increased angular set

\[ T_{n}^{\alpha} \delta(t) = \begin{cases} q_{i} - V_{j} = q_{j} - V_{j} & \text{for some pair } (i,j), \quad 0 \leq t \leq 2 \end{cases} \]

\[ T_{n}^{\delta} \rightarrow T_{n} \quad \text{uniformly on compact sets of } \mathbb{R} \setminus T_{n}^{\text{sing}}(t) \]

### 2.8 Landford's theorem

no boundary conditions \( \Lambda = \mathbb{R}^3 \)

sequence of probability measures \( \mu_{n} \) on \( T = U_{0}^{\infty} (\mathbb{R}^3 \times \mathbb{R}^3)^{n} \), correlation functions \( \Phi_{n}^{\delta} \)

rescaled correlation functions \( \Phi_{n}^{\delta} = c^{\frac{1}{n}} \Phi_{n}^{\epsilon} \)

all \( n = 1, 2, \ldots \)

\( c(1) \) uniform bound

\[ 0 \leq \Phi_{n}^{\delta}(x) \leq C n^{-1} \prod_{j=1}^{n} (1 + \Phi_{n}^{\epsilon}(V_{j})) \]

\( c(2) \) There exist continuous function \( \Phi_{n} : T_{n} \rightarrow \mathbb{R}^{+} \) such that uniformly on compacta of \( T_{n} \setminus T_{n}^{\text{sing}}(0) \)

\[ \lim_{n \rightarrow 0} \Phi_{n}^{\delta} = \Phi_{n} \]

Then \( 1 \) for \( 0 \leq t < t_{0} \) (\( t_{0} \) mean free time)

\[ \lim_{n \rightarrow 0} \Phi_{n}^{\delta}(t) = \Phi_{n}(t) \]

uniformly on compacta of \( T_{n} \setminus T_{n}^{\text{sing}}(t_{0}) \)

\[ \Phi_{n}(t) : T_{n} \rightarrow \mathbb{R}^{+}, \quad \text{are continuous and } \]
\[ \partial_t \tau_n(x) = - \sum_{j=1}^n \nabla \cdot \left( v_j \nabla q_j \right) \tau_n(x) + C_{n,1} \tau_n(x) \]

\[ (C_{n,n+1} g_j)(x_1, ..., x_n) = \sum_{j=1}^n a_j \int d\omega \int d\omega' \left( \omega \cdot (x_{n+1} - x_j) \right) \]

\[ g \left( \ldots, q_j, v_j', \ldots, q_{n-1}, v_{n-1} \right) - g \left( \ldots, q_j, v_j, \ldots, q_{n-1}, v_{n-1} \right) \]

(calcul Boltzmann hierarchy).

\( \text{What is the kinetic equation?} \)

We assume in addition law of large numbers

\[ \text{also } \text{molecular chaos } \Rightarrow \text{ actually different } \]

\[ \tau_2^{\text{eff}}(q_1, v_{1}, q_2, v_{2}) = \tau_2^{\text{eff}}(v_{1}, v_{2}) \]

for \( v_{1}, v_{2} \) incoming

\[ \text{then is very close to the singular set } \]

\[ \text{law of large number } \Rightarrow \text{ Boltzmann function, } \]

\[ \tau_1(x, t) = \tau_1(x_1) \tau_1(x) \]

\[ \tau_n(x, t) = \prod_{j=1}^n \tau_1(q_j, v_j, t) \]

\[ \text{Boltzmann function.} \]

\[ \tau_n(x, t) = \prod_{j=1}^n \tau_1(q_j, v_j, t) \]

\[ \text{in this case the relation is the Boltzmann hierarchy is} \]

\[ \tau_n(x, t) = \prod_{j=1}^n \tau_1(q_j, v_j, t) \]

and

\[ \partial_t \phi(x, v, t) = -v \cdot \nabla \phi + a^2 \int d\omega \int d\omega' \left( \omega \cdot (v - v') \right) \left( \phi(q, v') (\delta(q', v) - \phi(q', v)) \right) \]

\( x \in \mathbb{R}^3 \)
What happens if \( \tau_2 + \tau_3 \tau_4 \) is solved analytically

exchangeable measure, defined, Hewitt, Savage

In general, there is a decomposition measure \( dA(r) \) such that

\[
\tau_n(\frac{3}{2}) = \int dA(r) \prod_{i=1}^n \tau(x_i)
\]

e.g. probability \( p \) one has \( \tau \) and \( 1-p \) \( \bar{\tau} \)

Then solution to Boltzmann equation

\[
\tau_n(x, t) = \int dA(r) \prod_{i=1}^n \tau(x_i, t)
\]

solution to weak-mean transport equation

On the kinetic scale:
the initial profile is random, but evolves deterministically

2.9 Time reversibility, one-sided non

Lampert Theorem \( t > 0 \). For \( t < 0 \), some recall by macroscopic equation or

\[
\partial_t p = -v \cdot \nabla q \left( \mathcal{Q}(p, q) \right)
\]

collision operator

\( t > 0 \). Convergence at time \( t \) is weaker than at time \( 0 \), singular \( x \) is increasing not

\[
0 \quad \text{p.d}(t) \quad \text{p.d}(2t)
\]

Given an a-priori bound \( |\tau_n| \leq C \frac{n}{\prod_{i=1}^n q(r)} \)

(\text{NOT available})

one can reach arbitrary kinetic times

But at time \( t \) one cannot backtrack
at time \( t \)
\[ v_j \to -v_j \]
\[ f(x, v, t) \to f(x, -v, t) \]

The theorem does not apply
cut the singular set

The convergence cannot be \( L^2 \) or \( L^\infty \) both at \( t = 0 \) and \( t > 0 \)

\[ \text{Can be} \]
\[ \text{deterministic} \]
\[ \text{velocity reversal} \]

\[ \text{randomness in initial conditions is required} \]
\[ \text{how much difficult question} \]

example

\[ \text{Lattice, random velocities} \]
\[ f(x, v) = f(x, v) \]

\[ \text{law of large numbers, does transport equation hold?} \]

\[ \text{should be OK, not covered by theorem} \]

\subsection*{2.10 Local Poisson}

\[ t \]
\[ x \]

Reference point

Local statistics

\[ n^\varepsilon (x + \varepsilon \Delta x, t) \]

Corollary

Under the conditions of Lundstedt theorem \( \tau_2 = \tau_3 \tau_1 \cdots \), then in the sense of moments

\[ \lim_{\varepsilon \to 0} n^\varepsilon (x + \varepsilon \Delta x, t) = n_{\nu}(x, 0)(\Delta x, \Delta t) \]

Poisson field on \( \mathbb{R}^d \) with

intensity \( f(x, v, t) \, dv \, dt \)

Berouk

\( (x, t) \) reference point

\[ p = f(x, v, t) \nu(\Delta x, \Delta t) \]
\[ \mathbb{P}(h^2(x + \varepsilon^{\Delta_1}, \Delta_2, t)) = \sum_{x + \varepsilon^{\Delta_1}} \mathbb{P}(h^2(x + \varepsilon^{\Delta_2}, \Delta_1, t)) \]

\[ \sum_{x + \varepsilon^{\Delta_1}} \mathbb{P}(h^2(x + \varepsilon^{\Delta_2}, \Delta_1, t)) \rightarrow 1 \Delta \mathbb{P}(h^2(x + \varepsilon^{\Delta_2}, \Delta_1, t)) \]

Similarly for higher moments.

2.11 Smooth potential

Smooth potential, finite range a.

I. Gallagher, I. Saint-Raymond, and Torria, Benjamin

Zürich Lectures on Advanced Mathematics, 2014

PDE techniques, restrictive


Uniform bounds, oscillating

Theorem (Palatini, Saffiro, S Simonella). Uniform bounds, oscillating.

Potential $V$ satisfies

1) $V(x) = 0$ for $|x| > a$, radial.

2) $V \in C^2(\mathbb{R}^3)$ or $V \in C^2(\mathbb{R}^3 \setminus \{0\})$.

3) $V(x) \rightarrow \infty$ as $x \rightarrow \infty$.

(iii) The potential is stable (as for Gibbs measures) (too strong?)
\[
\sum_{i+3=1}^n V(q_i - q_i) \geq -B \; n.
\]

Then convergence as in Landau's theorem.

The "right" observables ( ).

Fixed \( N \) receive correlation functions

\[
T_N(x_1, \ldots, x_n) = \int_{\mathbb{R}^n} \Phi_N(x_1, \ldots, x_N) \; dx_1 \ldots dx_N
\]

Modified correlation

\[
\tilde{T}_N(x_1, \ldots, x_n) = \int_{S(x_1, x_n) \subset \mathbb{R}^n} dx_1 \ldots dx_N \tilde{\Phi}_N
\]

\( S(x_1, x_n) = \{ q, v \in \mathbb{R}^d \mid |q - q_i| \geq \varepsilon_i \; \text{for all} \; i = 1, \ldots, n \} \)

This results in hierarchy

\( \tilde{T}_n \) couples to \( \tilde{T}_{n+1} \); \( n = 1, 2, \ldots \).

But only \( \tilde{T}_{n+1} \) survives as \( \varepsilon \rightarrow 0 \).

Collision history as for hard spheres,

- add new sphere \( \hat{\omega}_1, \hat{V}_2 \);
- solve Newton's equation of motion for two particles.

Recollisions disappear as \( \varepsilon \rightarrow 0 \).

Limit to the Boltzmann hierarchy with mechanical scattering cross section.
major difficulty: collision time is unbounded

- omit

require suitable expansion
similar to cluster expansions in statistical mechanics

- omit

counter example of Uchiyama
discrete velocity space!
\( \text{dangerous because of recollisions} \)
dimension 2, particles are diamonds

\[ \text{Pour velocities, } \tilde{\mathbf{v}}_i = i \mathbf{v} \]

elastic collisions

\[ \text{Macroscopic equation (Broadwell equation)} \]
\[ \partial_t f(x,v) = -v \cdot \nabla_x f(x,v) + 4\varepsilon \left( f(x+rv) f(x-rv) - f(x,v) f(x-v,v) \right) \]
\[ v = \pm \ell_1, \pm \ell_2 \]
\[ R \text{ rotates } b \pi/2 \]

well-posed kinetic equation, H-Theorem, equilibrium

Grad limit: side length \( \varepsilon a \sqrt{2} \), \( N = \varepsilon^{-1} \)

law of large numbers \( \tau_i^k \) has a limit \( \tau \)
and \( \tau_2^k \to \tau \)

\( \text{from collision history} \)

\[ \text{Omit} \]

enough randomness

\( \text{KolmogorovHack} \)

\( \text{Limit remains independent} \)
however collisions in a single particle tree remain
example

\[ \begin{array}{c}
\text{collision with probability } > 0
\end{array} \]

- stochastic model with spatial structure

Rozakhonov, Terzer 1997
Caprino, Pulvirenti 1995

law of large numbers
arbitrary kinetic time

particles in \( \mathbb{R} \): free motion, collision at coincident
independent probabilities: \( 1 - \varepsilon \) continue to move freely
\( \varepsilon \) collision \( N \sim \varepsilon^{-1} \)

discrete velocities
\( \nu = -\nu \)

collision \( k(\nu, \nu | \nu', \nu') > 0 \)

symmetry in the mechanical system
\( k(\nu, \nu | \nu', \nu') = k(\nu', \nu | \nu, \nu') = k(\nu, \nu | \nu', \nu') \)

possibly nonconservation laws

\[ \begin{array}{c}
\text{Caprino Pulvirenti special collision rule}
\end{array} \]

\[ (1, 1, 2) \sim (1, -2) \]
\[ (1, -2) \sim (2, -2) \]

time reversibility \( k(\nu, \nu | \nu', \nu') = k(-\nu, \nu | \nu', \nu') \)