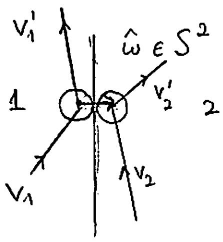
2.) change due to collisions

INSERT: hard sphere collisions



$T_{\hat\omega} : (v_1, v_2) \longmapsto (v_1', v_2')$

component $\parallel$ in $\hat\omega$ as 1D collision

$$v_1' = v_1 - \left((v_1 - v_2)\cdot\hat\omega\right)\hat\omega$$
$$v_2' = v_2 + \left((v_1 - v_2)\cdot\hat\omega\right)\hat\omega$$

conservation of momentum $v_1 + v_2 = v_1' + v_2'$

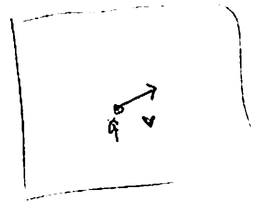
energy $v_1^2 + v_2^2 = v_1'^2 + v_2'^2$ $\quad T_{\hat\omega}$ is isometry

incoming velocities $(v_1 - v_2)\cdot\hat\omega > 0$ $\quad$ ($\underbrace{v_1\cdot\hat\omega > 0}_{v_2 = 0}$, plus Galilean)

---

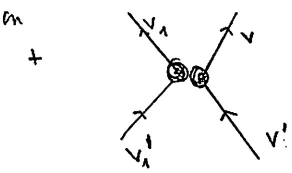
rate ansatz

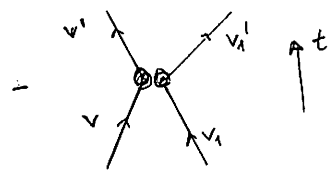
$$\underbrace{\partial_t f(x, v, t)}_{\text{suppress}} = \int dv' \, (\underbrace{K(v'|v) \, f(x, v')}_{\substack{v' \to v \\ \text{gain } +}} - \underbrace{k(v|v') \, f(x, v)}_{\substack{v \mapsto v' \\ \text{loss}}})$$



K depends itself on f : $\quad\|$ non-linear Markov jump process $\|$

notation





loss $\quad$ number of collisions in dt

$$\int_{x + \Delta(dt)} dv_1 \, dq_1 \, N \, f(q_1, v_1, t) = N \int dv_1 \, a^2 \int d\hat\omega \; \hat\omega\cdot(v - v_1)_+ \; f(q_1, v_1, t)$$

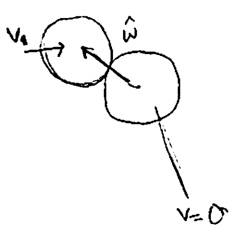
particles colliding in dt

$$a_+ := \begin{cases} a & \text{for } a \geq 0, \\ 0 & \text{for } a < 0. \end{cases}$$

$\leadsto$

loss $\quad - a^2 N \int dv_1 \int_{S^2} d\hat\omega \; (\hat\omega\cdot(v - v_1))_+ \, f(x, v_1) \, f(x, v)$

$a^2 v_n \, dt$



$$\int_{\Delta(t)} dq_1 = a^2 \int_{S^2} d\hat\omega \, (\hat\omega\cdot v_1)_+ \, dt$$

gain

$$a^2 N \int dv'\, dv_1' \int d\hat\omega\, (\hat\omega \cdot (v'-v_1'))_+ \, f(v_1')\, f(v')\, \delta(v - v(v', v_1', \hat\omega))$$

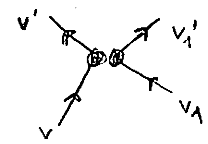
integrate against test function

$$a^2 N \int dv' dv_1' \int d\hat\omega\, (\hat\omega \cdot (v'-v_1'))_+ \, f(v_1')\, f(v')\, g(v(v', v_1', \hat\omega))$$ change (v', v_1') to (v, v_1)

isometry $dv'\, dv_1' = dv\, dv_1$, momentum conservation

$$a^2 N \int dv\, dv_1 \int d\hat\omega\, (\hat\omega \cdot (v-v_1))_+ \, f(v_1')\, f(v')\, g(v)$$

$\rightsquigarrow$ gain: $a^2 N \int dv_1 \int d\hat\omega\, (\hat\omega \cdot (v-v_1))_+ \, f(v_1')\, f(v')$

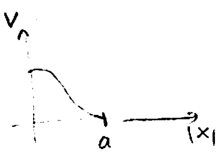


$\rightsquigarrow$ Boltzmann equation (1872)

$$\partial_t f(x,v) = - v \cdot \nabla_x f(x,v) + \underbrace{a^2 N}_{O(1)} \int_{\mathbb{R}^3} dv_1 \int_{S^2} d\hat\omega\, (\hat\omega \cdot (v-v_1))_+ \left(f(x,v_1') f(x,v') - f(x,v_1) f(x,v) \right)$$

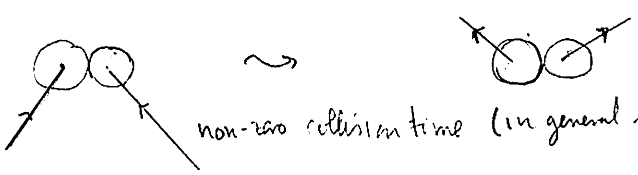
difficult, Villani

Note: rate under kinetic scaling $(\varepsilon a)^2 N = a^2 \varepsilon^2 N = O(1)$.



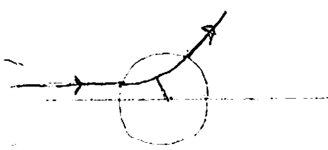
Remarks: • holds also for smooth potentials, finite range

)



non-zero collision time (in general unbounded)

center of mass coordinate $q_2(t) - q_1(t)$



differential cross section not so explicit

• random scattering subject to conservation laws ‖

– basis for stochastic models

e.g. Kac model velocities $v_1, \dots, v_N \in \mathbb{R}$ mean field model

$i \neq j$ $(v_i, v_j) \rightsquigarrow (v_i', v_j')$ random rotation

random walk on S^{N-1} $E = \frac{1}{2} \vec v^2$ is conserved

‖ particle in cell, lattice Boltzmann — wind tunnels ‖ $\Rightarrow$ 

2.3 Entropy increase

spatially homogeneous f depends only on v

entropy $S(f) = -\int dx\, f(v) \log f(v)$

large deviation formula of Boltzmann

entropy production

$$\frac{d}{dt} S(f(t)) = -\underbrace{\int dv\, \partial_t f}_{0} - \int dv\, (\partial_t f) \log f$$

$$= -a^2 N \int dv_1 dv_2 \int d\hat{\omega} \quad (\hat{\omega}\cdot(v_1 - v_2))_+ \log f(v_1) \left[f(v_1') f(v_2') - f(v_1) f(v_2) \right]$$

$v_1, v_2 \to v_2, v_1$ $\quad \omega \to -\omega$

$v_1, v_2 \to v_1', v_2'$

$\log f(v_1) + \log f(v_2) - \log f(v_1') - \log f(v_2')$

$$\rightsquigarrow \frac{d}{dt} S(f(t)) = +\frac{1}{4} a^2 N \int dv_1 dv_2 \int d\hat{\omega} \quad (\hat{\omega}\cdot(v_1 - v_2))_+ \left(f(v_1') f(v_2') - f(v_1) f(v_2) \right) \log \frac{f(v_1') f(v_2')}{f(v_1) f(v_2)}$$

$$(a - b) \log \frac{a}{b} \geq 0$$

$\frac{d}{dt} S(f(t)) \geq 0$

stationary $\sigma = 0$ ... $\mid$ $a = b$

$$\log f(v_1') + \log f(v_2') = \log f(v_1) + \log f(v_2)$$

$\log f$ is a collisional invariant. The set of solutions is

requires $\int f (1+v^2) < \infty$, $-\int f \log f < \infty$

$$\log f = c_0 + \vec{\alpha}\cdot\vec{v} - \beta \tfrac{1}{2} v^2$$

$\rightsquigarrow$ f is a shifted Gaussian

triumph of kinetic theory $\|$ equation must be correct. $\|$

2.4 Point processes

$\Lambda \subset \mathbb{R}^d$ (later on $q_j, p_j \in \Lambda \times \mathbb{R}^3$ $\quad x_j = (q_j, p_j) \in \Lambda \times \mathbb{R}^3$

points $(x_1, \dots, x_n)$

$$T = \bigcup_{n \geq 0} \Lambda^n \qquad n = 0 \quad \phi$$

probability measure point configurations density w.r.t. dx

$\{ f_n(x_1, \dots, x_n), n = 1, 2, \}$, f_n is symmetric, $f_n \geq 0$, f_0 probability of ϕ.

bound. $f_n(x_1, \dots, x_n) \leq C \prod_{j=1}^{n} h(x_j)$ $\qquad \int_\Lambda dx\, h(x) < \infty$

normalization

$$\sum_{n=0}^{\infty} \int_{\Lambda^n} dx_1 \dots dx_n \frac{1}{n!} f_n(x_1, \dots, x_n) = 1$$

empirical density $\qquad \sum_{j=1}^{n} \delta(x - x_j)\, dx = n(dx)$

moments

$$\mathbb{E}(n(dx)) = \sum_{n=0}^{\infty} \frac{1}{n!} \int dx_1 \dots dx_n\, f_{n+1}(x, x_1, \dots, x_n) \quad dx$$

$$\mathbb{E}(n(dx)\, n(dy)) = \sum_{n=0}^{\infty} \frac{1}{n!} \int dx_1 \dots dx_n\, f_n(x_1, \dots, x_n) \underbrace{\sum_{j,i=1}^{n} \delta(x_i - x)\, \delta(x_j - y)}_{\sum_{i \neq j = 1}^{n} \delta(x_i - x)\, \delta(x_j - y) + \sum_{j=1}^{n} \delta(x - x_j)\, \delta(y - x_j)\, dx\, dy} \quad dx\, dy$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \int dx_1 \dots dx_n\, f_{n+2}(x, y, x_1, \dots, x_n) \quad dx\, dy$$
$$+ \sum_{n=0}^{\infty} \frac{1}{n!} \int dx_1 \dots dx_n\, f_{n+1}(x, x_1, \dots, x_n) \quad \delta(x-y)\, dx\, dy$$

singular term, not so convenient

define n-th correlation function

$$\rho_0 = 1$$
$$\rho_n(x_1, \dots, x_n) = \sum_{m=0}^{\infty} \frac{1}{m!} \int dx_{n+1} \dots dx_{n+m}\, f_{n+m}(x_1, \dots, x_{n+m}) \qquad n = 1, 2,$$

- $\rho_n \geq 0$, ρ_n symmetric, $\rho_n(x_1, \dots, x_n) \leq C \prod_{j=1}^{n} h(x_j)$

Th.

• inversion formula

$$f_n(x_1,\dots,x_n) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \int dx_{n+1} \; dx_{n+m}\, \rho_{n+m}(x_1, \dots, x_{n+m})$$

• moments

$$\rho_m(y_1,\dots,y_m) = \mathbb{E}\Big(\sum_{\substack{j_1\dots j_m=1\\ j_1\neq \dots \neq j_m}}^{n} \prod_{\ell=1}^{m} \delta(y_\ell - x_{j_\ell})\Big)$$

• normalization

$$\int dx_1 \dots dx_n\, \rho_n(x_1 \dots x_n) = \mathbb{E}\big(N(N-1) \dots (N-n+1)\big)$$

↖ number as random variable

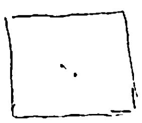
• number fixed $n = N$

measure $\begin{cases} f_N(x_1, \dots, x_N) \\ n \neq N \quad f_n = 0 \end{cases}$

• up to normalization ρ_n is the n-th marginal, can be misleading

• Poisson $\rho_n(x_1,\dots,x_n) = \prod_{j=1}^{n} \bar\rho(x_j)$ $\qquad f_n(x_1,\dots,x_n) = \Big(\prod_{j=1}^{n} \bar\rho(x_j) \Big) e^{-\int_\Lambda \bar\rho(x)dx}$

• law of large numbers



unit box $\Lambda \subset \mathbb{R}^d$ $\qquad$ typical distance ε

$\mathbb{E}(N) = \varepsilon^{-d}$

$$n^\varepsilon(dx) = \varepsilon^d \sum_{j=1}^{N} \delta(x - x_j)$$

pointwise

$$\langle n^\varepsilon(f) \rangle \xrightarrow[\varepsilon\to 0]{} \int_\Lambda dx\, f(x)\, r(x) \qquad \Big| \qquad \lim_{\varepsilon\to 0} \varepsilon^d \rho_1^\varepsilon(x_1) = r(x_1)$$

$$\langle n^\varepsilon(f)^2 \rangle \xrightarrow[\varepsilon\to 0]{} \Big(\int_\Lambda dx\, f(x)\, r(x) \Big)^2 \qquad \Big| \qquad \lim_{\varepsilon\to 0} \varepsilon^{2d} r_2^\varepsilon(x_1, x_2) = r(x_1) r(x_2)$$

in probability $\lim_{\varepsilon\to 0} n^\varepsilon(f) = \int_\Lambda dx\, f(x)\, r(x)$

⇝ We only need ρ_1 and ρ_2 for LLN.

2.5 Hierarchy of correlations

mass 1

n particles   smooth potential $x_j = (q_j, v_j) \in \mathbb{R}^6$ $\underline{x} = (x_1, .., x_n)$

$$\partial_t f_n(\underline{x}, t) = \sum_{j=1}^{n} \Big\{ - p_j \cdot \nabla_{q_j} - \sum_{\substack{i=1\\ i\neq j}}^{n} F(q_j - q_i) \cdot \nabla_{v_j} \Big\} f_n(\underline{x}, t) \qquad F = -\nabla V$$

$$\partial_t \rho_n(\underline{x}) = \sum_{m=0}^{\infty} \frac{1}{m!} \int \overset{\overset{q,p}{6}}{dx_{n+1}} \cdots dx_{n+m} \; \partial_t f_{n+m}(\underline{x})$$

$$= \sum_{m=0}^{\infty} \frac{1}{m!} \int dx_{n+1} \ldots dx_{n+m} \Big[- \sum_{j=1}^{n} v_j \cdot \nabla_{q_j} \underbrace{- \sum_{i \neq j = 1} F(q_j - q_i) \cdot \nabla_{v_j}}_{n\text{-particle}}$$

$$\underbrace{- \sum_{j=n+1}^{n+m} v_j \cdot \nabla_j}_{\substack{\sim\\ 0\\ \text{partial}}} \; \underbrace{- \sum_{i \neq j = n+1}^{n+m} F(q_j - q_i) \cdot \nabla_{v_j}}_{\sim\, 0} \; \underbrace{- \sum_{i=1}^{n} \sum_{j=n+1}^{n+m} F(q_j - q_i) \cdot \nabla_{v_j} - \sum_{i=n+1}^{n+m} \sum_{j=1}^{n} F(q_j - q_i) \cdot \nabla_{v_j}}_{\text{cross terms}} \Big] f$$

$$\begin{array}{cc} n & m \end{array}$$
$$\left(\begin{array}{c|c} \cancel{\;} & 0 \\ \hline & \cancel{\;} \;\; 0 \end{array} \right) \text{ partial}$$

$$\partial_t \rho_n(\underline{x}, t) = \underbrace{\Big(- \sum_{j=1}^{n} v_j \cdot \nabla_{q_j} - \sum_{i \neq j = 1}^{n} F(q_j - q_i) \cdot \nabla_{v_j} \Big) \rho_n(\underline{x}, t)}_{n\text{-particle dynamics}}$$

$$+ \sum_{j=1}^{n} \int dx_{n+1} \, F(q_j - q_{n+1}) \cdot \nabla_{p_j} \, \rho_{n+1}(\underline{x}, t) \qquad \text{interaction with outside}$$

BBGKY hierarchy n couples to $n+1$

$n = 1$

$$\partial_t \rho_1(q_1, p_1) = - v_1 \cdot \nabla_{q_1} \rho_1(q_1, p_1) \quad \text{(flow)}$$

$$- \int dq_2 \, dp_2 \, \underset{\substack{\uparrow\\ \text{collision}}}{F(q_1 - q_2)} \cdot \nabla_{v_1} \underbrace{\rho_2(q_1, p_1, q_2, p_2)}_{\cong\, \rho_1(q_1, p_1)\, \rho_1(q_2, p_2)}$$

lecture 2

DOES NOT WORK $q_1 \simeq q_2$ since F local

H. Grad 1958

C. Cercignani 1972

NEXT LECTURE   Sept. 26   10:00 – 11:30   Room 507