q_{-i} := \tfrac{1}{2}(\kappa_i + \varepsilon^1 \gamma_i), \quad i = 1, \ldots, m.$$

Non-nearest Neighbor Exclusion Processes

Assume $m = 3$.

Theorem (Dembo and Tsai, 2013)

Given any $\kappa_1, \kappa_2, \kappa_3 \in (0,1)$ *with* $\kappa_1 + \kappa_2 + \kappa_3 = 1$ *and* $\gamma > 0$, *for the following choice of asymmetry*

$$\gamma_j = \gamma\Big(\textstyle\sum_{i=j}^{3} \frac{2(i-j)}{j}\kappa_i + \kappa_j\Big) + O(\varepsilon), \quad j = 1, 2, 3. \qquad (\star)$$

Let $Z(t,x) := \exp(-\gamma\varepsilon^1 h(t,x) + c_\varepsilon t)$ *and* $Z_\varepsilon(t,x) := Z(\varepsilon^{-4}t, \varepsilon^{-2}x)$. *If* $Z_\varepsilon(0,\cdot)$ *satisfies the preceding moment conditions and if* $Z_\varepsilon(0,\cdot) \Rightarrow \mathcal{Z}_{ic}(\cdot)$, *then*

$$Z_\varepsilon(\cdot,\cdot) \Rightarrow \mathcal{Z}(\cdot,\cdot).$$

Microscopic eq'n for $Z(t,x) := \exp(-\gamma\varepsilon^1 h(t,x) + C_\varepsilon t)$

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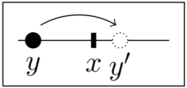
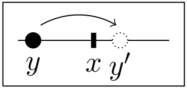
$$dZ(t,x) = \sum_{(y,y')\ni x} \Big((e^{-2\gamma\varepsilon} - 1) Z(t,x)$$



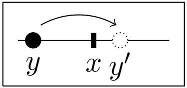
$$dZ(t,x) = \sum_{(y,y')\ni x} \Big((e^{-2\gamma\varepsilon} - 1) Z(t,x) \tfrac{1+\eta(y)}{2} \tfrac{1-\eta(y')}{2}$$



Microscopic eq'n for $Z(t,x) := \exp(-\gamma\varepsilon^1 h(t,x) + C_\varepsilon t)$

$$dZ(t,x) = \sum_{(y,y')\ni x} \Big((e^{-2\gamma\varepsilon} - 1) Z(t,x) \tfrac{1+\eta(y)}{2} \tfrac{1-\eta(y')}{2} q_{y'-y} dt$$

$$+ \, d\mathsf{MG}.$$



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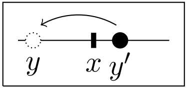
$$(e^{2\gamma\varepsilon} - 1) Z(t,x) \tfrac{1-\eta(y)}{2} \tfrac{1+\eta(y')}{2} q_{y-y'} dt \Big) + d\mathsf{MG}.$$



Microscopic eq'n for $Z(t,x) := \exp(-\gamma\varepsilon^1 h(t,x) + C_\varepsilon t)$

$$dZ(t,x) = \big(\varepsilon\mathcal{L} + \varepsilon^2\mathcal{Q}\big)dt + d\mathsf{MG},$$

where

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$$\text{where } \mathcal{L} := \sum_{i=1}^{3}\big(\tfrac{\gamma\kappa_i}{2} + O(\varepsilon)\big)\mathcal{L}_i,\ \mathcal{Q} := \sum_{k=1}^{3} b_k \sum_{(y,y')\ni x}\Big(\eta(y)\eta(y')Z(t,x)\Big).$$

$\mathcal{L}_i := (-\eta(x+\frac{1}{2}-i) - \ldots - \eta(x-\frac{1}{2}) + \eta(x+\frac{1}{2}) + \ldots + \eta(x-\frac{1}{2}+i))Z(t,x).$

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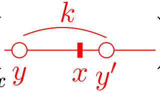
where $\mathcal{L} := \sum_{i=1}^{3}\big(\frac{\gamma\kappa_i}{2} + O(\varepsilon)\big)\mathcal{L}_i$, $\mathcal{Q} := \sum_{k=1}^{3} b_k \sum_{(y,y')\ni x}\Big($ 

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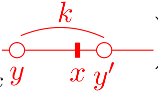
where $\mathcal{L} := \sum_{i=1}^{3}\big(\frac{\gamma\kappa_i}{2} + O(\varepsilon)\big)\mathcal{L}_i$, $\mathcal{Q} := \sum_{k=1}^{3} b_k \sum_{(y,y')\ni x}\Big($ 

$\Big)$.

- Wish to match $(\varepsilon\mathcal{L} + \varepsilon^2\mathcal{Q})$ to

$$\text{Laplacian} = \qquad (Z(t,x+i) + Z(t,x-i) - 2Z(t,x))$$

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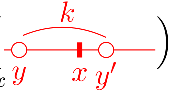
where $\mathcal{L} := \sum_{i=1}^{3}\big(\frac{\gamma\kappa_i}{2} + O(\varepsilon)\big)\mathcal{L}_i$, $\mathcal{Q} := \sum_{k=1}^{3} b_k \sum_{(y,y')\ni x}\Big($ 

$\Big)$.

- Wish to match $(\varepsilon\mathcal{L} + \varepsilon^2\mathcal{Q})$ to

$$\text{Laplacian} = \tfrac{1}{2}\sum_{i=1}^{3}\widetilde{\kappa}_i\big(Z(t,x+i) + Z(t,x-i) - 2Z(t,x)\big)$$

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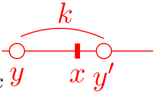
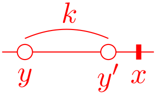
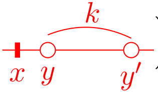
where $\mathcal{L} := \sum_{i=1}^{3}\big(\frac{\gamma\kappa_i}{2} + O(\varepsilon)\big)\mathcal{L}_i$, $\mathcal{Q} := \sum_{k=1}^{3} b_k \sum_{(y,y')\ni x}\Big($  $\Big)$.

- Wish to match $(\varepsilon\mathcal{L} + \varepsilon^2\mathcal{Q})$ to

Laplacian $= \sum_{i=1}^{3}\varepsilon\frac{\gamma\widetilde{\kappa}_i}{2}\mathcal{L}_i + \sum_{k=1}^{2}\varepsilon^2\widetilde{b}_k\Big(\sum_{y,y'\in(x-3,x)}$ 

$+ \sum_{y,y'\in(x,x+3)}$  $\Big) + \varepsilon^3\text{cubic} + \ldots$

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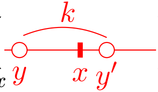
where $\mathcal{L} := \sum_{i=1}^{3}\big(\frac{\gamma\kappa_i}{2} + O(\varepsilon)\big)\mathcal{L}_i$, $\mathcal{Q} := \sum_{k=1}^{3} b_k \sum_{(y,y')\ni x}\Big($ 
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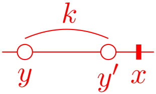

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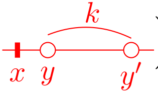

$$\Big) + \varepsilon^3\text{cubic} + \dots$$

- Match linear terms by choosing $\widetilde{\kappa}_i = \kappa_i + O(\varepsilon)$.

Gradient Condition

- How to match quadratic terms?

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Definition

We call $f(x)$ a **gradient term** if

$$f(x) = \sum_{|j|\leq 3} (g_j(x+j) - g_j(x)),$$

Gradient Condition

- How to match quadratic terms?

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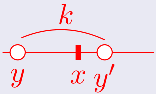
$$f(x) = \sum_{|j|\leq 3} (g_j(x+j) - g_j(x)),$$

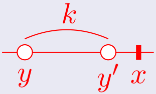
for some $g_j(x)$ of the form $g_j(x) = \eta(x+\ell_1)\cdots\eta(x+\eta_n)Z(t,x)$.

Gradient Condition

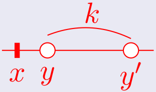
- How to match quadratic terms?

Proposition



$=$ 

$+$ *grad. term*

$=$ 

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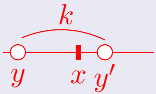

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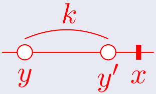

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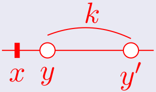

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- Matching the quadratic terms amounts to matching the coefficients: $kb_k = 2(3-k)\widetilde{b}_k$, which gives the condition $(\star)$.

Gradient Condition

- How to match quadratic terms?

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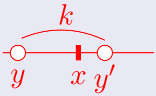
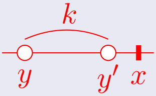


$=$ 

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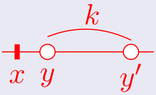
$=$ 

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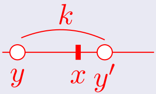
- Matching the quadratic terms amounts to matching the coefficients: $kb_k = 2(3-k)\widetilde{b}_k$, which gives the condition $(\star)$.
- Cubic terms: use $m \leq 3$.

Gradient Condition

- How to match quadratic terms?

Proposition



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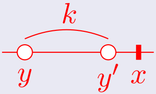

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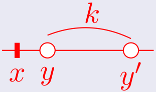

+ grad. term.

- Matching the quadratic terms amounts to matching the coefficients: $kb_k = 2(3-k)\widetilde{b}_k$, which gives the condition $(\star)$.
- Cubic terms: use $m \leq 3$.
- higher order terms: hydrodynamic limit argument.