Columbia University Fall 2017

(Large scale dynamics and their fluctuation theory)

lectures in recent years on stochastic particle systems, dynamics of interfaces, KPZ related topics

carrying coal to Newcastle
hot dogs to New York

other area of active interest

- dynamics of many classical particles ~ with random initial conditions (dynamical system)

dynamical system ⇔ probability

law of large numbers

goal: fluctuation theory

low density fluids || weakly nonlinear waves || one-dimensional fluids

nonlinear fluctuation theory

Gaussian Gaussian

large deviations?

Columbia-University Fall 2017

Large Scale Dynamics and Their Fluctuation Theory

1. A puzzle from Statistical Physics

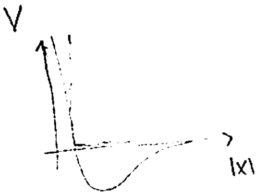
many-body mechanical system

positions $q_j \in \mathbb{R}^3$, momenta (= velocities) $p_j \in \mathbb{R}^3$, $j = 1, \dots, N$, mass m

Newton 1686, Maxwell, Boltzmann $\cong$ 1860

$$\frac{d}{dt} q_j = \frac{1}{m} p_j, \quad \frac{d}{dt} p_j = - \sum_{\substack{i=1 \\ i \neq j}}^{N} \nabla V(q_j - q_i), \quad j = 1, \dots, N$$ (Newton) + random initial condition

V is radial potential

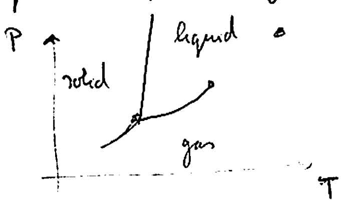


model potential

"true" potential comes from quantum mechanics

"simple fluid"

equilibrium phase diagram (Gibbs measures)



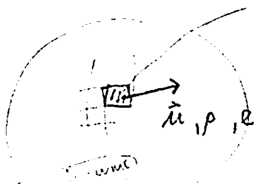
|| fluid ||

$$\frac{1}{Z} e^{-\beta \sum_{i \neq j} V(q_i - q_j)}$$ stationary measures

ρ density

$\vec{u}$ velocity

Euler 1757 continuum description



$\vec{u}, \rho, e$ total energy = kinetic + internal

$\rho(x,t)$, $x \in \mathbb{R}^3$, $t \geq 0$.

$$\partial_t \rho + \nabla_x \cdot (\rho \vec{u}) = 0$$

$$\partial_t m \rho u_\alpha + \nabla_x \cdot (m \rho u_\alpha \vec{u}) + \partial_{x_\alpha} P(\rho, e_{int}) = 0, \quad \alpha = 1,2,3$$ (Euler)

thermodynamics

+ initial conditions

$$\partial_t e + \partial_x j_e = 0$$

no pressure in Newton

|| Why do particles follow the solution of the PDE ?

physics: taken for granted || or study microscopic scale | hierarchical

fluid dynamics cutoff? | useful for guessing equations

Mark Kac 1959 Probability and related topics in physical sciences first hint   3 g of water has $N = 10^{23}$

$\Rightarrow$ law of large numbers

EXAMPLE (oversimplified)

only positions $x_j \in \mathbb{R}$, $j = 1, \ldots, N$

- static $t = 0$ $\{x_j\}$ Poisson point process intensity $\rho_0 \geq 0$ $\int dx\, \rho_0(x) < \infty$, N random

slow variation $\rho_0(\varepsilon x)$ $\varepsilon \ll 1$ $\mathbb{E}(N) = \varepsilon^{-1} \int dx\, \rho_0(x)$ interparticle 1

f test function   empirical density $\sum_{j=1}^{N} \delta(x - x_j)$

$$n^{\varepsilon}(f) = \varepsilon \sum_{j=1}^{N} f(\varepsilon x_j) =$$

(N random)

ε^{-1}

$$\lim_{\varepsilon \to 0} n^{\varepsilon}(f) = \int dx\, \rho_0(x) f(x) \quad \text{a.s.}$$

$T(x' - x)\,dx'$ $T(x) = T(-x)$

x

microscopic length scale

- dynamics $x_j(t)$ symmetric random walk i.i.d.

empirical at time t $\sum_{j=1}^{N} \delta(x - x_j(t))$

then $$\lim_{\varepsilon \to 0} \varepsilon (x_j(\varepsilon^{-2} t) - x_j(0)) = \sqrt{D}\, B(t)$$ BM

$$n^{\varepsilon}(f, t) = \varepsilon \sum_{j=1}^{N} f(\varepsilon x_j(\varepsilon^{-2} t))$$

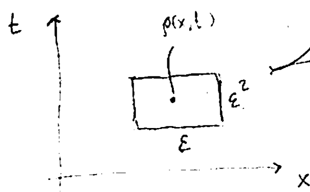
$$\lim_{\varepsilon \to 0} n^{\varepsilon}(f, t) = \int dx\, \rho(x,t) f(x) \quad \text{a.s.} \qquad \partial_t \rho = \tfrac{1}{2} \partial_x D \partial_x \rho \qquad \rho(x, 0) = \rho_0(x).$$

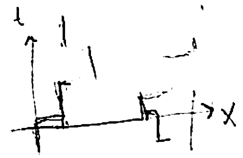
three lessons

$\Rightarrow$ law of large numbers

$\Rightarrow$ local stationarity (macroscopic scale)

"macroscopic scale"





stationary $\{x_j(t)\}$ process with density $\rho(x,t)$

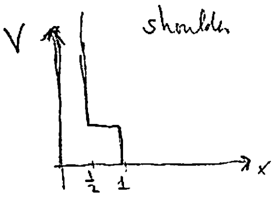
is self-generated by the dynamics

$\Rightarrow$ fluctuations   CLT   linear fluctuation theory   OU process

critics

- macroscopic equation is linear, nonlinear interactions, HERE $D \to D(\rho)$
  Varadhan, Yau, …   hydrodynamic limit
- physical evolution is deterministic !!
  BUT with random initial data !
- fluid is not local Poisson, equilibration poorly understood for deterministic systems

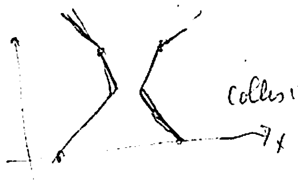
molecular dynamics
work with C. Mendl



$d = 1$

$N = 4000$

density $\frac{1}{2}$



collision-to-collision

$$\frac{1}{Z} e^{-\beta \left(\sum_j V(q_{j+1} - q_j) \right)}$$

particle positions equilibrated, velocities i.i.d. not Gaussian

t = 2000 collisions per particle, study local statistics, one particle velocity distribution

depends on β

i.i.d Gaussian ~~are~~ observed (3 body collisions required)
the denser the better

currently: Euler is like Mt Everest, no trail

Olla, Varadhan, Yau 1993
still needs stochastic collisions
best available, no follow up

HERE: small parameter,

1) low density

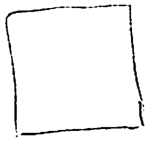
Lanford 1976
much recent work.

J. Lukkarinen, HS. 2005–2009

2) weakly nonlinear wave equation

2. Low density gas

2.1 Kinetic scaling

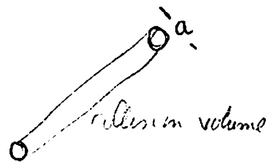


box $\Lambda \subset \mathbb{R}^3$ kinetic scale (macroscopic)

N hard spheres diameter εa $\quad \varepsilon \ll 1$

$\mathbb{E}(N) \to \infty$

- velocities are $O(1)$
- N and ε ? mean free path ℓ fixed



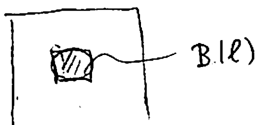
$$(\varepsilon a)^2 \ell N = |\Lambda|$$

$\rightsquigarrow \quad \varepsilon^2 N = O(1)$

volume occupied $(\varepsilon a)^3 \varepsilon^{-2} = \varepsilon a^3 \to 0$ as $\varepsilon \to 0$

ideal gas thermodynamics

Gibbs measure $\frac{1}{Z} \frac{1}{N!} dq_1 \ldots dq_N \exp\left[-\frac{1}{2}\sum_{i\neq j=1}^{N} V((q_i - q_j)/\varepsilon)\right] = \mu^\varepsilon$



box with a finite number of particles

$\ell^3 N = 1 \quad \ell = \varepsilon^{2/3}$ $\qquad \mu^\varepsilon|_{B(\ell)} \xrightarrow[\text{rescaled}]{}$ Poisson process

$\delta > 0$

$N = \varepsilon^{-2-\delta}$ strong interactions

$N = \varepsilon^{-2+\delta}$ non-interacting limit

free motion / collisions on the same scale.

end of lect. 1

2.2 The rate equation

one-particle phase space $\Lambda \times \mathbb{R}^3$, Boltzmann f function $f : \Lambda \times \mathbb{R}^3 \to \mathbb{R}_+$, infinite volume $\Lambda = \mathbb{R}^3$

$$N \int_\Delta dx\, dv\, f(x,v,t) \cong \text{number of particles in box } \Delta \subset \Lambda \times \mathbb{R}^3$$

notation $x \in \Lambda, v \in \mathbb{R}^3$

change in time of f

1) free streaming $\quad \partial_t f = -v \cdot \nabla_x f$