2.6 hard sphere dynamics | fixed number N $\quad \Lambda = \mathbb{R}^3$

diameter a

phase space

| switch to velocities v_j | $\quad$ mass 1

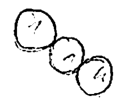
$\Gamma = \{ \underline{q}, \underline{v} \mid \quad |q_i - q_j| \geq a \quad$ all pairs $i,j \}$

boundary $\quad \partial\Gamma = \{ \underline{q}, \underline{v} \mid$ at least one pair $|q_i - q_j| = a, \ (v_i - v_j)\cdot \hat{\omega}_{ij} \geq 0 \}$

incoming

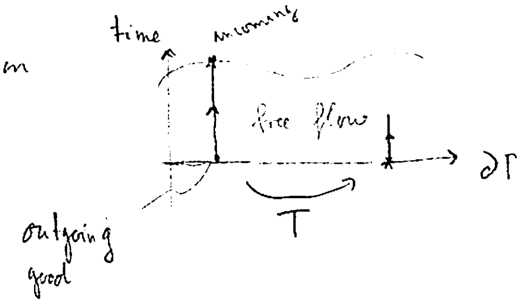
$\partial\Gamma$ has bad points

$\Longrightarrow$ multiple contacts



for bad points the dynamics remains undefined

flow under a function



on $\partial\Gamma$ one has the induced Lebesgue measure, T invariant pair for collision i,j),

$$|\hat{\omega}\cdot(v_i - v_j)| \, d\hat{\omega} \, dq_i \, dv_i \, dv_j \prod_{\ell \neq i,j} dq_\ell \, dv_\ell$$

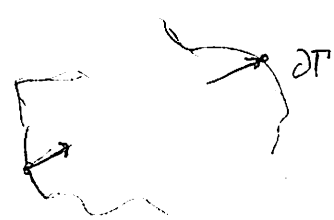
T is almost surely defined.

Implies:

Theorem (Alexander 1975) There is a set $\Gamma_0 \subset \Gamma$ (of Lebesgue measure 0) such that on $\Gamma \setminus \Gamma_0$ the hard sphere flow $\{T_t, t \in \mathbb{R}\}$ is well-defined with the properties

$$T_t \, T_s = \quad {}_{t+s}, \quad T_0 = 1$$

and T_t leaves the Lebesgue measure invariant.



The generator is $\quad - \sum_{j=1}^{n} v_j \cdot \nabla_{q_j} \quad$ with reflecting b.c. functions which are C^1 through a collision

2.7 Pseudo trajectories (collision histories), a branching "process"

Omit

$$\partial_t \rho_1(q_1,v_1,t) = -\sum_{n=0}^{\infty} \frac{1}{n!} \int dx_2 \, dx_{n+1} \sum_{j=1}^{n+1} v_j \cdot \nabla_{q_j} f_{n+1}(x_1,\dots,x_{n+1},t)$$

$$= - \sum_{n=0}^{\infty} \frac{1}{n!} \int dx_2 \, dx_{n+1} \, v_1 \cdot \nabla_{q_1} f_{n+1} \qquad (i)$$

$$- \sum_{n=0}^{\infty} \frac{1}{n!} \int dx_2 \, dx_{n+2} \, v_2 \cdot \nabla_{q_2} f_{n+2} \qquad (ii)$$

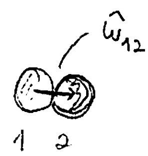
omit partial integrations

---

integral identity

$$v_1 \cdot \nabla_{q_1} \int_{\{|q_1-q_2|\geq a\}} dq_2 \, g(q_1,q_2) = \int_{\{|q_1-q_2|\geq a\}} dq_2 \, v_1 \cdot \nabla_{q_1} g(q_1,q_2)$$

$$= a^2 \int d\hat{\omega}_{12} \, v_1 \cdot \hat{\omega}_{12} \, g(q_1, q_1 + a\hat{\omega}_{12})$$



$$\partial_t \rho_1(q_1,v_1) = -v_1 \cdot \nabla_{q_1} \rho(q_1,v_1) - a^2 \int d\hat{\omega}_{12} \, dv_2 \, v_1 \cdot \hat{\omega}_{12} \, \rho_2(q_1,v_1,q_1+a\hat{\omega}_{12},v_2) \qquad (i')$$

$$- \int_{|q_1-q_2|\geq a} dq_2 \, dv_2 \, v_2 \cdot \nabla_{q_2} \rho_2(q_1,v_1,q_2,v_2)$$

$$(ii)$$

$$- a^2 \int dq_2 \int d\hat{\omega}_{23} \int dv_2 \, dv_3 \, v_2 \cdot \hat{\omega}_{23} \, \rho_3(q_1,v_1,q_2,v_2,q_2+a\hat{\omega}_{23},v_3)$$

---

integral identity

$$\int_{|q_1-q_2|\geq a} dq_2 \, \nabla_{q_2} f(q_2) = - \int d\hat{\omega} \, \hat{\omega} \, f(q_1+a\hat{\omega})$$

$$\partial_t \rho_1(q_1,v_1) = -v_1 \cdot \nabla_{q_1} \rho(q_1,v_1) + a^2 \int d\hat{\omega}_{12} \, dv_2 \, (v_2-v_1)\cdot\hat{\omega}_{12} \, \rho_2(q_1,v_1,q_2+a\hat{\omega},v_2) - I$$

$$I = a^2 \int dq_2 \int d\hat{\omega}_{23} \int dv_2 \, dv_3 \, v_2 \cdot \hat{\omega}_{23} \, \rho_3(q_1,v_1,q_2,v_2,q_2+a\hat{\omega}_{23},v_3)$$

$v_2 \leftrightarrow v_3$, interchange outgoing and incoming, continuity through collision, 2 and 3,

$(v_2-v_3)\cdot\hat{\omega}_{23}$, integration v_2', v_3', continuity through collision

longer

Isometry

$$- a^2 \int dq_2 \int d\hat{\omega}_{23} \int dv_2' dv_3' (v_2'-v_3')\hat{\omega}_{23} \, \rho_3(q_1,v_1,q_2,v_2',q_2+a\hat{\omega}_{23},v_3') \qquad I = -I$$

outgoing

$$0$$

result for $n=1$ $\quad \partial_t \rho_1 = -v_1\cdot\nabla_{q_1}\rho_1 + a^2\, \mathcal{C}_{12}\,\rho_2$

integrated version

1 particle flow

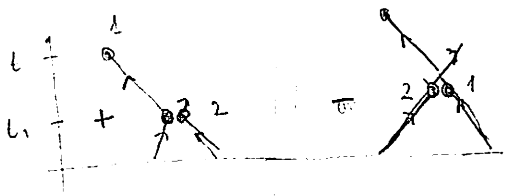
initial datum

$$\rho_1(q_1,v_1;t) = S_1(t)\,\tilde{\rho}_1(q_1,v_1) + \int_0^t dt_1\, S_1(t-t_1)\, a^2 \int dv_2 \int_{S^2} d\hat{\omega}_{12}\, (v_2-v_1)\cdot\hat{\omega}_{12}$$

$$\rho_2(q_1,v_1,q_1+a\hat{\omega}_{12},v_2,t_1)$$ — 2 particle phase space

set of non-zero Lebesgue

graphical representation



both loss and gain

$\hat{\omega} = \hat{\omega}_{j\,n+1}$ $\qquad X_j = (q_j, v_j)$

We need the full hierarchy

$$\partial_t \rho_n(x_1,\ldots,x_n,t) = -\sum_{j=1}^n v_j\cdot\partial_{q_j}\rho_n + \sum_{j=1}^n \int dv_{n+1}\int d\hat{\omega}\;\hat{\omega}\cdot(v_{n+1}-v_j)$$

$$\underbrace{\rho_{n+1}(x_1,\ldots,x_n,q_j+a\hat{\omega},v_{n+1})}_{\mathcal{C}_{n,n+1}}$$

critique, the boundary values could be ill-defined

integrated

n-particle flow

$$\rho_n(t) = S_n(t)\,\rho_n(0) + \int_0^t dt_1\, S_n(t-t_1)\,\mathcal{C}_{n,n+1}\,\rho_{n+1}(t_1)$$

iterate

$\leadsto$

$$\rho_n(x_1,\ldots,x_n,t) = \sum_{m=0}^{\infty} \int_{0\le t_m \le \ldots \le t_1 \le t} \left(S_n(t-t_1)\,\mathcal{C}_{n,n+1} \ldots \mathcal{C}_{n+m-1,n+m}\,\tilde{\rho}_{n+m}\right)(x_1,\ldots,x_n)$$

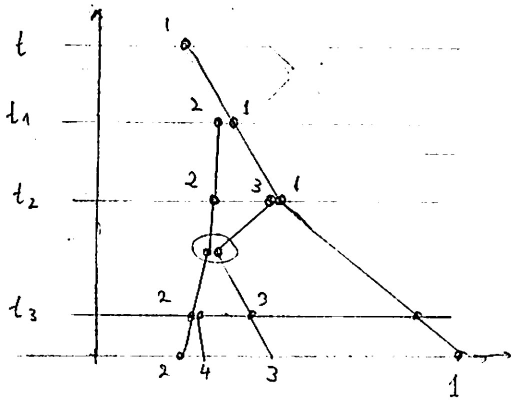
This is a correct identity $\quad$ Simonella Ph.D 2012

proof is direct, always integrating over sets of full measure

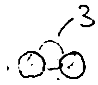
The result can be read as branching process

collision histories, pseudo trajectories graphical representation

ρ_1



recollision



overlap is not allowed

set of full measure at time $t=0$

"weight" of collision history

$dt_1 \quad dt_m$ ordered

$v_1 \quad v_m$ i.i.d

$d\omega_1 \quad d\omega_m$ i.i.d but with a constraint

!! weights do not have a definite sign !!

2.7 kinetic scaling and kinetic limit

$a \rightsquigarrow \varepsilon a \qquad E(N) = \varepsilon^{-2}$

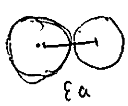
rescaled correlation functions

$\tau_n^\varepsilon = \varepsilon^{2n} \rho_n$

$\rightsquigarrow$ all ε-factors cancel

$$\partial_t \varepsilon^{2n} \rho_n = (\varepsilon a)^2 \varepsilon^{2n} \rho_{n+1} = a^2 \varepsilon^{2n+2} \rho_{n+1}$$

ONLY new additional sphere is add at εa



space-, time is already kinetic scale.

$$\partial_t \tau_n^\varepsilon = \sum_{j=1}^{n} v_j \cdot \nabla_{q_j} \tau_n^\varepsilon + a^2 C^\varepsilon_{n,n+1} \tau^\varepsilon_{n+1}$$

diameter $a\varepsilon$

$\lim_{\varepsilon \to 0} r_n^\varepsilon \cancel{(t)}$?

I) uniform bound

naive bound $\qquad |\cdot|$, assume that velocities are bounded

$\sim \underbrace{c^m}_{\text{correlation functions}} \underbrace{\frac{t^m}{m!}}_{\text{time integrations}} \underbrace{n \dots (n+m-1)}_{\text{number of terms (branching)}}$

$\rightsquigarrow \quad |t| < t_0 \qquad t_0 = O(1)$. This is in fact correct

independent of n

same problem for Boltzmann equation $\qquad Q(f,f)$

absolute value $\qquad ff - ff \rightsquigarrow ff + ff$

$\frac{d}{dt} x(t) = x(t)^2 \qquad$ blows up in finite time

---

improved version $\qquad h_\beta \quad$ Maxwellian $\quad \frac{1}{(2\pi/\beta)^{3/2}} e^{-\frac{1}{2}\beta v^2}$

initial correlations

$$|r_n(x_1 \dots x_n)| \le \chi_n(q_1, \dots, q_n) \prod_{j=1}^n z h_\beta(v_j) \qquad \beta > 0$$

C

$\chi = \begin{cases} 0 & \text{if one pair overlaps} \\ 1 & \text{otherwise} \end{cases}$

- invariant measure for n hard spheres $\qquad$ removes the flow S_n

- energy conservation $\qquad \sum_{j=1}^n |v_j| \le \left(n \sum_{i=1}^n v_j^2\right)^{1/2} \qquad$ removes the factors p

invariant under $S_n(t)$

iterative bound using both invariance properties

$\beta = \beta_0 > \beta_1 \qquad > \beta_m \qquad$ optimize the choice

$z = z_0 < z_1 \qquad < z_m$

$\rightsquigarrow \quad t < \left(\frac{1}{5}\right)$ mean free time $= \frac{1}{5} \frac{1}{\underbrace{\pi a^2 z}_{\text{mean free path}}} \sqrt{\beta/3}$ typical velocity

this precise number is of no importance

The finite radius of convergence is one of the central open problems !!

(no good idea around!)

Γ_n n-particle phase space

II) term by term convergence

typical initial measure is $\square$ box Λ

$$x_n(\underline{q}) \prod_{j=1}^{n} f(q_j, v_j) \frac{1}{n!} d\underline{q}\, d\underline{v} \quad \text{on } \Gamma_n = (\Lambda \times \mathbb{R}^3)^n \text{ with } 0 \leq f(q,v) \leq z h_\beta(v)$$

Then

$$\lim_{n\to\infty} \tau_n^\varepsilon(\underline{x}) = \prod_{j=1}^{n} f(q_j, v_j)$$

uniformly on compact sets of $\Gamma_n \setminus \Gamma_n^{sing}(0)$ ← singular set

$$\Gamma_n^{sing}(0) = \{ q_i = q_j \text{ for at least one pair } (i,j) \} \times \mathbb{R}^{3n}$$

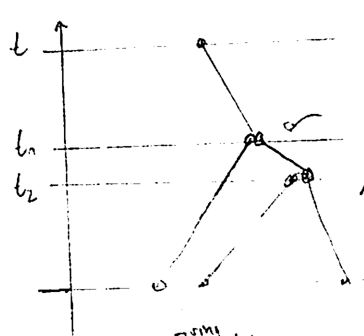
Let us assume.

$\{\tau_n^\varepsilon\}$ uniform bound

limit functions τ_n continuous on Γ_n

$\lim_{\varepsilon\to 0} \tau_n^\varepsilon = \tau_n$ uniformly on compacta of $\Gamma_n \setminus \Gamma_n^{sing}(0)$

τ_1^ε



$\lim_{\varepsilon\to 0}$ yields loss + gain term

to have a recollision: $\hat{\omega}$ is arbitrary, direction of velocity has to be focused.

area ε^2

recollisions give a contribution of order $\varepsilon^2 \to 0$ as $\varepsilon \to 0$

$\tau_1^\varepsilon(t) \to \tau_1(t)$ uniformly on compacta