3. Kinetic fluctuation theory

3.1 Background

kinetic scale, test function ϕ, empirical measure on $T_1 = \Lambda \times \mathbb{R}^3$, hard spheres diameter εa

$$n^{\varepsilon}(\phi,t) = \varepsilon^2 \sum_{j=1}^{N} \phi(q_j(t), v_j(t)) \qquad \mathbb{E}(N) = \varepsilon^{-2},$$

small ε

$$n^{\varepsilon}(\phi,t) \cong \int dq\, dv\, \phi(x,v)\, \underbrace{f(x,v,t)}_{\text{solution to B-equation}} + \varepsilon\, \underbrace{\xi(\phi,t)}_{\text{fluctuations}}$$

fluid $\overset{\text{why}}{\smile}$ density fluctuations

$\Longleftrightarrow$ light scattering

equilibrium, non-equilibrium

T_1 || T_2

How to guess the structure of $\xi(\phi,t)$?

guess of transport equation is easy, fluctuations, less clear

one expects: (i) CLT Gaussian infinite-dimensional

(ii) Markov property in t

REMINDER: $\mathbb{R}^n$ Gauß-Markov

stationary

$$d\xi_t = A\xi_t\, dt + B\, dW_t$$

$\xi_t \in \mathbb{R}^n$, $A \in M_n$, $B \in M_n$, $\{W_{t,j} \mid j \in 1,\dots,n\}$ i.i.d. standard BM, "white noise"

- stationary measure,

$$\xi_t(\phi) = \xi_0(e^{A^T t}\phi) + \int_0^t ds\, \langle e^{A^T(t-s)}\phi, B\, dW_s\rangle$$

$$\mathbb{E}(\xi_t(\phi)\xi_t(\psi)) = \mathbb{E}(\xi_0(e^{A^T t}\phi)\, \xi_0(e^{A^T t}\psi)) + \int_0^t ds\, \langle e^{A^T(t-s)}\phi, BB^T e^{A^T(t-s)}\psi\rangle$$

$\|e^{At}\| \le e^{-\gamma t}$, $\gamma > 0$, strict contraction

Then

$$\lim_{t\to\infty} \mathbb{E}(\xi_t(\phi)\xi_t(\psi)) = \int_0^\infty dt \langle \phi, e^{At} B B^T e^{A^T t} \psi \rangle = \langle \phi, C \psi \rangle$$

covariance of Gaussian invariant measure

$\Leftrightarrow$

$$-(AC + CA^T) = BB^T .$$

In our application

$A = R - D$ and

$RC = -CR^T$ reversible part

$DC = CD^T$ dissipative part $\Longrightarrow$ $2DC = BB^T$

Onsager (1931) Nobel prize not his words

A = linearized macroscopic equation, hence $A(t)$

first step stationary case, thermal equilibrium, macroscopic equation trivializes, but fluctuations do not

$\leadsto$ C equilibrium, A linearization $\leadsto$ noise strength B.

apply to low density gas $\Lambda = \mathbb{R}^3$

thermal equilibrium ; Gibbs measure $\cong$ Poisson intensity $\varepsilon^{-2}\rho$ $\rho > 0$

fluctuation field

$$\xi^\varepsilon(\phi,t) = \varepsilon^{-1}\Big(\varepsilon^2 \sum_j \phi(q_j(t), v_j(t)) - \int dq\,dv\, \rho\, h_\beta(v)\, \phi(q,v)\Big)$$

$\xi^\varepsilon(\phi,t)$ is space-time stationary

(a little bit formal) existence of infinite volume Gibbs measure, o.k. at low density

existence of infinite volume dynamics in $\mathbb{R}$ ok, $\mathbb{R}^2$ ok, higher dimensions difficult

Presutti, 1976 equilibrium dynamics L^2-theory

Dobrushin, Sinai, Sukhov, 1989,

Poisson

$$\lim_{\varepsilon\to 0} \mathbb{E}\left(\xi^\varepsilon(\phi,0)\, \xi^\varepsilon(\psi,0)\right) = \int dx\, dv\, \rho\, h_\beta(v)\, \phi(x,v)\, \psi(x,v) = \langle \phi, C\psi \rangle$$

linearization

$$\rho h_\beta(v) + \phi(q,v)$$

linear order

$$\partial_t(\rho h_\beta + \phi) = -v\cdot\nabla_x(\rho h_\beta+\phi) + Q(\rho h_\beta+\phi, \rho h_\beta+\phi) \overset{\checkmark}{\simeq}$$

$$= -v\cdot\nabla_x \phi + a^2 \int dv_1 \int d\hat\omega\, (\hat\omega\cdot(v-v_1))_+\, \rho \left[h_\beta(v_1')\,\phi(v') + h_\beta(v')\,\phi(v_1') \right.$$

$$\left. - h_\beta(v_1)\,\phi(v) - h_\beta(v)\,\phi(v_1) \right]$$

$$= R\phi - D\phi$$

check $\quad RC = -CR^T, \quad DC = CD^T$

and

$$\langle \psi, BB^T\phi\rangle = \frac{1}{2}\int dx \int dv_1\, dv_2 \int d\hat\omega\, (\hat\omega\cdot(v_1-v_2))_+\, \rho h_\beta(v_1)\, \rho h_\beta(v_2)$$

$$\left[\psi(x,v_1') + \psi(x,v_2') - \psi(x,v_1) - \psi(x,v_2) \right]$$

$$\left[\phi(x,v_1') + \psi(x,v_2') - \psi(x,v_1) - \psi(x,v_2) \right]$$

- noise is δ-correlated in time (by definition)
- δ-correlated in space
- correlated in velocities

BB^T has five eigenvalues 0 in $L^2(\mathbb{R}^3, dv)$

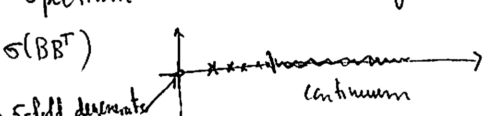
$1,\ \vec{v},\ \frac{1}{2}v^2$ these are the conserved fields

their fluctuations are surface-like, much smaller

spectrum has been analysed $\mathcal{H} = L^2(\mathbb{R}^3, dv\ \ )$

|| structure is $BB^T = \underset{\substack{\uparrow\\ \text{compact}}}{K} + \underset{\substack{\uparrow\\ \text{multiplication}}}{V}$

$\sigma(BB^T)$



H. Grad

Cercignani, Illner, Pulvirenti

I do not know a simple formula for B by itself

time-dependent profile $f(x,v,t)$

linearization as $f(x,v,t) + \phi(x,v,t)$

noise-strength is time-dependent $B(t)\, B(t)^T$

local Poisson, natural guess

$\rho h_\beta(v_1)\ \rho h_\beta(v_2)$ is replaced by $f(x,v_1,t)\, f(x,v_2,t)$.

In our context Onsager's insight can be partially proved.

3.2 Time-dependent covariance in equilibrium

see van Beijeren, Lanford, Lebowitz, HS 1980 finite kinetic time

T. Bodineau, I. Gallagher, L. Saint-Raymond, 2016 dimension $d=2$
all kinetic times (even longer)

Theorem hard spheres, either $d \geq 2$, $|t| \leq \frac{1}{\varepsilon} t_0$, or $d = 2$; all t

$\phi, \psi \in L^2(\mathbb{R}^3, \rho h_\beta(v)\, dv)$. Then

$$\lim_{\varepsilon \to 0} \mathbb{E}\big(\xi^\varepsilon(\phi,t)\, \xi^\varepsilon(\psi,0)\big) = \langle \phi, e^{Rt - D|t|} C \psi \rangle_{L^2}$$

Proof:

$$\mathbb{E}\big(\xi^\varepsilon(\phi,t)\, \xi^\varepsilon(\psi,0)\big) = \mathbb{E}\Big[\varepsilon^2 \sum_j \phi(q_j(t), v_j(t)) \underbrace{\Big(\sum_j \psi(q_j, v_j) - \mathbb{E}\big(\sum_j \psi(q_j,v_j)\big)\Big)}_{\text{defines a signed measure}} \Big]$$

with rescaled correlation functions

average
first correlation function

$$r_n^\varepsilon(x_1,\dots,x_n) = \sum_{j=1}^n \psi(x_j)\, \tau^\varepsilon_{eq,n}(x_1,\dots,x_n)$$

decay as $e^{-|q|/\varepsilon}$ range of V

$$+ \varepsilon^{-2}\int dx_{n+1}\, \psi(x_{n+1})\Big(\tau^\varepsilon_{eq,n+1}(x_1,\dots,x_{n+1}) - \tau^\varepsilon_{eq,n}(x_1,\dots,x_n)\,\tau^\varepsilon_{eq,1}(x_{n+1})\Big)$$

we write $\tau^\varepsilon_{eq,n}(x_1,\dots,x_n) = (q_1,\dots,q_n)\prod_{j=1}^n h_\beta(v_j)$

From the low density expansion (see book by Ruelle)

$$\sup_{\vec q\in\mathbb{R}^{3n}} \rho\varepsilon^{-2}\int_{\mathbb{R}^3} dq\, \big|\tau^\varepsilon_{eq,n+1}(\vec q, q) - \tau^\varepsilon_{eq,n}(\vec q)\,\tau^\varepsilon_{eq,1}(q)\big| \le C\varepsilon$$

uniformly in the volume

⇝ uniform bound for $0\le t\le \varepsilon^{-1}t_0$

⇝ term by term convergence

limit correlation functions satisfy Boltzmann hierarchy with initial conditions

$$r_n(x_1,\dots,x_n) = \sum_{j=1}^n \psi(x_j)\prod_{j=1}^n \rho\, h_\beta(v_j)$$

$$\cong \Big(e^{\sum_{j=1}^n \psi(x_j)} - 1\Big)\prod_{j=1}^n \rho\, h_\beta(v_j)$$

to linear order

⇝

$$r_n(x_1,\dots,x_n,t) = \sum_{j=1}^n \big(e^{(A-D)t}\psi\big)(x_j)\prod_{j=1}^n \rho\, h_\beta(v_j)$$

$n=1$ yields the desired limit. □

- A similar argument confirms also the time-dependent covariance.
- as stochastic process, open problem
- dynamical correlations on the kinetic scale ≫ interparticle distance.

3.3 Beyond the kinetic time scale

novel theme, but with preexamples, e.g. Erdős, Yau on quantum particle in a random potential

simple Example random Lorentz gas

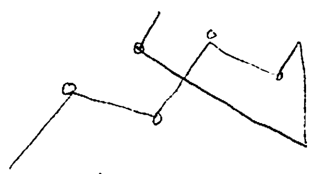
$\mathbb{R}^d$ Poisson point process, intensity $\varepsilon^{-(d-1)}$ $\{q_j\}$

$\hat{=}$ center of scatterers, diameter of scatterer is ε

mechanical motion in this random potential

$$\frac{d^2}{dt^2} q^\varepsilon(t) = \sum_j F_\varepsilon(q^\varepsilon(t) - q_j), \quad q^\varepsilon(0)=0 \qquad F^\varepsilon = -\nabla V_\varepsilon, \quad V_\varepsilon(x) = V(x/\varepsilon)$$

V radial
here hard core



$$q^\varepsilon(t) = \int_0^t ds\, v^\varepsilon(s) \qquad v = \dot{q}$$

specific observable

$\lim \varepsilon \to 0$

$0 \le t \le T$ as a stochastic process

$$\lim_{\varepsilon \to 0} v^\varepsilon(t) = v(t)$$

$v(t)$ is a random jump process, $|v(t)| = |v(0)| = 1$,
e.g. $d=3$
$v(t)$ exponential waiting, uniformization of velocities over the sphere.

holds for all T.

Q: Over which time span remains $q^\varepsilon(t)$ close to $q(t) = \int_0^t ds\, v(s)$
$\uparrow$ mechanical $\uparrow$ random flight

more sharply: take $\alpha > 0$, consider

$$\frac{1}{\sqrt{\varepsilon^{-\alpha} t}}\, q^\varepsilon(\varepsilon^{-\alpha} t)$$

If this stays close, then $q(t)$ is already Brownian motion $b(t)$. from Markov limit

Problem: For which α

$$\lim_{\varepsilon \to 0} \frac{1}{\sqrt{\varepsilon^{-\alpha} t}}\, q^\varepsilon(\varepsilon^{-\alpha} t) = b(t)$$

recent result of B. Toth claims $0 < \alpha < 1$, presumably $0 < \alpha < 2$.

BGS 2016 prove convergence to linearized Boltzmann for

t fixed, $N = \varepsilon^{-1}\gamma$, $\gamma \to \infty$ but $\gamma \ll \sqrt{\log\log\log N}$

on that scale one observes already the hydrodynamic approximation to the linearized Boltzmann equation

$$r_1^{(N)}(x,v,0) = h_\beta(v)\, \psi(x,v,0)$$

with

mean zero, internal energy, $d=2$

$$\psi(x,v,0) = \rho_0(x) + u_0(x)\cdot v + \theta_0(x)\left(\beta \tfrac{1}{2} v^2 - 1\right)$$

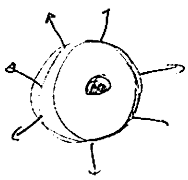
Then

$$\lim_{\substack{N\to\infty \\ \gamma\to\infty}} r_1^{(N)}(x,v,t) = h_\beta(v)\left(\rho(x,t) + \vec{u}(x,t)\cdot v + \theta(x,t)\left(\beta \tfrac{1}{2} v^2 - 1\right)\right)$$

and ρ, u, θ solves

$$\partial_t \rho + \nabla_x \cdot \vec{u} = 0$$
$$\partial_t \vec{u} + \nabla_x(\rho + \theta) = 0$$
$$\partial_t \theta + \nabla_x \cdot u = 0$$

(sound and energy waves)



(linearized hydrodynamics).

nonlinear waves

Lecture 6, Oct 31, 10:00