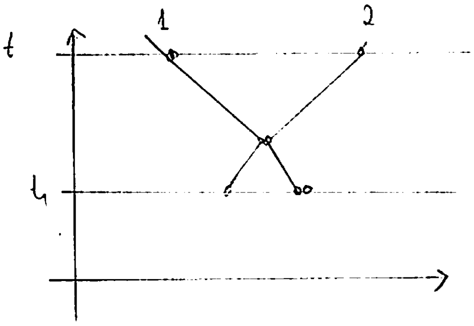
$\underline{\tau_n^\varepsilon}$



increased singular set

$$T_n^{sing}(t) = \{\, q_i - v_i s = q_j - v_j s \quad \text{for some pair } (i,j)\,,\ 0 < s < t \,\}$$

$\tau_n^\varepsilon \longrightarrow \tau_n$ uniformly on compact sets of $T_n \setminus T_n^{sing}(t)$

2.8 Lanford's theorem

no boundary conditions $\Lambda = \mathbb{R}^3$ (more generally point processes on $\mathbb{R}^6$)

sequence of probability measures μ_ε on $T = \bigcup_{n \geq 0} (\mathbb{R}^3 \times \mathbb{R}^3)^n$, correlation functions ρ_n^ε

rescaled correlation functions $\tau_n^\varepsilon = \varepsilon^{2n} \rho_n^\varepsilon$

all $n = 1, 2, \ldots$

(C1) uniform bound.

$$0 \leq \tau_n^\varepsilon(\underline{x}) \leq C \prod_{j=1}^{n} (\rho\, h_\beta(v_j))$$

(C2) There exist continuous functions $\tau_n : T_n \to \mathbb{R}_+$ such that uniformly on compacta of $T_n \setminus T_n^{sing}(0)$

$$\lim_{\varepsilon \to 0} \tau_n^\varepsilon = \tau_n$$

Then, for $0 \leq t < t_0$ ($= \frac{1}{5}$ mean free time)

$$\lim_{\varepsilon \to 0} \tau_n^\varepsilon(t) = \tau_n(t)$$

uniformly on compacta of $T_n \setminus T_n^{sing}(t_0)$

$\tau_n(t) : T_n \to \mathbb{R}_+$ are continuous and satisfy.

free motion

$$\partial_t \tau_n(t) = -\sum_{j=1}^{n} v_j \cdot \nabla_{q_j} \tau_n(t) + C_{n,n+1} \tau_{n+1}(t)$$

$$(C_{n,n+1} g)(x_1,..,x_n) = \sum_{j=1}^{n} a^2 \int dv_{n+1} \int d\hat{\omega} \quad (\hat{\omega}\cdot(v_{n+1} - v_j))_+$$

$$\Big(g(\ldots, \underbrace{q_j, v_j', \ldots, q_j, v_{n+1}'} \;- g(\ldots \underbrace{q_j, v_j, \ldots, q_j, v_{n+1}})\Big).$$

(called Boltzmann hierarchy).

- Where is the kinetic equation?

We assume in addition law of large numbers

also molecular chaos ⟸ is actually different (refers to incoming collisions)

$$\tau_2^\varepsilon(q_1, v_1, q_2+\varepsilon a, v_2) = \tau_1^\varepsilon(q_1,v_1)\,\tau_1^\varepsilon(q_2+\varepsilon a, v_2)$$

for v_1, v_2 incoming

|| this is very close to the singular set || not covered by law of large numbers

omit omit

law of large number, on limit function,

$$\tau_2(x_1,x_2) = \tau_1(x_1)\,\tau_1(x_2)$$

Boltzmann f function.

$$\rightsquigarrow \quad \tau_n(\underline{x}) = \prod_{j=1}^{n} f(q_j, v_j)$$

in this case the solution to the Boltzmann hierarchy is

$$\tau_n(\underline{x}, t) = \prod_{j=1}^{n} f(q_j, v_j, t)$$

and

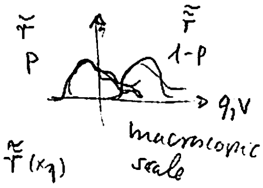
$$\partial_t f(x,v,t) = -v\cdot\nabla_x f + a^2 \int dv_1 \int d\hat{\omega} \; (\hat{\omega}\cdot(v_1 - v))_+ \big(f(x,v_1')f(x,v') - f(x,v_1)f(x,v)\big) \qquad x \in \mathbb{R}^3$$

What happens if $\tau_2 \neq \tau_1 \tau_1$? riddle analytically

exchangeable measure, de Finetti, Hewitt, Savage

In general, there is a decomposition measure $dQ(\tau)$ such that

$$\tau_n^{x_1,\dots,x_n}(\vec{x}) = \int dQ(\tau) \prod_{j=1}^{n} \tau(x_j)$$



e.g.: probability p one has $\tilde{\tau}$ and $1-p$ $\tilde{\tilde{\tau}}$ $\tau(x_1) = p\,\tilde{\tau}(x_1) + (1-p)\,\tilde{\tilde{\tau}}(x_1)$

Then solution to Boltzmann hierarchy

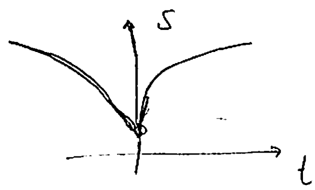
$$\tau_n^{x_1,\dots,x_n}(\underline{x},t) = \int dQ(\tau) \prod_{j=1}^{n} \underbrace{\tau(x_j,t)}_{\text{solution to nonlinear transport equation}}$$

On the kinetic scale:

the initial profile is random, but evolves deterministically

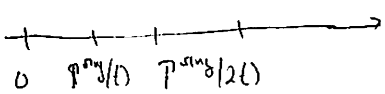
2.9 Time reversibility, one-sidedness

Lanford Theorem $t > 0$. For $t < 0$, same result by macroscopic equation is



$$\partial_t f = -v \cdot \nabla_q f - \underbrace{Q(f,f)}_{\text{collision operator}}$$

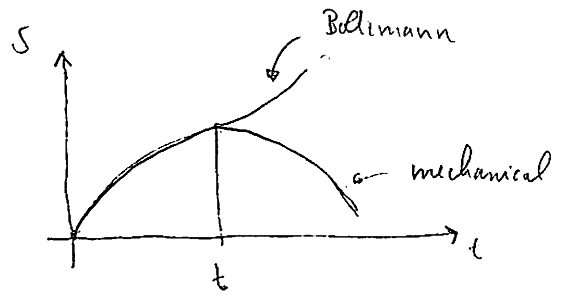
$t > 0$. Convergence at time t is weaker than at time 0, singular set is increasing in t



Given an a-priori bound $|\tau_n| \le C \prod_{j=1}^{n}(\dots)$ (NOT available)

one can reach arbitrary kinetic times

But at time t one cannot back track



theorem does not apply
Cut the singular set

at time t $\quad v_j \rightsquigarrow -v_j \qquad f(x,v,t) \rightsquigarrow \underbrace{f(x,-v,t)}_{\text{new initial data}}$

The convergence cannot be $\underbrace{L^2 \text{ (or } L^\infty)}_{\text{this convergence is invariant under } v \rightsquigarrow -v}$ both at $t=0$ and $t>0$

randomness in initial conditions is required

|| how much difficult question ||

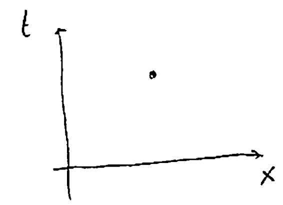
Cannot be deterministic velocity reversal

example

lattice, i.i.d velocities $\quad f(x,v) = \rho\, h(v)$

law of large numbers, does transport equation hold?
should be ok, not covered by theorem

2.10 Local Poisson



reference point
local statistics

$$n^\varepsilon((x+\varepsilon^{2/3}\Delta_1)\times\Delta_2, t)$$

<u>Corollary</u>: Under the conditions of Lanford's theorem $T_2 = T_1 T_1 \ldots$, then in the sense of moments

$$\lim_{\varepsilon\to 0} n^\varepsilon((x+\varepsilon^{2/3}\Delta_1)\times\Delta_2, t) = n_{f(x,v,t)}(\Delta_1\times\Delta_2)$$

Poisson field on $\mathbb{R}^c$ with intensity $f(x,v,t)\, dq\, dv$

(x,t) reference point

Lebesgue

$\rho = \int f(x,v,t)\,dv$

Proof: $\mathbb{E}\left(n^{\varepsilon}(x + \varepsilon^{2/3}\Delta_1, \Delta_2, t) \right) = \int_{x+\varepsilon^{2/3}\Delta_1} dq_1 \int_{\Delta_2} dv_1 \; \rho_1^{\varepsilon}(q_1, v_1, t)$ (omit

$$= \int_{\Delta_1} dq_1 \int_{\Delta_2} dv_1 \; \underbrace{\varepsilon^2 \rho_1^{\varepsilon}(x + \varepsilon^{2/3} q_1, v_1, t)}_{\tau_1^{\varepsilon}(x+\varepsilon^{2/3} q_1, v_1, t)} \longrightarrow |\Delta_1| \int_{\Delta_2} dv_1 \, f(x, v_1, t)$$

$$\mathbb{E}\left(n^{\varepsilon}(x+\varepsilon^{2/3}\Delta_1, \Delta_2, t) \, n^{\varepsilon}(x+\varepsilon^{2/3}\Delta_1', \Delta_2', t) \right)$$

$$= \int_{\Delta_1} dq_1 \int_{\Delta_1'} dq_2 \int_{\Delta_2} dv_1 \int_{\Delta_2'} dv_2 \;\; \varepsilon^2 \rho_2^{\varepsilon}(x + \varepsilon^{2/3} \underbrace{q_1, v_1}, x + \varepsilon^{2/3} q_2, v_2, t)$$

close to the singular set
not close enough

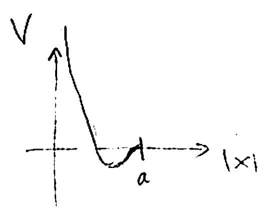
$$+ \int_{\Delta_1 \cap \Delta_1'} dq_1 \int_{\Delta_2 \cap \Delta_2'} dv_1 \;\; \varepsilon^2 \rho_1 (x + \varepsilon^{2/3} q_1, v_1, t)$$

$$\underset{\varepsilon \to 0}{\longrightarrow} |\Delta_1| \, |\Delta_1'| + \int_{\Delta} dv_1 \int_{\Delta'} dv_2 \; f(x, v_1, t) f(x, v_2, t) + |\Delta_1 \cap \Delta_1'| \int_{\Delta_2 \cap \Delta_2'} dv_1 \; f(x, v_1, t)$$

omit

Similarly for higher moments. $\square$

2.11 Smooth potential



smooth potential, finite range a

I. Gallagher, L. Saint-Raymond, B. Texier
Benjamin

Zürich Lectures in Advanced Mathematics 2014 PDE techniques, restrictive

Pulvirenti, C. Saffirio, S. Simonella, Rev Math. Phys. 2014
collision histories

Theorem (Pulvirenti, Saffirio, S. Simonella). Uniform bounds, $0 \le t \le t_0$ as in Lanford

Potential V satisfies

(i) $V(x) = 0$ for $|x| > a$, radial

(ii) $V \in C^2(\mathbb{R}^3)$ or $V \in C^2(\mathbb{R}^3 \setminus \{0\})$ $V(x) \to \infty$ as $x \to 0$

(iii) The potential is stable (as for Gibbs measures) (too strong?)

meaning $\sum_{i\neq j=1}^{n} V(q_i - q_j) \geq -Bn$.

Then convergence as in Lanford's theorem

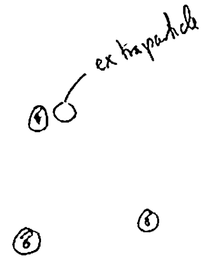
The "right" observables ( )

Fixed N naive correlation functions

$$\tau_n(x_1,\dots,x_n) = \int_{\mathbb{R}^{6(N-n)}} f_N(x_1,\dots,x_N)\, dx_{n+1}\dots dx_N \qquad \int f_N = 1$$

modified correlations

$$\tilde{\tau}_n(x_1,\dots,x_n) = \int_{S(x_1,\dots,x_n)^{\otimes(N-n)}} dx_{n+1} \dots dx_N\, f_N$$



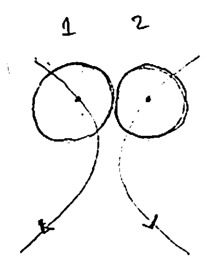
$$S(x_1,\dots,x_n) = \{ q,v \in \mathbb{R}^6 \mid |q-q_j| > \varepsilon,\ \text{all } j=1,\dots,n \}$$

⇝ results in hierarchy

$\tilde{\tau}_n$ couples to $\tilde{\tau}_{n+j}$, $j=1,2,\dots$.

but only $\tilde{\tau}_{n+1}$ survives as $\varepsilon \to 0$

collision history as for hard spheres

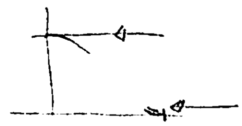


- add new sphere $\hat{\omega}, v_2$
- solve Newton's equation of motion for two particles

recollisions disappear as $\varepsilon \to 0$

limit is the Boltzmann hierarchy with mechanical scattering cross section

major difficulty   collision time is unbounded
grazing collisions



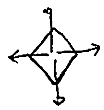
Omit

requires suitable expansions
similar to cluster expansions in statistical mechanics

- counter example of Uchiyama

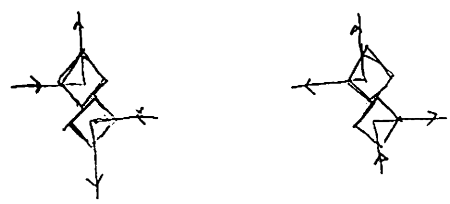
discrete velocity space !   (dangerous because of recollisions)

dimension 2, particles are diamonds.



four velocities $\vec{v}_i = \pm e_j$

elastic collisions



macroscopic equation (Broadwell equation)

$$\partial_t f(x,v) = -v \cdot \nabla_x f(x,v) + 4a\Big(f(x, Rv) f(x,-Rv) - f(x,v) f(x,-v) \Big)$$

$v = \pm e_1, \pm e_2$

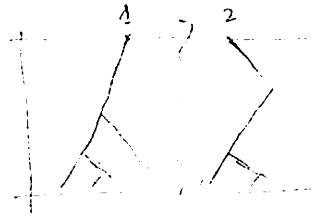
R rotates by $\pi/2$.

well-posed kinetic equation, H-theorem, equilibrates

Grad limit   side length $\varepsilon a/\sqrt{2}$, $N = \varepsilon^{-1}$ (because of two dimensions)

law of large numbers τ_1^ε has a limit τ from collision histories
and $\tau_2^\varepsilon \to \tau\tau$

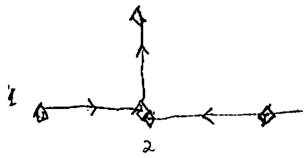
Omit



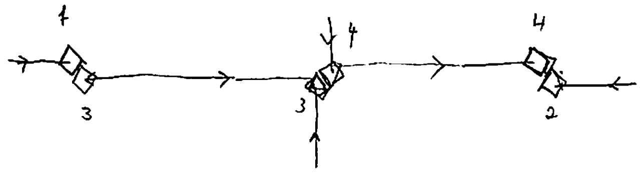
enough randomness
that the two backwards
histories remains independent

however recollisions in a single particle tree remain

example



1 and 2 are correlated



recollision with probability > 0

~~omit~~

perhaps omit

∘ stochastic models with spatial structure

Rezakhanlou, Tarver 1997 — law of large numbers

Caprino, Pulvirenti 1995 — arbitrary kinetic time

particles in $\mathbb{R}$, free motion, collision at coinciden.

independent probabilities $1-\varepsilon$ continue to move freely

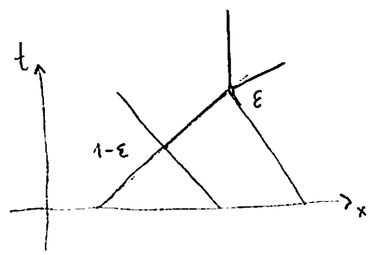
ε collision $\qquad N = \varepsilon^{-1}$

discrete velocities set $v = -v$ $\qquad$ collision $K(v_1, v_2 | v_1', v_2') \geq 0$

symmetries as the mechanical system

$$k(v_1, v_2 | v_1', v_2') = K(v_2, v_1 | v_1', v_2') = k(v_1, v_2 | v_2', v_1') \quad \times$$

possibly conservation laws



Caprino Pulvirenti special collision rule

$(\pm 1, \pm 2)$ $\qquad (2,-1) \rightsquigarrow (1,-2)$

$\qquad\qquad\qquad (1,-2) \rightsquigarrow (2,-1)$

time reversibility $k(v_1, v_2 | v_1', v_2') = k(-v_1', -v_2' | -v_1, -v_2)$