5. Nonlinear wave equations in one dimension

goal: wave equation

$$\partial_t^2 u = \partial_x (V'(\partial_x u)) \quad , \quad x \in \mathbb{R} ,$$

with random initial data $\qquad$ energy $\int dx \{ \frac{1}{2}\pi^2 + V(\partial_x u) \}$

$\Rightarrow$ belong to the KPZ universality class

5.1 Landau-Lifshitz fluctuation theory

$\mathbb{R}^d$, ρ scalar

$$\partial_t \rho + \nabla \cdot j(\rho) = 0$$

add fluctuating currents

white

$$\partial_t \rho + \nabla \cdot (j(\rho) - \underbrace{D(\rho)\nabla \rho + B(\rho)\xi}) = 0$$

$$2DC = B^2 \qquad C \text{ static covariance}$$

$\rho = \rho_0 + u$

$$\partial_t u + \nabla \cdot (\underbrace{j'(\rho_0) u - D \nabla u}_{\text{current } j(x,t)} + B \xi) = 0$$

stationary $\quad \int_{\mathbb{R}^d} dx \, \langle j(x,t) j(0,0) \rangle_c = C D \delta(t)$

$\Rightarrow$ stability $\quad$ expand to second order. leading $\quad j(\rho_0 + u) = j(\rho_0) + j'(\rho_0) u + \frac{1}{2} j''(\rho_0) u^2$

$d=1$ stochastic Burgers

$d \geq 2$ NOT KPZ

$$\partial_t u + \nabla \cdot (j'(\rho_0) u + \underbrace{\tfrac{1}{2} j''(\rho_0) u^2}_{\text{perturbation}} - D \nabla u + B \xi) = 0$$

heat kernel

$$\int dx \langle j(x,t) j(0,0) \rangle = C D \delta(t) + C^2 j''(\rho_0)^2 \, p_t(0,0) \cong \delta(t) + t^{-d/2}$$

$d=3$ $\quad$ D is modified $\qquad$ large scale diffusive

$d=2$ $\quad$ marginal

$d=1$ $\quad$ $t^{-1/2}$ unstable $\qquad$ Resibois Pomeau 1985 $\qquad$ $t^{-2/3}$ $\qquad$ (partial truth only)

$d=1$ $\qquad \partial_t u + \partial_x \left(u^2 + \partial_x u + \xi \right) = 0$

stochastic Burgers, KPZ, stationary $\qquad$ BCFV 2015

prove $t^{-2/3}$

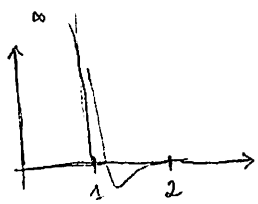
5.2 1D fluids, anharmonic chains, discrete waves

classical particles on the line

$$H = \frac{1}{2} \sum_j p_j^2 + \frac{1}{2} \sum_{i \neq j} V(q_i - q_j)$$

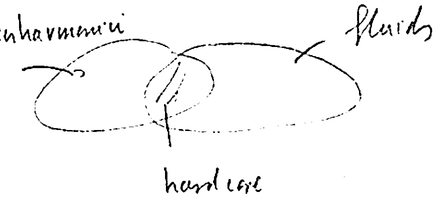
short range, stable potential

simplify



then only pair q_j, q_{j+1}, ordered

$$H = \sum \left\{ \tfrac{1}{2} p_j^2 + V(q_{j+1} - q_j) \right\}$$



now allow $V(q) = q^2 + q^3 + q^4$

Fermi - Pasta - Ulam

condition: $\int dx \, e^{-\beta (V(x) + Px)} < \infty$

$\beta > 0$, some range of P.

$$\frac{d^2}{dt^2} q_j = V'(q_{j+1} - q_j) - V'(q_j - q_{j-1})$$

discrete wave equation $\quad \partial_t^2 u = \partial_x \left(V'(\partial_x u) \right)$

- equilibrium measures

stretch $r_j = q_{j+1} - q_j$

$$\frac{1}{Z} \prod_j e^{-\beta \left(\frac{1}{2}(p_j - v)^2 + V(r_j) + P r_j \right)} \, dp_j \, dr_j \qquad \text{i.i.d.}$$

another reason for anharmonic chains.

β, v, P

$$\mathcal{P} = -\frac{1}{Z}\int V'(x)\, e^{-\beta(V(x)+\mathcal{P}x)}\, dx \quad = \text{average force, pressure}$$

- conserved fields

$$e_j = \frac{1}{2}p_j^2 + V(r_j)$$

$$\frac{d}{dt} r_j = p_{j+1} - p_j$$

$$\frac{d}{dt} p_j = V'(r_j) - V'(r_{j-1})$$

r , p , e

thermodynamic dual variables are

$$\frac{d}{dt} e_j = p_{j+1}V'(r_j) - p_j V'(r_{j-1})$$

$\mathcal{P}$ $\quad v \quad$ β

5.3 Time correlations

$\langle\cdot\rangle_{\mathcal{P},\beta}$ equilibrium measure $\quad v = 0$

conserved fields $\quad \vec{g}_j = \begin{pmatrix} r_j \\ p_j \\ e_j \end{pmatrix}$

time correlator

$$S_{\alpha\alpha'}(j,t) = \langle g_{\alpha,j}(t)\, g_{\alpha',0}(0)\rangle^c_{\mathcal{P},\beta}$$

claim: There is a transformation matrix R, independent of j,t, such that for large j,t

$$(R\, S\, R^T)(j,t) \simeq \begin{pmatrix} f_-(j,t) & 0 & 0 \\ 0 & f_0(j,t) & 0 \\ 0 & 0 & f_+(j,t) \end{pmatrix}$$

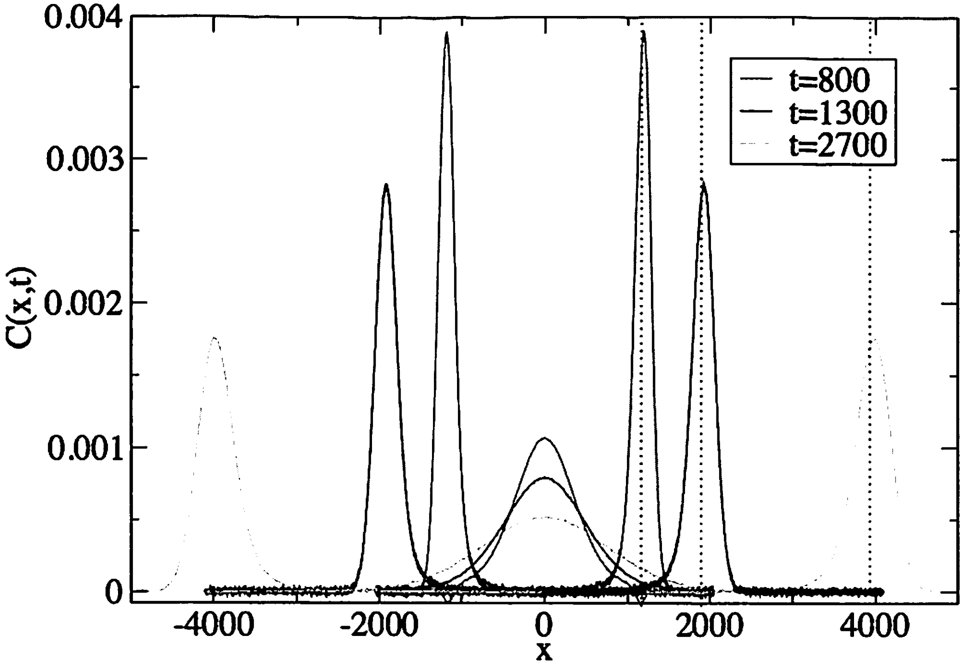
sound | heat | sound

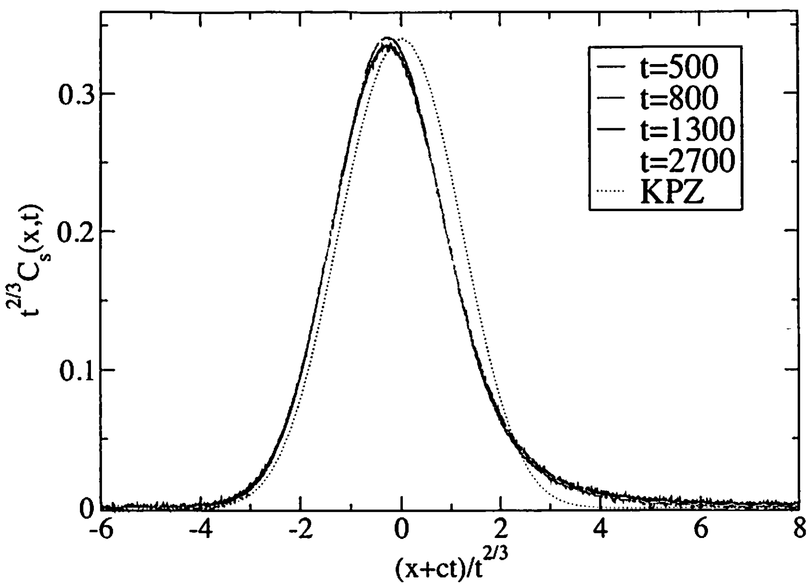
$$f_\pm(j,t) = \chi_\pm (\Gamma|t|)^{-2/3} f_{KPZ}\left((\Gamma|t|)^{-2/3}(j \mp ct)\right) \qquad c > 0 \ \text{ adiabatic sound speed}$$

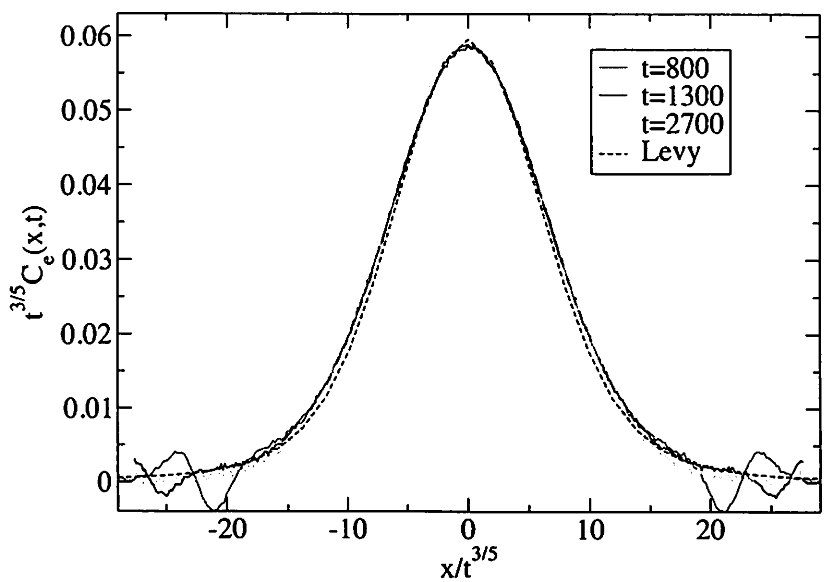
$$f_0(j,t) = \chi_0 (\Gamma_0|t|)^{-3/5} f_L\left((\Gamma_0|t|)^{-3/5} j\right)$$

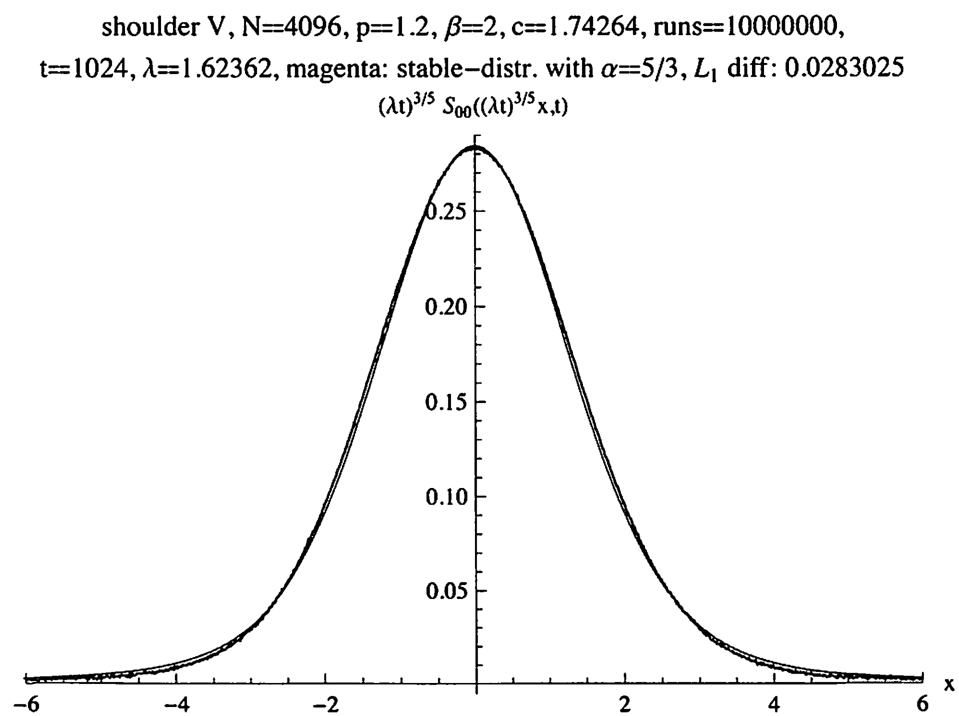
Γ, Γ_0, c are not universal

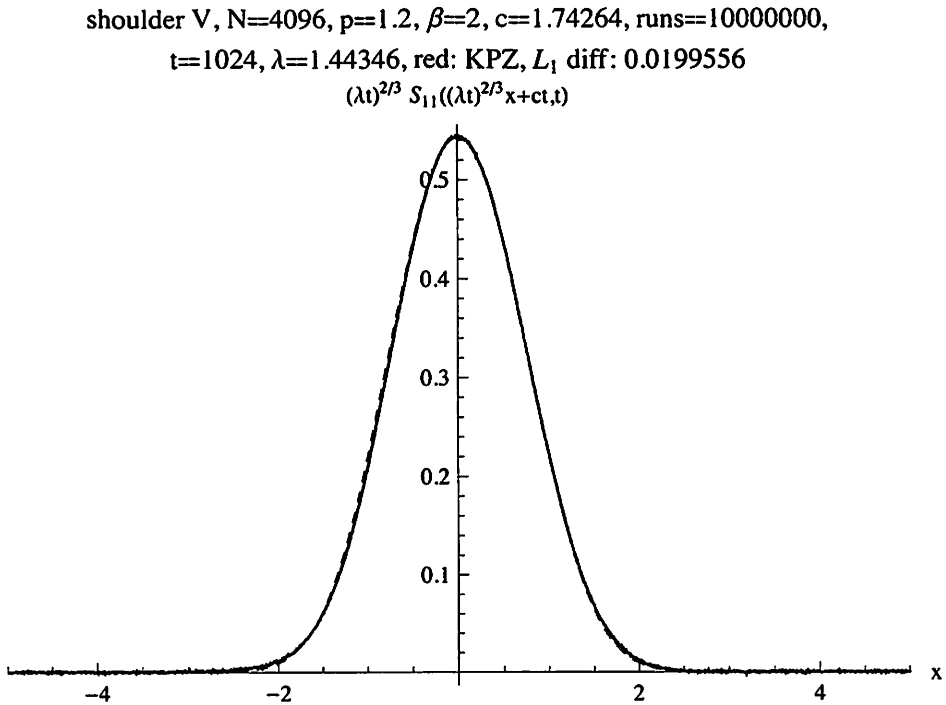
$\chi_\pm, \chi_0,$

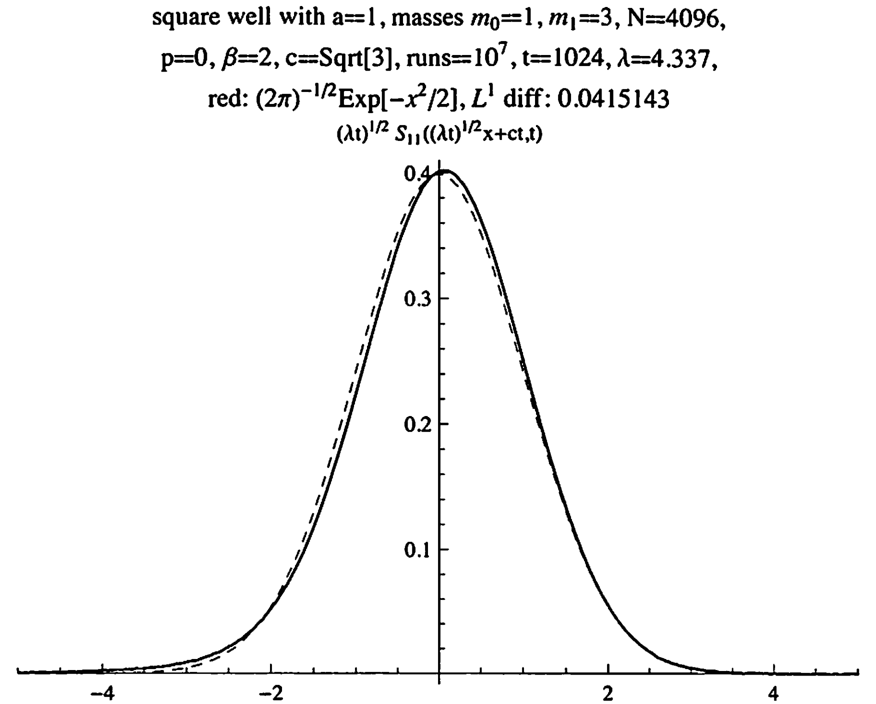

0.004
0.003
0.002
0.001
0
C(x,t)
t=800
t=1300
t=2700
-4000
-2000
0
2000
4000
x

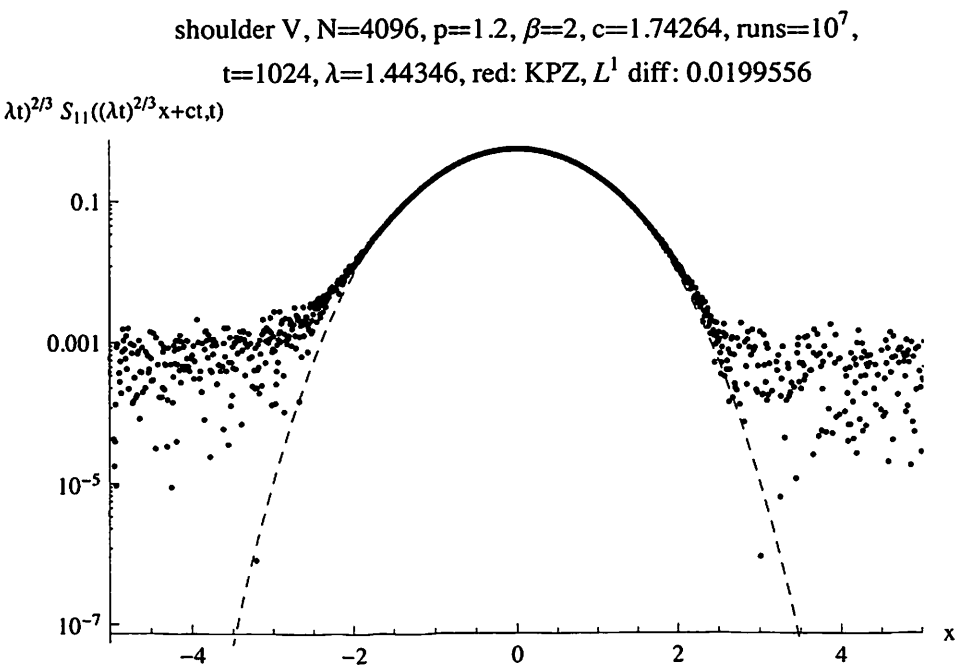

0.3
0.2
0.1
0
$t^{2/3}C_s(x,t)$
t=500
t=800
t=1300
t=2700
KPZ
-6
-4
-2
0
2
4
6
8
$(x+ct)/t^{2/3}$

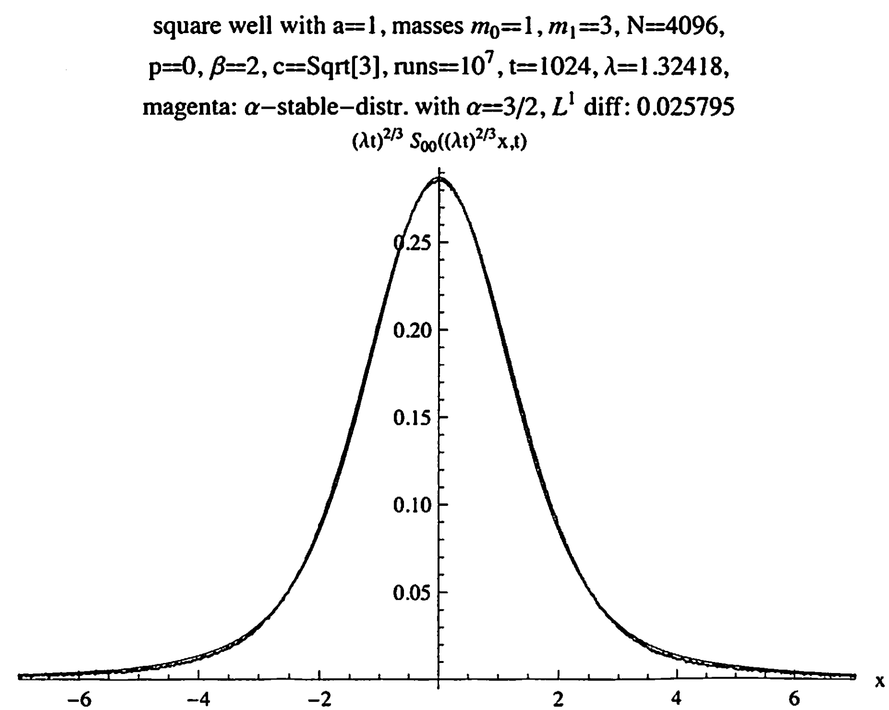

0.06
0.05
0.04
0.03
0.02
0.01
0
$t^{3/5}C_e(x,t)$
t=800
t=1300
t=2700
Levy
-20
-10
0
10
20
$x/t^{3/5}$


shoulder V, N==4096, p==1.2, β==2, c==1.74264, runs==10000000,
t==1024, λ==1.62362, magenta: stable−distr. with α==5/3, L_1 diff: 0.0283025
$(\lambda t)^{3/5}\ S_{00}((\lambda t)^{3/5}x,t)$
0.25
0.20
0.15
0.10
0.05
−6
−4
−2
2
4
6
x


shoulder V, N=4096, p=1.2, β=2, c=1.74264, runs=10000000,
t=1024, λ=1.44346, red: KPZ, L_1 diff: 0.0199556
$(\lambda t)^{2/3}\, S_{11}((\lambda t)^{2/3}x+ct,t)$
0.5
0.4
0.3
0.2
0.1
−4
−2
2
4
x


square well with a=1, masses m_0=1, m_1=3, N=4096,
p=0, β=2, c=Sqrt[3], runs=10^7, t=1024, λ=4.337,
red: $(2\pi)^{-1/2}$Exp[$-x^2/2$], L^1 diff: 0.0415143
$(\lambda t)^{1/2}\, S_{11}((\lambda t)^{1/2}x+ct,t)$
0.4
0.3
0.2
0.1
−4
−2
2
4
x


shoulder V, N=4096, p=1.2, β=2, c=1.74264, runs=10^7,
t=1024, λ=1.44346, red: KPZ, L^1 diff: 0.0199556
$\lambda t)^{2/3}\, S_{11}((\lambda t)^{2/3}x+ct,t)$
0.1
0.001
10^{-5}
10^{-7}
−4
−2
0
2
4
x


square well with a=1, masses m_0=1, m_1=3, N=4096,
p=0, β=2, c=Sqrt[3], runs=10^7, t=1024, λ=1.32418,
magenta: α−stable−distr. with α=3/2, L^1 diff: 0.025795
$(\lambda t)^{2/3}\, S_{00}((\lambda t)^{2/3}x,t)$
0.25
0.20
0.15
0.10
0.05
−6
−4
−2
2
4
6
x