- f_{KPZ} is the stationary KPZ scaling function $\langle u(x,t)u(0,0)\rangle \cong (\Gamma t)^{-2/3} f_{KPZ}((\Gamma t)^{-2/3}x)$ Burgers

  f_{KPZ} is the long time asymptotics of second class particle

  $f_{KPZ} > 0$, even, $\int f_{KPZ}(x)\,dx = 1$, tails $e^{-0.295|x|^3}$

- $\hat{f}_0(k) = e^{-|k|^{5/3}}$ α-stable distribution [Baik-Rains]

5.4 Euler equations

ASSUMPTION: no further conservation laws

known exceptions are harmonic $V(x) = x^2$

$V(x) = e^{-x}$

Calogero-Moser $\frac{1}{x^2}$ is not nearest neighbor, long range

3 conservation laws $\Leftrightarrow$ infinite number

local equilibrium

$$\frac{1}{Z}\exp\Big[-\sum_j \beta(\varepsilon j)\big(\tfrac{1}{2}(p_j - u(\varepsilon j))^2 + V(q_{j+1}-q_j)\big) + P(\varepsilon j)\, r_j\Big]$$

| as for kinetic limit hard to prove

maintained by flow by updating the parameters

space-time ε^{-1}

$$\partial_t \langle \vec{g}_j(t)\rangle_{\vec{\mu}} + \nabla\langle \vec{J}_j(t)\rangle_{\vec{\mu}} = 0 \qquad \vec{\mu} = (P, v, \beta)$$

$$\partial_t \underbrace{\langle g_j\rangle_{\vec{\mu}(t)}} + \nabla \underbrace{\langle J_j\rangle_{\vec{\mu}(t)}} = 0$$

static expectation

macroscopic fields ℓ, u, e_{tot}

$$\partial_t \ell - \partial_x u = 0\,, \quad \partial_t u + \partial_x P(\ell, e_{tot} - \tfrac{1}{2}u^2) = 0\,, \quad \partial_t e_{tot} + \partial_x\big(u\, P(\ell, e_{tot} - \tfrac{1}{2}u^2)\big) = 0$$

$P, \beta \Leftrightarrow \ell, e_{int}$

$$\ell = \frac{1}{Z}\int e^{-\beta(V(x)+Px)}\, x\, dx \quad , \quad e_{int} = \frac{1}{2\beta^2} + \frac{1}{Z}\int e^{-\beta(V(x)+Px)}\, V(x)\, dx$$

linearization $\ell + u_1, \ 0 + u_2, \ e_{tot} + u_3$

$$\partial_t \vec{u} + \partial_x A \vec{u} = 0$$

$$A = \begin{pmatrix} 0 & -1 & 0 \\ \partial_\ell P & 0 & \partial_e P \\ 0 & P & 0 \end{pmatrix}$$

eigenvalues $0, \pm c$, $c > 0$

c speed of sound

static correlator

$$C = \sum_j S(j,0)$$

sum rule $\quad C = \sum_j S(j,t)$ $\qquad$ conservation law

Landau - Lifshitz

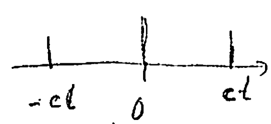
$$\partial_t u + \partial_x \left(Au - \partial_x D u + B \xi \right) = 0$$

initial condition $\quad u(x,0)$ white noise , mean zero

$$\langle u_\alpha(x,0)\, u_{\alpha'}(x',0)\rangle = \delta(x-x')\, C_{\alpha\alpha'}$$

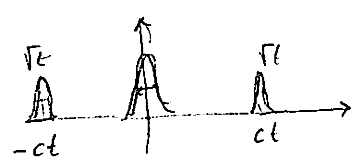
solution with $D = B = 0$

$$\langle u_\alpha(x,t)\, u_{\alpha'}(0,0)\rangle = \left(e^{-t\partial_x A}\, C\right)_{\alpha\alpha'}(x,0)$$



add D, B

peaks broaden



empirically wrong

We have to do better!

5.5 Nonlinear fluctuating hydrodynamics

- expand currents to second order
- linear term is dominating (Au), switch to basis in which A is diagonal

linear transformation (normal modes)

$$\vec{\phi} = R\vec{u}\,, \qquad R A R^{-1} = \mathrm{diag}(-c, 0, c)\,, \qquad R C R^{T} = \mathbb{1}$$

$$\partial_t \phi_\alpha + \partial_x\Big(c_\alpha \phi_\alpha + \langle \phi, G^\alpha \phi\rangle - \partial_x (\mathbb{D}\phi)_\alpha + (B\xi)_\alpha\Big) = 0 \qquad (*)$$

$$\underbrace{\alpha = -1, 0, 1}_{\text{eigenvalue label}}$$

$\vec{c} = (-c, 0, c)$, G^α transformed Hessians,

$$R^{-1}\mathbb{D}R = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \gamma_1 & 0 \\ \tilde{\nu}_2 & 0 & \nu_2 \end{pmatrix}, \qquad BB^T = 2\mathbb{D}$$

(*) NFH, multi-component Burgers

- A nondegenerate, $A = 0$ Ertas, Kadar 1993 not so much explored
- NO Cole-Hopf, Hairer scheme has been worked out

  Perkowski and Kupiainen, Marcozzi 2016

- stationary measures Funaki (2016)

  Gaussian white noise iff cyclicity

  $$\| \; G^\alpha_{\beta\gamma} = G^\alpha_{\gamma\beta} = G^\gamma_{\alpha\beta} \; \|$$

never satisfied in physical models

5.6 Decoupling and mode-coupling

Decoupling hypothesis (A nondegenerate). If $G^\alpha_{\alpha\alpha} \neq 0$, then mode α is governed by the stochastic Burgers equation as $t, x \to \infty$.

Implies

$$\langle \phi_\alpha(x,t)\, \phi_\alpha(0,0)\rangle \cong (\Gamma_s t)^{-2/3} f_{KPZ}\left((\Gamma_s t)^{-3/2} (x - c_\alpha t)\right)$$

and $\quad \Gamma_s = 2\sqrt{2}\, | G^\alpha_{\alpha\alpha} |$

anharmonic chains

generically $\quad G^\alpha_{\alpha\alpha} \neq 0, \quad \alpha = \pm 1$

BUT

$G^0_{00} = 0$ always

One-loop: approximate equation for S (3×3 matrix)

$$\partial_t S(x,t) = (-A\partial_x + D\partial_x^2) S(x,t) + \partial_x^2 \int_0^t ds \int dy\, \underbrace{M(y,s)}_{\text{memory kernel}} S(x-y, t-s)$$

$$M_{\alpha\alpha'}(x,t) = 2\, \mathrm{tr}\left((SG^\alpha)^T (SG^{\alpha'})\right)(x,t)$$

⇝ heat mode: diagonal $S_{\alpha\beta} = \delta_{\alpha\beta} f_\alpha$

sound modes scale as KPZ determines the memory kernel

linear equation for f_0 ⇝

$$\hat{f}_0(k) = e^{-|k|^{5/3}}$$

input	$e^{ickt}\, \hat{g}(t k^\gamma)$	$c \neq 0$
output	$e^{-\lvert k\rvert^{1+\frac{1}{\gamma}}}$	
KPZ	$t^{2/3} k = (t k^{3/2})^{2/3}$	$\frac{1}{\gamma} = \frac{2}{3} \quad 1 + \frac{1}{\gamma} = \frac{5}{3}$

5.7 Phase diagram

based on decoupling and mode-coupling

general discussion Schmidt, Schütz 2016

special case $(-c, 0, c)$ $G^0_{00} = 0$

relevant couplings are the diagonals (normal mode)

diag G^σ $\sigma = \pm 1, 0$

symmetries

diag $G^1 = (a_1, a_2, a_3)$, diag $G^{-1} = -(a_3, a_2, a_1)$

diag $G^0 = (-a_4, 0, a_4)$ and $a_4 > 0$

Phase 1 : $a_1 \neq 0$ KPZ + Levy $5/3$

Phase 2 : $a_1 = 0$ $a_2 = 0$, $a_3 = 0$ diffusive + Levy $\frac{3}{2}$ $(= 1 + \frac{1}{2})$

see picture: this is the only proved case !

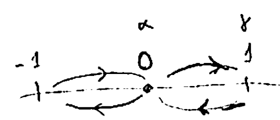
Jara, Komorowski, Olla 2014

Bernardin, Gonçalves, Jara

standard example $V(x) = V(-x)$, $P = 0$

out of the blue Lee-Dadswell 2015 $\beta = 1$, $P = 0.59$, $V = -\frac{2}{3}x^3 + \frac{1}{4}x^4$

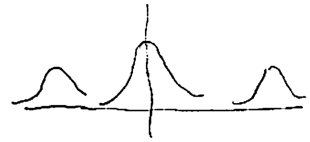
Phase 3 : $a_3 = 0$, but not $a_1 = 0 = a_2$



$\alpha = 1 + \frac{1}{\gamma}$
$\gamma = 1 + \frac{1}{\alpha}$ $\rightsquigarrow$ $\gamma = \alpha$ $\gamma = 1 + \frac{1}{\gamma}$ golden mean

all three Levy golden mean $\frac{1}{2}(1 + \sqrt{5})$

this cannot be the case power law



towards $|x| \to \infty$

unphysical

correct result is sound: maximally asymmetric Levy $\frac{1}{2}(1+\sqrt{5})$

heat: symmetric Levy $\frac{1}{2}(1+\sqrt{5})$

∘ only studied example C. Mendl, A. Dhar (numerics)

$$V(x) = \frac{1}{2}x^2 + \cos\left(\pi\left(x - \frac{1}{3}\right)\right) + \frac{1}{8}x^4$$

$$\beta = 1, \quad P = 2.214$$