Summary of previous lecture

DNLS on lattice $\mathbb{Z}^d$, wave field $\psi : \mathbb{Z}^d \to \mathbb{C}$

$$H = \frac{1}{2} \sum_{x,y} \alpha(x-y)\, \psi(y)^* \psi(x) + \frac{1}{4} \lambda \sum_x |\psi(x)|^4$$

$\lambda \geq 0$ $\quad$ $\lambda \ll 1$

$\alpha$ compact support

$$i \frac{d}{dt} \psi(x) = \sum_y \alpha(x-y)\, \psi(y) - \lambda |\psi(x)|^2 \psi(x)$$

$\hat{\alpha} = 2\pi \omega$, $\omega$ dispersion relation

random initial data $\quad$ Gauss, mean zero, $\mathbb{E}_G(\psi(x)\psi(y)) = 0$

random Wigner function $\quad W_\psi(x,p) = \sum_{y \in \mathbb{Z}^d} e^{i 2\pi p \cdot y}\, \psi^*(x + \frac{1}{2}y)\, \psi(x - \frac{1}{2}y)$

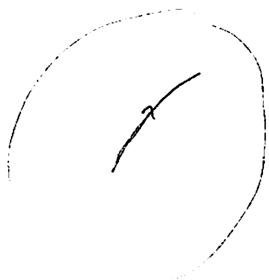
$\lambda = 0$, large scale, slow spatial variation

law of large numbers $(t=0)$ then

$$\lim_{\varepsilon \to 0} W_\psi(\lfloor \varepsilon^{-1} r \rfloor, p, \varepsilon^{-1} t) = W(r,p,t) \quad \text{in probability}$$

$$\partial_t W(r,p,t) = - \underbrace{\nabla \omega}_{\text{velocity}} \cdot \nabla_r W(r,p,t)$$

semi classical approximation



$\lambda = 0$
slow variation

next step spatially homogeneous

$\mathbb{E}(\psi(x)^* \psi(y)) = C(x-y)$ $\quad$ average Wigner $\quad W(p) = \hat{C}(p)$

$\lambda$ small

## 4.3 Collision operator, kinetic conjecture

- spatially homogeneous Gauss measure $\mathbb{E}_G$

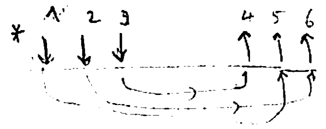
$$\mathbb{E}_G\left(\psi^*(x)\psi(y)\right) = \int dk\, e^{i2\pi k(x-y)} \underbrace{W(k)}_{\text{Wigner function}}$$

Fourier space (as distributions)

$$\mathbb{E}_G\left(\hat\psi^*(k)\hat\psi(k')\right) = \delta(k-k')\,W(k)$$

multipoint $\mathbb{E}_G\left(\prod_{j=1}^{n} \hat\psi^*(k_j)\,\hat\psi(k_j')\right) = \sum_{\pi} \prod_{j=1}^{n} W(k_j)\,\delta(k_j - k_{\pi(j)})$ — all pairings

$\psi^* \downarrow$ , $\psi \uparrow$




- equations of motion

$$\frac{d}{dt}\hat\psi(k_1) = -i\omega(k_1)\hat\psi(k_1) - i\lambda \int_{\mathbb{T}^{3d}} dk_2\,dk_3\,dk_4\, \delta(k_1+k_2-k_3-k_4)$$

translation invariance

$$\underbrace{\hat V(k_2-k_3)}_{=1 \text{ in our case}}\, \hat\psi(k_2)^*\,\hat\psi(k_3)\,\hat\psi(k_4)$$

$$\hat\phi(k,t) := e^{i\omega t}\hat\psi(k,t)$$

$\omega(k_j) = \omega_j$

$$\frac{d}{dt}\hat\phi(k_1,t) = -i\lambda \int_{(\mathbb{T}^d)^3} dk_2\,dk_3\,dk_4\, \delta(k_1+k_2-k_3-k_4) \quad \text{mod } 1$$

$$e^{+it(\omega_1+\omega_2-\omega_3-\omega_4)}\, \hat\phi(k_2,t)^*\,\hat\phi(k_3,t)\,\hat\phi(k_4,t)$$

$$\hat\phi(k_1,t) = \hat\phi(k_1,0) - i\lambda\int_0^t ds \int dk_2\,dk_3\,dk_4\,\delta(k_1+k_2-k_3-k_4)\, e^{i(\omega_1+\omega_2-\omega_3-\omega_4)s}\,\hat\phi(k_2,s)^*\,\hat\phi(k_3,s)\,\hat\phi(k_4,s)$$

Want to compute average Wigner function at time $t$

$$\mathbb{E}_G\left(\hat\psi(k_0,t)^*\psi(k_1,t)\right) = \delta(k_0-k_1)\,W_\lambda(k_1,t) = \mathbb{E}_G\left(\hat\phi(k_0,t)^*\hat\phi(k_1,t)\right)$$

We expand in $\lambda$ (Duhamel series)

iterate above

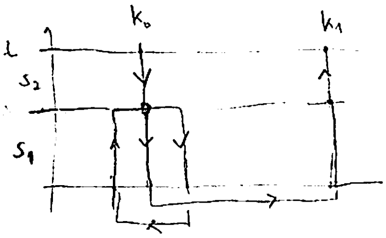
use diagrams to shorten the notation.

$n=0$ $\quad W(k_1)$ $\qquad \downarrow \phi^*$ $\quad \uparrow \phi$

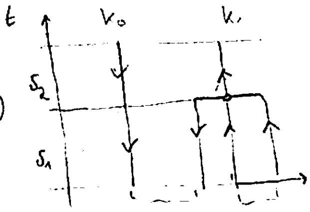
$n=1$

$(+i\lambda)$

$\delta(s_1+s_2-t)$




$(-i\lambda)$

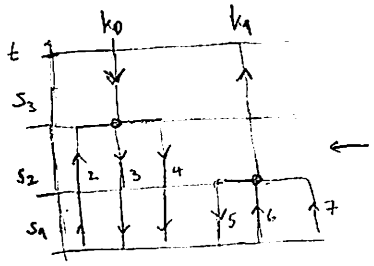



$\uparrow e^{i\omega t}$ $\downarrow e^{-i\omega t}$

2 trees $\times$ 2 pairings $= 4$ diagrams $\quad$ all phase factors cancel

$i\lambda - i\lambda = 0$

$n=2$

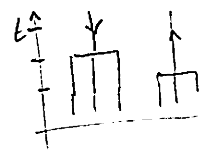



$\delta(s_1+s_2+s_3-t)$

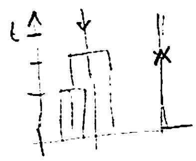
$\leftarrow$ product of phase factors $\quad e^{\pm i\,\omega_\alpha s_\beta}$

- pairings
- translation invariance $\quad \delta(k_1+k_2-k_3-k_4)$, $\quad \delta(k_1+k_5-k_6-k_7)$ $\qquad$ Kirchhoff rule
- phase factor $\quad s_1$ and $s_3$ $\quad$ phase factor $=0$

trees




$\underline{2}$ $\quad +$



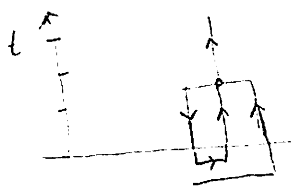

$\underline{6}$ $\quad = 8$

$= 48$ diagrams

pairings $\quad 6$

zero momentum loop

$O(t^2)$




phase factors cancel

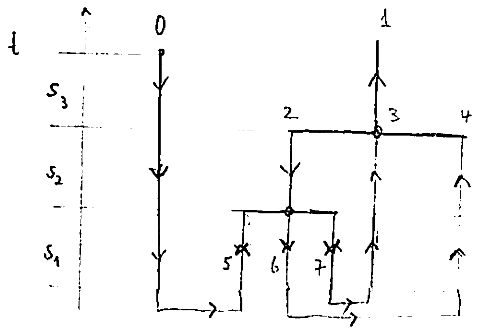
$t^2\,\delta(k_0-k_1)\,W(k_1)\int dk_1 dk_2\, W(k_1)\,W(k_2)$

sum exactly to 0

16 diagrams are of order $\lambda^2 t$

only 4 classes

I explain only one diagram




$$\lambda^2 \int W_1 \; W_4 \cdot W_3 \quad \delta_{05}\, \delta_{46}\, \delta_{37} \quad \delta(k_0+k_2-k_3-k_4)\, \delta(k_2+k_5-k_3-k_7) \quad e^{i s_2(-\omega_1-\omega_2+\omega_3+\omega_4)}$$

$$= \lambda^2 W_1 \cdot \int dk_2 dk_3 dk_4 \; \delta(k_1+k_2-k_3-k_4) \int_0^t ds_2 \int_0^{s_2} ds_1 \; e^{i(s_2-s_1)(-\omega_1-\omega_2+\omega_3+\omega_4)} \; W_3 W_4 \; \delta(k_0-k_1)\, \delta(k_1+k_2-k_3-k_4)$$

<u>assumption</u>

$$p_t(x) = \int_{\mathbb{T}^d} dk \; e^{i\omega(k)t} \, e^{i2\pi x\cdot k}$$

$l_3$-dispersivity

$$\| p_t \|_3^3 = \sum_{x\in\mathbb{Z}^d} | p_t(x) |^3 \leq C \langle t \rangle^{-1-\delta} \qquad \delta > 0$$

<u>Lemma</u>. $\sigma, \sigma' = \pm 1$, $f \in l_1(\mathbb{Z}^{3d})$ $\quad \hat{f}(k, k', k_0-k-k')$

$$\Big| \int_{(\mathbb{T}^d)^2} dk\, dk' \; e^{it(\omega(k) + \sigma\omega(k') + \sigma'\omega(k_0-k-k'))} \Big| \leq C \, \|f\|_1 \langle t \rangle^{-1-\delta}$$

long / short

With this lemma we conclude that for large $t$

$$\lambda^2 W_1 \int dk_2 dk_3 dk_4 \; \delta(k_1+k_2-k_3-k_4) \int_0^t ds_2 \int_0^{s_2} ds_1 \; \cos\big((s_2-s_1)(\omega_1+\omega_2-\omega_3-\omega_4)\big) \; W_1 W_3 W_4$$

$$\cong \lambda^2 t \int dk_2 dk_3 dk_4 \; \delta(k_1+k_2-k_3-k_4)\, \delta(\omega_1+\omega_2-\omega_3-\omega_4) \; W_1 W_3 W_4$$

where $\quad \delta(x) = \lim_{\beta\to 0_+} \frac{\beta}{\pi} \frac{1}{x^2+\beta^2}$

interchange

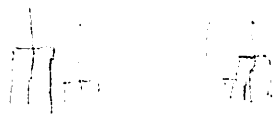
<u>Proof of Lemma</u>. Back to position space

$$\sum_{x_1,x_2,x_3\in\mathbb{Z}^d} f(x_1,x_2,x_3) \sum_{y\in\mathbb{Z}^d} e^{-i2\pi k_0(y+x_3)} \; p_{-t}(y+x_3-x_1)\; p_{-\sigma' t}(y+x_3-x_2)\; p_{-\sigma t}(y)$$

Hölder $\quad \|g_1 g_2 g_3\|_1 \leq \|g_1\|_3 \|g_2\|_3 \|g_3\|_3$

$$\leq \sum_{x_1,x_2,x_3 \in \mathbb{Z}^d} |f(x_1,x_2,x_3)| \quad \|p_t\|_3^3 \qquad \leq C \langle t\rangle^{-1-\delta}.$$

all diagrams yield
to second order



$$\lambda^2 t \int dk_2 dk_3 dk_4\ \delta(k_1+k_2-k_3-k_4)\,\delta(\omega_1+\omega_2-\omega_3-\omega_4)\big(W_2W_3W_4 - W_1(W_2W_3+W_2W_4-W_3W_4)\big)$$

$$W_1W_2W_3W_4\left(\frac{1}{W_1}+\frac{1}{W_2}-\frac{1}{W_3}-\frac{1}{W_4}\right)$$

$$= \lambda^2 t\ \mathcal{C}(W)(k_1)$$

This leads to the

<u>Conjecture (kinetic limit)</u>. The limit

$$\lim_{\lambda\to 0} W_\lambda(k,\lambda^{-2}t) = W(k,t).$$

exists and $W(k,t)$ is solution of

$$\partial_t W(k,t) = \mathcal{C}(W(t))(k) \qquad W(k,0) = W_0.$$

<u>4.4 Did we find the right equation</u>

- no momentum conservation

umklapp $\quad k_1+k_2 = k_3+k_4 \mod 1 \qquad$ Umklapp processes

- number and energy conservation

$$\partial_t \int dk\, W(k,t) = \int dk_1\, \mathcal{C}(W(t))(k_1) = 0$$

$$\partial_t \int dk\, \omega(k) W(k,t) = \int dk_1\, \omega(k_1)\, \mathcal{C}(W(t))(k_1)$$

$$= \int dk_1 dk_2 dk_3 dk_4\ \delta(\underline{k})\,\delta(\underline{\omega})\ W_1W_2W_3W_4\left(\frac{1}{W_1}+\frac{1}{W_2}-\frac{1}{W_3}-\frac{1}{W_4}\right)\underbrace{\omega_1}_{\frac{1}{4}(\omega_1+\omega_2-\omega_3-\omega_4)}$$

$$= 0$$

- the collisions are defined through the solution

$$\omega(k_1)+\omega(k_2) = \omega(k_3)+\omega(k_1+k_2-k_3)$$

can have many solution branches
classical particles unique solution labelled by $\hat{\omega}$.