LECTURES ON MEAN FIELD GAMES:
II. CALCULUS OVER WASSERSTEIN SPACE, CONTROL OF
McKEAN-VLASOV DYNAMICS, AND THE MASTER
EQUATION

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**The Analytic (PDE) Approach to MFGs**

For fixed $\mu = (\mu_t)_t$, the value function

$$V^\mu(t, x) = \inf_{(\alpha_s)_{t \leq s \leq T}} \mathbb{E}\left[ \int_t^T f(s, X_s, \mu_s, \alpha_s) \, ds + g(X_T, \mu_T) \mid X_t = x \right]$$

solves a HJB (backward) equation

$$\partial_t V^\mu(t, x) + \inf_{\alpha} [b(t, x, \mu_t, \alpha) \cdot \partial_x V^\mu(t, x) + f(t, x, \mu_t, \alpha)]$$

$$\frac{1}{2} \text{trace} [\sigma(t, x)\sigma(t, x) \partial_{xx} V^\mu(t, x)] = 0$$

with terminal condition $V^\mu(T, x) = g(x, \mu_T)$

The fixed point step is implemented by requiring that $t \rightarrow \mu_t$ solves the (forward) Kolmogorov equation

$$\partial_t \mu_t = \mu_t \mathcal{L}_t^\dagger$$

This is also a **nonlinear PDE** because $\mu_t$ appears in $b$ ......

System of **strongly coupled nonlinear PDEs**! Time goes in both directions
HJB EQUATION FROM ITÔ’S FORMULA

Classical Optimal Control set-up ($\mu$ fixed)

Dynamic Programming Principle

\[ t \mapsto V^{\mu}(t, X_t) \text{ is a martingale when } (X_t)_{0 \leq t \leq T} \text{ is optimal} \]

Classical Itô formula to compute:

\[ d_t V^{\mu}(t, X_t) \]

when \((t, x) \mapsto V^{\mu}(t, x)\) is smooth and

\[ dX_t = b(t, X_t, \hat{\alpha}_t)dt + \sigma(t, X_t, \hat{\alpha}_t)dW_t \]

is optimal to

- set the drift to 0
- get HJB
MFG COUNTERPART

- MFG is not an optimization problem per-se
- Optimal control arguments (for $\mu$ fixed) affected by fixed point step
- What is the effect of last step substitution $\mu_t = \mathbb{P} X_t$?
- In equilibrium, do we still have:
  - Dynamic Programming Principle?
  - Martingale property of $t \mapsto V^\mu(t, X_t)$
- What would be the right Itô formula to compute:
  $$d_t V^\mu(t, X_t)$$

when
$$dX_t = b(t, X_t, \hat{\alpha}_t) dt + \sigma(t, X_t, \hat{\alpha}_t) dW_t$$
is optimal and $\mu_t = \mathbb{P} X_t$?
MORE REASONS TO DIFFERENTIATE FUNCTIONS OF MEASURES

Back to the \textit{N-player games} (with reduced or distributed controls):
\[
dx^i_t = b(t, X^i_t, \bar{\mu}^N_t, \phi(t, X^i_t, \bar{\mu}^N_t)) \, dt + \sigma(t, X^i_t, \bar{\mu}^N_t, \phi(t, X^i_t, \bar{\mu}^N_t)) \, dW^i_t, \quad t \in [0, T],
\]

Propagation of Chaos
\begin{itemize}
\item \(X^1_t, \ldots, X^k_t, \ldots\) become independent in the limit \(N \to \infty\)
\item \(X^i = (X^i_t)_{0 \leq t \leq T} \implies X = (X_t)_{0 \leq t \leq T}\) solution of the McKean–Vlasov equation:
\[
dx_t = b(t, X_t, \mathbb{P}_{X_t}, \phi(t, X_t, \mathbb{P}_{X_t})) \, dt + \sigma(t, X_t, \mathbb{P}_{X_t}, \phi(t, X_t, \mathbb{P}_{X_t})) \, dW_t, \quad t \in [0, T],
\]
where \(W = (W_t)_{0 \leq t \leq T}\) is a standard Wiener process.
\end{itemize}

Expected Costs:
\[
J^i(\phi) = \mathbb{E} \left[ \int_0^T f(t, X^i_t, \bar{\mu}^N_t, \phi(t, X^i_t, \bar{\mu}^N_t)) \, dt + g(X^i_T, \bar{\mu}^N_T) \right],
\]
converge to:
\[
J(\phi) = \mathbb{E} \left[ \int_0^T f(t, X_t, \mathbb{P}_{X_t}, \phi(t, X_t, \mathbb{P}_{X_t})) \, dt + g(X_T, \mathbb{P}_{X_T}) \right].
\]

Optimization after the limit: Control of McKean-Vlasov equations!
\textbf{Taking Stock}

SDE State Dynamics
for \( N \) players
\quad \Downarrow \quad \text{Fixed Point}
\quad \lim_{N \to \infty}
\quad \text{McKean Vlasov\nDynamics}

\quad \xrightarrow{\text{Optimization}}

\quad \Downarrow \quad \text{Fixed Point}
\quad \lim_{N \to \infty}

\text{Nash Equilibrium}
for \( N \) players

\text{Mean Field Game?}
\text{Controlled McKean-Vlasov SDE?}

\text{Is the above diagram commutative?}
CONTROLLED MCKEAN-VLASOV SDEs

\[
\inf_{\alpha=(\alpha_t)_{0\leq t\leq T}} \mathbb{E} \left[ \int_0^T f(t, X_t, P_{X_t}, \alpha_t) \, dt + g(X_T, P_{X_T}) \right]
\]

under dynamical constraint \( dX_t = b(t, X_t, P_{X_t}, \alpha_t) \, dt + \sigma(t, X_t, P_{X_t}, \alpha_t) \, dW_t \).

- State \((X_t, P_{X_t})\) infinite dimensional
- State trajectory \( t \mapsto (X_t, \mu_t) \) is a very thin submanifold due to constraint \( \mu_t = P_{X_t} \)
- Open loop form: \( \alpha = (\alpha_t)_{0\leq t\leq T} \) adapted
- Closed loop form: \( \alpha_t = \phi(t, X_t, P_{X_t}) \)

Whether we use

- Infinite dimensional HJB equation
- Pontryagin stochastic maximum principle with Hamiltonian

\[
H(t, x, \mu, y, z, \alpha) = b(t, x, \mu, \alpha) \cdot y + \sigma(t, x, \mu, \alpha) \cdot z + f(t, x, \mu, \alpha)
\]

and introduce the adjoint equations,

WE NEED TO DIFFERENTIATE FUNCTIONS OF MEASURES !
DIFFERENTIABILITY OF FUNCTIONS OF MEASURES

\( \mathcal{M}(\mathbb{R}^d) \) space of **signed** (finite) measures on \( \mathbb{R}^d \)

- Banach space (dual of a space of continuous functions)
- Classical differential calculus available
- If
  \[
  \mathcal{M}(\mathbb{R}^d) \ni m \mapsto \phi(m) \in \mathbb{R}
  \]
  "\( \phi \) is differentiable" has a meaning
- For \( m_0 \in \mathcal{M}(\mathbb{R}^d) \) one can define
  \[
  \frac{\delta \phi(m_0)}{\delta m}(\cdot)
  \]
  as a function on \( \mathbb{R}^d \) in **Fréchet** or **Gâteaux** sense

**Bensoussan-Frehe-Yam** alternative is to work only with measures with **densities** and view \( \phi \) as a function on \( L^1(\mathbb{R}^d, dx) \)!
Measures appearing in MFG theory are probability distributions of random variables !!!

Wasserstein space

\[ \mathcal{P}_2(\mathbb{R}^d) = \left\{ \mu \in \mathcal{P}(\mathbb{R}^d); \int_{\mathbb{R}^d} |x|^2 d\mu(x) < \infty \right\} \]

Metric space for the 2-Wasserstein distance

\[ W_2(\mu, \nu) = \inf_{\pi \in \Pi(\mu, \nu)} {\left[ \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^2 \pi(dx, dy) \right]}^{1/2} \]

where \( \Pi(\mu, \nu) \) is the set of probability measures coupling \( \mu \) and \( \nu \).

Topological properties of Wasserstein space well understood as following statements are equivalents

- \( \mu^N \rightarrow \mu \) in Wasserstein space
- \( \mu^N \rightarrow \mu \) weakly and \( \int |x|^2 \mu^N(dx) \rightarrow \int |x|^2 \mu(dx) \)
Glivenko-Cantelli in Wasserstein Space

$X^1, X^2, \cdots$, i.i.d. random variables in $\mathbb{R}^d$ with common distribution $\mu$ s.t.

$$M_q(\mu) = \int_{\mathbb{R}^d} |x|^q \mu(dx) < \infty.$$ 

If $q = 2$,

$$\mathbb{P}\left[ \lim_{N \to \infty} W_2(\mu^N, \mu) = 0 \right] = 1.$$

where $\mu^N = \frac{1}{N} \sum_{i=1}^N \delta_{X_i}$ is a (random) empirical measure. Standard LLN!

**Crucial Estimate: Glivenko-Cantelli** If $q > 4$ for each dimension $d \geq 1$, $\exists C = C(d, q, M_q(\mu))$ s.t. for all $N \geq 1$:

$$\mathbb{E}\left[ W_2(\mu^N, \mu)^2 \right] \leq C \begin{cases} 
N^{-1/2}, & \text{if } d < 4, \\
N^{-1/2} \log N, & \text{if } d = 4, \\
N^{-2/d}, & \text{if } d > 4.
\end{cases} \quad (1)$$
What does it mean "\( \phi \) is differentiable" or "\( \phi \) is convex" for

\[
\mathcal{P}_2(\mathbb{R}^d) \ni \mu \mapsto \phi(\mu) \in \mathbb{R}
\]

Wasserstein space \( \mathcal{P}_2(\mathbb{R}^d) \) is a metric space for \( W_2 \)

- Optimal transportation (Monge-Ampere-Kantorovich)
- Curve length and shortest paths (geodesics)
- Notion of **convex function** on \( \mathcal{P}_2(\mathbb{R}^d) \)
- **Tangent spaces** and differential geometry on \( \mathcal{P}_2(\mathbb{R}^d) \).
- Differential calculus on Wasserstein space

Brenier, Benamou, Ambrosio, Gigli, Otto, Caffarelli, Villani, Carlier, ....
**Differentiability in the sense of P.L. Lions**

If $\mathcal{P}_2(\mathbb{R}^d) \ni \mu \mapsto \phi(\mu) \in \mathbb{R}$ is "differentiable" on Wasserstein space what about $\mathbb{R}^{dN} \ni (x^1, \ldots, x^N) \mapsto u(x^1, \ldots, x^N) = \phi \left( \frac{1}{N} \sum_{j=1}^{N} \delta_{x_j} \right)$?

How does $\partial \phi(\mu)$ relate to $\partial_{x_i} u(x^1, \ldots, x^N)$?

**Lions’ Solution**

- **Lift** $\phi$ up to $L^2(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ into $\tilde{\phi}$ defined by $\tilde{\phi}(X) = \phi(\tilde{\mathbb{P}}X)$
- **Use** Fréchet differentials on **flat space** $L^2$

**Definition of L-differentiability**

$\phi$ is differentiable at $\mu_0$ if $\tilde{\phi}$ is Fréchet differentiable at $X_0$ s.t. $\tilde{\mathbb{P}}X_0 = \mu_0$

- **Check** definition is **intrinsic**
Properties of L-differentials

- $\partial \phi(\mu_0) = D\tilde{\phi}(X_0) \in L^2(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$
- The distribution of the random variable $\partial \phi(\mu_0)$ depends only on $\mu_0$, NOT ON THE RANDOM VARIABLE $X_0$ used to represent it
- $\exists \xi : \mathbb{R}^d \mapsto \mathbb{R}^d$ uniquely defined $\mu_0$ a.e. such that $\partial \phi(\mu_0) = D\tilde{\phi}(X_0) = \xi(X_0)$
- we use $\partial \phi(\mu_0)(\cdot) = \xi$

Examples

\[
\phi(\mu) = \int_{\mathbb{R}^d} h(x) \mu(dx) \implies \partial \phi(\mu)(\cdot) = \partial h(\cdot)
\]

\[
\phi(\mu) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} h(x - y) \mu(dx) \mu(dy) \implies \partial \phi(\mu)(\cdot) = [2\partial h(\cdot) \ast \mu](\cdot)
\]

\[
\phi(\mu) = \int_{\mathbb{R}^d} \varphi(x, \mu) \mu(dx) \implies \partial \phi(\mu)(\cdot) = \partial_x \varphi(\cdot, \mu) + \int_{\mathbb{R}^d} \partial_\mu \varphi(x', \mu)(\cdot) \mu(dx')
\]
**Two More Examples**

Assume $\phi : \mathcal{P}_2(\mathbb{R}^d) \mapsto \mathbb{R}$ is L-differentiable and define

$$
\phi^N : \mathbb{R}^d \times \cdots \times \mathbb{R}^d \ni (x^1, \ldots, x^N) \mapsto \phi^N(x^1, \ldots, x^N) = \phi\left(\frac{1}{N} \sum_{i=1}^{N} \delta_{x^i}\right)
$$

$$
\partial_{x^i} \phi^N(x^1, \ldots, x^N) = \frac{1}{N} \partial_{\mu} \phi\left(\frac{1}{N} \sum_{i=1}^{N} \delta_{x^i}\right)(x^i)
$$

Assume $\phi : \mathcal{M}_2(\mathbb{R}^d) \mapsto \mathbb{R}$ has a linear functional derivative (at least in a neighborhood of $\mathcal{P}_2(\mathbb{R}^d)$ and that $\mathbb{R}^d \ni x \mapsto [\delta \phi / \delta m](m)(x)$ is differentiable and the derivative

$$
\mathcal{M}_2(\mathbb{R}^d) \times \mathbb{R}^d \ni (m, x) \mapsto \partial_x \left[ \frac{\delta \phi}{\delta m} \right](m)(x) \in \mathbb{R}^d
$$

is jointly continuous in $(m, x)$ and is of linear growth in $x$, then $\phi$ is L-differentiable and

$$
\partial_{\mu} \phi(\mu)(\cdot) = \partial_x \frac{\delta \phi}{\delta m}(\mu)(\cdot), \quad \mu \in \mathcal{P}_2(\mathbb{R}^d).
$$
**Convex Functions of Measures**

\( \phi : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R} \) is said to be **L-convex** if

\[
\forall \mu, \mu' \quad \phi(\mu') - \phi(\mu) - \mathbb{E}[\partial_\mu \phi(\mu)(X) \cdot (X' - X)] \geq 0,
\]

whenever \( P_X = \mu \) and \( P_{X'} = \mu' \).

**Example 1**

\[
\mu \mapsto \phi(\mu) = g\left( \int_{\mathbb{R}^d} \zeta(x)d\mu(x) \right),
\]

- for \( g : \mathbb{R} \to \mathbb{R} \) is non-decreasing convex differentiable
- and \( \zeta : \mathbb{R}^d \to \mathbb{R} \) convex differentiable with derivative of at most of linear growth

**Example 2**

\[
\mu \mapsto \phi(\mu) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g(x, x')d\mu(x)d\mu(x')
\]

- If \( g : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \) is convex differentiable (\( \partial g \) linear growth)

**A sobering counter-example.** If \( \mu_0 \in \mathcal{P}_2(E) \) is fixed, the square distance function

\[
\mathcal{P}_2(E) \ni \mu \to W_2(\mu_0, \mu)^2 \in \mathbb{R}
\]

may not be convex or even L-differentiable!
BACK TO THE CONTROL OF McKean-Vlasov EQUATIONS

\[
\inf_{\alpha=(\alpha_t)_{0 \leq t \leq T}} \mathbb{E} \left[ \int_0^T f(t, X_t, \mathbb{P}_{X_t}, \alpha_t) \, dt + g(X_T, \mathbb{P}_{X_T}) \right]
\]

under the dynamical constraint

\[
dX_t = b(t, X_t, \mathbb{P}_{X_t}, \alpha_t) \, dt + \sigma(t, X_t, \mathbb{P}_{X_t}, \alpha_t) \, dW_t.
\]
EXAMPLE: POTENTIAL MEAN FIELD GAMES

Start with **Mean Field Game à la Lasry-Lions**

\[
\inf_{\alpha=(\alpha_t)_{0 \leq t \leq T}, \, dX_t=\alpha_t \, dt + \sigma \, dW_t} \mathbb{E} \left[ \int_0^T \left[ \frac{1}{2} |\alpha_t|^2 + f(t, X_t, \mu_t) \right] dt + g(X_T, \mu_T) \right]
\]

s.t. \( f \) and \( g \) are differentiable w.r.t. \( x \) and there exist differentiable functions \( F \) and \( G \)

\[
\partial_x f(t, x, \mu) = \partial_{\mu} F(t, \mu)(x) \quad \text{and} \quad \partial_x g(x, \mu) = \partial_{\mu} G(\mu)(x)
\]

Solving this **MFG** is equivalent to solving the **central planner** optimization problem

\[
\inf_{\alpha=(\alpha_t)_{0 \leq t \leq T}, \, dX_t=\alpha_t \, dt + \sigma \, dW_t} \mathbb{E} \left[ \int_0^T \left[ \frac{1}{2} |\alpha_t|^2 + F(t, \mathbb{P} X_t) \right] dt + G(\mathbb{P} X_T) \right]
\]

Special case of **McKean-Vlasov optimal control**
The Adjoint Equations

Lifted Hamiltonian

\[ \tilde{H}(t, x, \tilde{X}, y, \alpha) = H(t, x, \mu, y, \alpha) \]

for any random variable \( \tilde{X} \) with distribution \( \mu \).

Given an admissible control \( \alpha = (\alpha_t)_{0 \leq t \leq T} \) and the corresponding controlled state process \( X^\alpha = (X^\alpha_t)_{0 \leq t \leq T} \), any couple \( (Y_t, Z_t)_{0 \leq t \leq T} \) satisfying:

\[
\begin{cases}
    dY_t = -\partial_x H(t, X^\alpha_t, \mathbb{P}X^\alpha_t, Y_t, \alpha_t)dt + Z_t dW_t \\
    Y_T = \partial_x g(X^\alpha_T, \mathbb{P}X^\alpha_T) + \tilde{E}[\partial_\mu H(t, \tilde{X}_t, X, \tilde{Y}_t, \tilde{\alpha}_t)|X = X^\alpha_t] dt
\end{cases}
\]

where \((\tilde{\alpha}, \tilde{X}, \tilde{Y}, \tilde{Z})\) is an independent copy of \((\alpha, X^\alpha, Y, Z)\), is called a set of adjoint processes

BSDE of Mean Field type according to Buckhdan-Li-Peng !!!

Extra terms in red are the ONLY difference between MFG and Control of McKean-Vlasov dynamics !!!
PONTRYAGIN MAXIMUM PRINCIPLE (SUFFICIENCY)

Assume
1. Coefficients continuously differentiable with bounded derivatives;
2. Terminal cost function $g$ is convex;
3. $\alpha = (\alpha_t)_{0 \leq t \leq T}$ admissible control, $X = (X_t)_{0 \leq t \leq T}$ corresponding dynamics, $(Y, Z) = (Y_t, Z_t)_{0 \leq t \leq T}$ adjoint processes and

$$(x, \mu, \alpha) \mapsto H(t, x, \mu, Y_t, Z_t, \alpha)$$

is $dt \otimes d\mathbb{P}$ a.e. convex,

then, if moreover

$$H(t, X_t, \mathbb{P}X_t, Y_t, Z_t, \alpha_t) = \inf_{\alpha \in A} H(t, X_t, \mathbb{P}X_t, Y_t, \alpha), \quad \text{a.s.}$$

Then $\alpha$ is an optimal control, i.e.

$$J(\alpha) = \inf_{\beta \in A} J(\beta).$$
**Particular Case: Scalar Interactions**

\[ b(t, x, \mu, \alpha) = \tilde{b}(t, x, \langle \psi, \mu \rangle, \alpha) \quad \sigma(t, x, \mu, \alpha) = \tilde{\sigma}(t, x, \langle \phi, \mu \rangle, \alpha) \]

\[ f(t, x, \mu, \alpha) = \tilde{f}(t, x, \langle \gamma, \mu \rangle, \alpha) \quad g(x, \mu) = \tilde{g}(x, \langle \zeta, \mu \rangle) \]

- \( \psi, \phi, \gamma \) and \( \zeta \) differentiable with at most quadratic growth at \( \infty \),
- \( \tilde{b}, \tilde{\sigma} \) and \( \tilde{f} \) differentiable in \( (x, r) \in \mathbb{R}^d \times \mathbb{R} \) for \( t, \alpha \) fixed
- \( \tilde{g} \) differentiable in \( (x, r) \in \mathbb{R}^d \times \mathbb{R} \).

Recall that the adjoint process satisfies

\[ Y_T = \partial_x g(X_T, \mathbb{P} X_T) + \tilde{\mathbb{E}}[\partial_\mu g(\tilde{X}_T, \mathbb{P} \tilde{X}_T)(X_T)]. \]

but since

\[ \partial_\mu g(x, \mu)(x') = \partial_r \tilde{g}(x, \langle \zeta, \mu \rangle) \partial_\zeta(x'), \]

the terminal condition reads

\[ Y_T = \partial_x \tilde{g}(X_T, \mathbb{E}[\zeta(X_T)]) + \tilde{\mathbb{E}}[\partial_r \tilde{g}(\tilde{X}_T, \mathbb{E}[\zeta(X_T)])] \partial_\zeta(X_T) \]

**Convexity** in \( \mu \) follows convexity of \( \tilde{g} \).
Scalar Interactions (cont.)

\[ H(t, x, \mu, y, z, \alpha) = \tilde{b}(t, x, \langle \psi, \mu \rangle, \alpha) \cdot y + \tilde{\sigma}(t, x, \langle \phi, \mu \rangle, \alpha) \cdot z + \tilde{f}(t, x, \langle \gamma, \mu \rangle, \alpha). \]

\[ \partial_\mu H(t, x, \mu, y, z, \alpha) \] can be identified with

\[ \partial_\mu H(t, x, \mu, y, z, \alpha)(x') = \left[ \partial_r \tilde{b}(t, x, \langle \psi, \mu \rangle, \alpha) \cdot y \right] \partial_\mu \psi(x') \]
\[ + \left[ \partial_r \tilde{\sigma}(t, x, \langle \phi, \mu \rangle, \alpha) \cdot z \right] \partial_\mu \phi(x') \]
\[ + \partial_r \tilde{f}(t, x, \langle \gamma, \mu \rangle, \alpha) \partial_\mu \gamma(x') \]

and the adjoint equation rewrites:

\[ dY_t = -\left\{ \partial_x \tilde{b}(t, X_t, \mathbb{E}[\psi(X_t)], \alpha_t) \cdot Y_t + \partial_x \tilde{\sigma}(t, X_t, \mathbb{E}[\phi(X_t)], \alpha_t) \cdot Z_t \right\} dt + Z_t dW_t \]
\[ -\left\{ \tilde{E}[\partial_r \tilde{b}(t, \tilde{X}_t, \mathbb{E}[\psi(\tilde{X}_t)], \tilde{\alpha}_t) \cdot \tilde{Y}_t] \partial_\mu \psi(X_t) + \tilde{E}[\partial_r \tilde{\sigma}(t, \tilde{X}_t, \mathbb{E}[\phi(\tilde{X}_t)], \tilde{\alpha}_t) \cdot \tilde{Z}_t] \partial_\mu \phi(X_t) \right\} dt \]

Anderson - Djehiche
Solution of the McKV Control Problem

Assume

\[ b(t, x, \mu, \alpha) = b_0(t) \int_{\mathbb{R}^d} xd\mu(x) + b_1(t)x + b_2(t)\alpha \]

with \( b_0, b_1 \) and \( b_2 \) is \( \mathbb{R}^{d \times d} \)-valued and are bounded.

\[ f \text{ and } g \text{ as in MFG problem.} \]

Then there exists a solution \((X, Y, Z) = (X_t, Y_t, Z_t)_{0 \leq t \leq T}\) of the McKean-Vlasov FBSDE

\[
\begin{aligned}
    dX_t &= b_0(t)\mathbb{E}(X_t)dt + b_1(t)X_t dt + b_2(t)\hat{\alpha}(t, X_t, \mathbb{P}_{X_t}, Y_t)dt + \sigma dW_t, \\
    dY_t &= -\partial_x H(t, X_t, \mathbb{P}_{X_t}, Y_t, \hat{\alpha}_t) dt \\
     &\quad - \mathbb{E}[\partial_\mu \tilde{H}(t, \tilde{X}_t, X_t, \tilde{Y}_t, \tilde{\alpha}_t)] dt + Z_t dW_t.
\end{aligned}
\]

with \( Y_t = u(t, X_t, \mathbb{P}_{X_t}) \) for a function

\[ u : [0, T] \times \mathbb{R}^d \times \mathcal{P}_1(\mathbb{R}^d) \ni (t, x, \mu) \mapsto u(t, x, \mu) \]

uniformly of Lip-1 and with linear growth in \( x \).
For $N$ independent Brownian motions $(W^1, \ldots, W^N)$ and for a square integrable exchangeable process $\beta = (\beta^1, \ldots, \beta^N)$, consider the system

$$dX^i_t = \frac{1}{N} b_0(t) \sum_{j=1}^N X^j_t + b_1(t)X^i_t + b_2(t)\beta^i_t + \sigma dW^i_t, \quad X^i_0 = \xi^i_0,$$

and define the common cost

$$J^N(\beta) = \mathbb{E} \left[ \int_0^T f(s, X^i_s, \bar{\mu}^N_s, \beta^i_s) ds + g(X^i_T, \bar{\mu}^N_T) \right], \quad \text{with } \bar{\mu}^N_t = \frac{1}{N} \sum_{i=1}^N \delta_{X^i_t}.$$

Then, there exists a sequence $(\epsilon_N)_{N \geq 1}$, $\epsilon_N \searrow 0$, s.t. for all $\beta = (\beta^1, \ldots, \beta^N)$,

$$J^N(\beta) \geq J^N(\alpha) - \epsilon_N,$$

where, $\alpha = (\alpha^1, \ldots, \alpha^N)$ with

$$\alpha^i_t = \hat{\alpha}(s, \tilde{X}^i_t, u(t, \tilde{X}^i_t), \mathbb{P}_{X^i_t})$$

where $X$ and $u$ are from the solution to the controlled McKean Vlasov problem, and $(\tilde{X}^1, \ldots, \tilde{X}^N)$ is the state of the system controlled by $\alpha$, i.e.

$$d\tilde{X}^i_t = \frac{1}{N} \sum_{j=1}^N b_0(t)\tilde{X}^j_t + b_1(t)\tilde{X}^i_t + b_2(t)\hat{\alpha}(s, \tilde{X}^i_s, u(s, \tilde{X}^i_s), \mathbb{P}_{X^i_s}) + \sigma dW^i_t, \quad \tilde{X}^i_0 = \xi^i_0.$$
APPLICATION #2: CHAIN RULE

Assume
\[ dX_t = b_t dt + \sigma_t dW_t, \quad X_0 \in L^2(\Omega, \mathcal{F}, \mathbb{P}), \]
where
- \( W = (W_t)_{t \geq 0} \) is a \( \mathbb{F} \)-Brownian motion with values in \( \mathbb{R}^d \)
- \( (b_t)_{t \geq 0} \) and \( (\sigma_t)_{t \geq 0} \) are \( \mathbb{F} \)-progressive processes in \( \mathbb{R}^d \) and \( \mathbb{R}^{d \times d} \)
- Assume
\[ \forall T > 0, \quad \mathbb{E} \left[ \int_0^T (|b_t|^2 + |\sigma_t|^4) dt \right] < +\infty. \]

Then for any \( t \geq 0 \), if \( \mu_t = \mathbb{P}X_t \), and \( a_t = \sigma_t \sigma_t^\dagger \) then:
\[
    u(\mu_t) = u(\mu_0) + \int_0^t \mathbb{E}[\partial_\mu u(\mu_s)(X_s) \cdot b_s] ds + \frac{1}{2} \int_0^t \mathbb{E}[\partial_V (\partial_\mu u(\mu_s))(X_s) \cdot a_s] ds.
\]
Problem: if \( f : \mathcal{P}_2(\mathbb{R}^d) \mapsto \mathbb{R} \), minimize

\[
J(\alpha) = \int_0^T f(\mathbb{P}_{X_t^{\alpha}}) \, dt + \mathbb{E} \left[ \int_0^T \frac{1}{2} |\alpha_t|^2 \, dt \right]
\]

under the constraint:

\[
dX_t^{\alpha} = \alpha_t \, dt + dW_t, \quad 0 \leq t \leq T,
\]

Verification Argument: Assume \( u : [0, T] \times \mathcal{P}_2(\mathbb{R}^d) \rightarrow \mathbb{R} \) is \( C^{1,2} \), and satisfies

\[
\partial_t u(t, \mu) - \frac{1}{2} \int_{\mathbb{R}^d} |\partial_\mu u(t, \mu)(v)|^2 \, d\mu(v) + \frac{1}{2} \text{trace} \left[ \int_{\mathbb{R}^d} \partial_v \partial_\mu u(t, \mu)(v) d\mu(v) \right] + f(\mu) = 0,
\]

then, the McKean-Vlasov SDE

\[
d\hat{X}_t = -\partial_\mu u(t, \mathbb{P}_{\hat{X}_t})(\hat{X}_t) \, dt + dW_t, \quad 0 \leq t \leq T,
\]

has a unique solution \((\hat{X}_t)_{0 \leq t \leq T}\) satisfying \( \mathbb{E}[\sup_{0 \leq t \leq T} |\hat{X}_t|^2] < \infty \) which is the unique optimal path since \( \hat{\alpha}_t = -\partial_\mu u(t, \mathbb{P}_{\hat{X}_t})(\hat{X}_t) \) minimizes the cost:

\[
J(\hat{\alpha}) = \inf_{\alpha \in \mathcal{A}} J(\alpha).
\]
For a generic admissible control $\alpha = (\alpha_t)_{0 \leq t \leq T}$, set $X_t^\alpha = X_0 + \int_0^T \alpha_s ds + W_t$ and apply the chain rule:

$$du(t, \mathbb{P}_{X_t^\alpha})$$

$$= \left[ \partial_t u(t, \mathbb{P}_{X_t^\alpha}) + \mathbb{E} \left[ \partial_\mu u(t, \mathbb{P}_{X_t^\alpha}) (X_t^\alpha) \cdot \alpha_t \right] + \frac{1}{2} \mathbb{E} \left[ \text{trace} \left[ \partial_\nu \partial_\mu u(t, \mathbb{P}_{X_t^\alpha}) (X_t^\alpha) \right] \right] \right] dt$$

$$= \left[ -f(\mathbb{P}_{X_t^\alpha}) + \frac{1}{2} \mathbb{E} \left[ |\partial_\mu u(t, \mathbb{P}_{X_t^\alpha}) (X_t^\alpha)|^2 \right] + \mathbb{E} \left[ \partial_\mu u(t, \mathbb{P}_{X_t^\alpha}) (X_t^\alpha) \cdot \alpha_t \right] \right] dt$$

$$= \left[ -f(\mathbb{P}_{X_t^\alpha}) - \frac{1}{2} \mathbb{E} [\alpha_t^2] + \frac{1}{2} \mathbb{E} \left[ |\alpha_t + \partial_\mu u(t, \mathbb{P}_{X_t^\alpha}) (X_t^\alpha)|^2 \right] \right] dt$$

where we used the PDE satisfied by $u$, and identified a perfect square. Integrate both sides and get:

$$J(\alpha) = u(0, \mathbb{P}_{X_0^\alpha}) + \frac{1}{2} \mathbb{E} \left[ \int_0^T \left[ |\alpha_t + \partial_\mu u(t, \mathbb{P}_{X_t^\alpha}) (X_t^\alpha)|^2 \right] dt \right],$$

which shows that $\alpha_t = -\partial_\mu u(t, \mathbb{P}_{X_t^\alpha}) (X_t^\alpha)$ is optimal.
Joint Chain Rule

- If $u$ is smooth
- If $d\xi_t = \eta_t dt + \gamma_t dW_t$
- If $dX_t = b_t dt + \sigma_t dW_t$ and $\mu_t = \mathbb{P}X_t$

\[
\begin{align*}
    u(t, \xi_t, \mu_t) &= u(0, \xi_0, \mu_0) + \int_0^t \partial_x u(s, \xi_s, \mu_s) \cdot (\gamma_s dW_s) \\
    &\quad + \int_0^t \left( \partial_t u(s, \xi_s, \mu_s) + \partial_x u(s, \xi_s, \mu_s) \cdot \eta_s + \frac{1}{2} \text{trace} \left[ \partial_{xx} u(s, \xi_s, \mu_s) \gamma_s \gamma_s^\top \right] \right) ds \\
    &\quad + \int_0^t \tilde{E} \left[ \partial_\mu u(s, \xi_s, \mu_s)(\tilde{X}_s) \cdot \tilde{b}_s \right] ds + \frac{1}{2} \int_0^t \tilde{E} \left[ \text{trace} \left( \partial_v \left[ \partial_\mu u(s, \xi_s, \mu_s) \right] (\tilde{X}_s) \tilde{\sigma}_s \tilde{\sigma}_s^\top \right) \right] ds
\end{align*}
\]

where the process $(\tilde{X}_t, \tilde{b}_t, \tilde{\sigma}_t)_{0 \leq t \leq \tau}$ is an independent copy of the process $(X_t, b_t, \sigma_t)_{0 \leq t \leq \tau}$, on a different probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$.
Deriving the Master Equation

If \((t, x, \mu) \mapsto \mathcal{U}(t, x, \mu)\) is the master field

\[
\left( \mathcal{U}(t, X_t, \mu_t) - \int_0^t f(s, X_s, \mu_s, \hat{\alpha}(s, X_s, \mu_s, Y_s)) \, ds \right)_{0 \leq t \leq T}
\]

is a martingale whenever \((X_t, Y_t, Z_t)_{0 \leq t \leq T}\) is the solution of the forward-backward system characterizing the optimal path under the flow of measures \((\mu_t)_{0 \leq t \leq T}\). So if we compute its Itô differential, the drift must be 0.
An Example of Derivation

\[ dX_t = b(t, X_t, \mu_t, \alpha_t)dt + dW_t \]
\[ H(t, x, \mu, y, \alpha) = b(t, x, \mu, \alpha) \cdot y + f(t, x, \mu, \alpha) \]
\[ \hat{\alpha}(t, x, \mu, y) = \arg\inf_{\alpha} H(t, x, \mu, y, \alpha) \]

Itô’s Formula with \( \mu_t = \mathbb{P}X_t \)
(set \( \hat{\alpha}_t = \hat{\alpha}(t, X_t, \mu_t, \partial U(t, X_t, \mu_t)) \) and \( b_t = b(t, X_t, \mu_t, \hat{\alpha}_t) \))

\[ dU(t, X_t, \mu_t) = \]
\[ \left( \partial_t U(t, X_t, \mu_t) + b_t \cdot \partial_x U(t, X_t, \mu_t) + \frac{1}{2} \text{tr}[\partial_{xx}^2 U(t, X_t, \mu_t)] + f(t, x, \mu, \hat{\alpha}_t) \right)dt \]
\[ + \mathbb{E} \left[ b_t \cdot \partial_{\mu} U(t, X_t, \mu_t)(X_t) + \frac{1}{2} \partial_{\nu} \partial_{\mu} U(t, X_t, \mu_t)(X_t) \right] dt + \partial_x U(t, X_t, \mu_t)dW_t \]
**The Actual Master Equation**

\[
\begin{align*}
\partial_t U(t, x, \mu) &+ b(t, x, \mu, \alpha(t, x, \mu, \partial U(t, x, \mu))) \cdot \partial_x U(t, x, \mu) \\
&+ \frac{1}{2} \text{trace} \left[ \partial^2_{xx} U(t, x, \mu) \right] + f(t, x, \mu, \alpha(t, x, \mu, \partial U(t, x, \mu))) \\
&+ \int_{\mathbb{R}^d} \left[ b(t, x', \mu, \alpha(t, x, \mu, \partial U(t, x, \mu))) \cdot \partial_\mu U(t, x, \mu)(x') \\
&\quad + \frac{1}{2} \text{trace} \left( \partial_v \partial_\mu U(t, x, \mu)(x') \right) \right] d\mu(x') = 0,
\end{align*}
\]

for \((t, x, \mu) \in [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)\), with the **terminal** condition \(V(T, x, \mu) = g(x, \mu)\).