On a scaling limit of the parabolic Anderson model with exclusion interaction

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Joint work in progress with Martin Hairer
Setup

- We consider equations of the type

\[ \partial_t u = \Delta u + F(u, \xi), \]

where \( F \) is non-linear in \( u \), affin in \( \xi \) and \( \xi \) is an irregular input.
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- Naïve approach: Look at
  \[ \partial_t u_\varepsilon = \Delta u_\varepsilon + F(u_\varepsilon, \xi_\varepsilon), \]
  \( \xi_\varepsilon \) smoothened version of \( \xi \),
  \[ \rightarrow \text{ often does not converge as } \varepsilon \downarrow 0. \]
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  \( \rightarrow \) often does not converge as \( \varepsilon \downarrow 0 \).

- Solution: Use regularity structures to renormalise the equation.
Directions within regularity structures: White noise

- Look at convergence of the sequence of equations

\[ \partial_t u_\varepsilon = \Delta u_\varepsilon + F(u_\varepsilon, \xi_\varepsilon), \quad \xi_\varepsilon \text{ mollified white noise}. \]

- Recall: \( \xi \in S'(\mathbb{R}^d) \) is called white noise if \( \xi \) is Gaussian and

\[ \mathbb{E}[\xi(x)\xi(y)] = \delta_x(y). \]
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- **Goal:** Find an algorithmic way to deal with a large class of equations of the above type.
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- **Goal:** Find an algorithmic way to deal with a large class of equations of the above type.

- Example: Generalized KPZ (Bruned, Hairer, Zambotti)
  \[ \partial_t u = \Delta u + f(u)(\partial_x u)^2 + k(u)\partial_x u + h(u) + g(u)\xi \]
  \( \rightarrow \) 120 terms need to be controlled \( \Rightarrow \) not doable by hands.
Directions within regularity structures: non gaussian approximation

- Look at the convergence of the sequence of equations

\[ \partial_t u_\varepsilon = \Delta u_\varepsilon + F(u_\varepsilon, \xi_\varepsilon), \]

where \( \xi_\varepsilon \) is a non gaussian, smooth, strongly mixing field that approximates white noise.
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Examples:

- KPZ (Shen/Hairer)

\[
\partial_t u_\epsilon = \Delta u_\epsilon + (\partial_x u_\epsilon)^2 + \xi_\epsilon
\]

- "generalised" \( \Phi^4_3 \) equation (Shen/Xu)

\[
\partial_t u_\epsilon = \Delta u_\epsilon + V(u_\epsilon) + \xi_\epsilon,
\]

for some suitable polynomial \( V \).
Directions within regularity structures: the discrete case

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where time and space may be discrete.
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  where time and space may be discrete.
  
- The theory of regularity structures needs to be adapted to the discrete setting.

- Successfully dealt with by Hairer/Matetski who studied
  \[ \partial_t u_\varepsilon(x, t) = \Delta u_\varepsilon(x, t) - u_\varepsilon(x, t)^3 + \xi_\varepsilon(x, t), \]
  where \( x \in (\mathbb{Z}/\varepsilon\mathbb{Z})^3, \ t \geq 0, \) and \( \xi_\varepsilon \) is a gaussian approximation of white noise.
The PAM equation

\[
\begin{cases}
\frac{\partial u}{\partial t}(x, t) = \Delta^d u(x, t) + \xi(x, t)u(x, t), & x \in \mathbb{Z}^d, t \geq 0 \\
u(x, 0) = u_0(x).
\end{cases}
\]

- \(\Delta^d u(x, t) = \sum_{y:\|y-x\|=1}[u(y, t) - u(x, t)]\) is the discrete Laplacian.
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- The solution is given via the Feynman Kac formula

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u(x, t) = E_x \left[ \exp \left\{ \int_0^t \xi(X(s), t - s) \, ds \right\} u_0(X(t)) \right],
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where \(X\) is a simple random walk that starts in \(x\) under \(E_x\).
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**Goal:** Investigate the above equation under an appropriate space-time scaling.
The simple symmetric exclusion process

The simple symmetric exclusion process is a Markov process on \( \{0, 1\}^{\mathbb{Z}^d} \) whose generator acts on local function \( f : \{0, 1\}^{\mathbb{Z}^d} \to \mathbb{R} \) via

\[
(Lf)(\eta) = \sum_{||u-v||=1} \eta(u)[1 - \eta(v)][f(\eta^{u,v}) - f(\eta)],
\]

where

\[
\eta^{u,v}(z) = \begin{cases} 
\eta(z) & \text{if } z \neq u, v, \\
\eta(u) & \text{if } z = v, \\
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Fact: Let \( \rho \in (0, 1) \), then \( \nu_\rho = \bigotimes_{x \in \mathbb{Z}^d} \text{Ber}(\rho) \) is an invariant and reversible measure for \( \eta \). We always start \( \eta \) from \( \nu_\rho \).
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- \( \xi = \eta - \rho \).
Graphical representation of the exclusion process

Let $X_y$ be a particle starting at time $t$ at position $y$ following the arrows downwards. Then, $\eta_t(y) = \eta_0(X_y t)$.

Remark: The law of $X_y$ is that of a simple random walk!
Graphical representation of the exclusion process

Let $X^y$ be a particle starting at time $t$ at position $y$ following the arrows downwards. Then,

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**Remark:** The law of $X^y$ is that of a simple random walk!

$$\Rightarrow \quad \mathbb{E}[\xi(x, 0)\xi(y, t)] = \mathbb{E}[(\eta_0(x) - \rho)(\eta_0(X^y_t) - \rho)]$$
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$$\mathbb{E}[\xi(x, 0)\xi(y, t)] = \mathbb{E}[(\eta_0(x) - \rho)(\eta_0(X^y_t) - \rho)]$$
$$= \sum \rho_t(y, z)\mathbb{E}[(\eta_0(x) - \rho)(\eta_0(z) - \rho)]$$
$$= \rho_t(y, x)\rho(1 - \rho).$$
With the help of the Markov property we may even deduce that

$$\mathbb{E}[\xi(x, s)\xi(y, t)] = \rho(1 - \rho)p_{t-s}(x, y).$$
Fluctuations of the exclusion process

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Moreover,

$$p_{2^N(t-s)}(2^Nx, 2^Ny) \approx (4\pi 2^{2N}(t - s))^{-d/2} e^{-||x-y||^2/(t-s)}.$$
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Reasonable (and correct) guess:

- $2^{Nd/2}\xi(2^Nx, 2^Ny)$ converges to a Gaussian process $\Phi$ such that
  $$\mathbb{E}[\Phi(x, s)\Phi(y, t)] = \rho(1 - \rho)(4\pi(t - s))^{-d/2} e^{-||x-y||^2/(t-s)}.$$
Need for renormalisation

Scaling time by $2^{2N}$, space by $2^N$ and $\xi$ by $2^{Nd/2}$ we obtain

$$u(2^N x, 2^{2N} t) \approx E_{2^N x} \left[ \exp \left\{ 2^{Nd/2} \int_0^t \xi(2^N X(2^{2N} s), 2^{2N} s) \, ds \right\} \right]$$

$$= 1 + 2^{Nd/2} E_{2^N} \left[ \int_0^t \xi(2^N X(2^{2N} s), 2^{2N} s) \, ds \right]$$

$$+ 2^{Nd-1} E_{2^N x} \left[ \int_{0 \leq s_1, s_2 \leq t} \prod_{i=1}^2 \xi(2^N X(2^{2N} s_i), 2^{2N} s_i) \, ds_i \right] + \cdots$$
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Taking expectation with respect to $\xi$ of the last term we get approximately

$$E_{2^N x} \left[ \int_{0 \leq s_1, s_2 \leq t} \frac{1}{|s_1 - s_2|^{d/2}} e^{-||X(2^N s_1) - X(2^N s_2)||^2/(s_1 - s_2)} \, ds_1 \, ds_2 \right]$$

$\rightarrow$ problematic as soon as $d \geq 2$. 

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The result

- Let $\mathbb{T}_N^d = [0, 2^N)^d \cap \mathbb{Z}^d$ and define $\xi_N(x, t) = 2^{Nd/2} \xi(2^N x, 2^2N t)$ for $x \in \mathbb{T}_N^d 2^{-N}$. Let $u_N$ be the solution to

$$\frac{\partial u_N}{\partial t}(x, t) = 2^{2N} \Delta^d u_N(x, t) + [\xi_N(x, t) - C_N] u_N(x, t)$$

on $\mathbb{T}_N^d = [0, 2^N)^d \cap \mathbb{Z}^d$. 

Main result.

Let $d \in \{2, 3\}$. There is a sequence of constants $C_N$ tending to infinity and $T > 0$ such that $w_N$ converges in distribution in $C_\alpha, \alpha/2(T^d \times [0, T])$ with $\alpha < 2 - d/2$. The limit $w$ formally satisfies

$$\frac{\partial w}{\partial t}(x, t) = \Delta^d w(x, t) + (\Phi(x, t) - \infty) w(x, t).$$
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Let $T_N^d = [0, 2^N)^d \cap \mathbb{Z}^d$ and define $\xi_N(x, t) = 2^{Nd/2} \xi(2^N x, 2^{2N} t)$ for $x \in T_N^d 2^{-N}$. Let $u_N$ be the solution to

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$$\frac{\partial w}{\partial t}(x, t) = \Delta w(x, t) + (\Phi(x, t) - \infty) w(x, t).$$
Given an index set $A$ and a collection of random variables $\{X_a\}_{a \in A}$. We write for $B \subseteq A$

$$X_B = \{X_a : a \in B\} \quad \text{and} \quad X^B = \prod_{a \in B} X_a.$$ 

$\mathcal{P}(B)$ denotes the set of all partitions of $B$. 

**Definition** 

Fix a finite subset $B \subseteq A$. We define the cumulant $E_c(X_B)$ via

$$E_c(X_B) = \sum_{\pi \in \mathcal{P}(B)} (|\pi| - 1)! (-1)^{|\pi| - 1} \prod_{\bar{B} \in \pi} E(X_{\bar{B}}).$$
Introduction to cumulants I

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Introduction to cumulants II

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Examples:

\[ E_c[ X_{\{1,2\}} ] = E[X_1 X_2] - E[X_1]E[X_2]. \]
\[ E_c[ X_{\{1,2,3\}} ] = E[\prod_{i=1}^{3} X_i] - \sum_{i=1}^{3} E[X_i]E[\prod_{j \neq i} X_j] + 2 \prod_{i=1}^{3} E[X_i]. \]

Remark:

Cumulants are a mean to measure joint interaction of all random variables involved. If the \( X_i \)'s are gaussian, then \( E_c(X_B) = 0 \) unless \( |B| \leq 2 \).
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Remark:

- Cumulants are a mean to measure joint interaction of all random variables involved.
- If the \( X_i \)'s are gaussian, then \( Ec(X_B) = 0 \) unless \( |B| \leq 2 \).
Introduction to Wick products

Definition

The Wick product $X_A$ is recursively defined via $X_\emptyset = 1$ and

$$X^A = \sum_{B \subseteq A} : X_B : \sum_{\pi \in \mathcal{P}(A \setminus B)} \prod_{\bar{B} \in \pi} E_c(X_{\bar{B}}).$$

Examples:

$X^1 = X_1 - EX_1$, and if the $X_i$'s have mean zero, $X^1 X^2 = X^1 X^2 - EX_1 X_2$,

$X^1 X^2 X^3 = \prod_{i=1}^3 X_i - \sum_{i=1}^3 X_i E \prod_{j \neq i} X_j$.

Remark: Expectations of Wick products can be expressed in terms of cumulants.
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Remark: Expectations of Wick products can be expressed in terms of cumulants.
Application to the PAM equation

Recall that

\[
2^{Nd-1} \mathbb{E} \left( E_{2N} \left[ \int_{0 \leq s_1, s_2 \leq t} \prod_{i=1}^{2} \xi(2^N X(2^N s_i), 2^N s_i) ds_i \right] \right)
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causes problems as soon as \( d \geq 2 \).
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**Idea:** Replace \( \prod_{i=1}^{2} \xi(2^N X(2^{2N} s_i), 2^{2N} s_i) \) by its Wick product. → need to control cumulants of \( \xi \).
How to obtain cumulants of $\xi$}

Let $X_{y_1}, \ldots, X_{y_n}$ be a collection of random walks that jump according to the exponential clocks from the graphical construction, then

$$(\eta_t(y_1), \ldots, \eta_t(y_n)) = (\eta_0(X_{y_1}t), \ldots, \eta_0(X_{y_n}t)),$$

Given a measurable function $f$, then for any initial state $\eta_0$,

$$E_{\eta_0}[f(\eta_t(y_1), \ldots, \eta_t(y_n))] = E_{\eta_0}[f(\eta_0(X_{y_1}t), \ldots, \eta_0(X_{y_n}t))] = \sum_{z_1, \ldots, z_n} p_{y_1, \ldots, y_n} t(z_1, \ldots, z_n) E_{\eta_0}[f(\eta_0(z_1), \ldots, \eta_0(z_n))].$$
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Let $X^{y_1}, \ldots, X^{y_n}$ be a collection of random walks that jump according to the exponential clocks from the graphical construction, then $(\eta_t(y_1), \ldots, \eta_t(y_n)) = (\eta_0(X_t^{y_1}), \ldots, \eta_0(X_t^{y_n})).$

Given a measurable function $f$, then for any initial state $\eta_0$,

$$\mathbb{E}_{\eta_0}[f(\eta_t(y_1), \ldots, \eta_t(y_n))] = \mathbb{E}_{\eta_0}[f(\eta_0(X_t^{y_1}), \ldots, \eta_0(X_t^{y_n}))]$$
How to obtain cumulants of $\xi$ 

Let $X^{y_1}, \ldots, X^{y_n}$ be a collection of random walks that jump according to the exponential clocks from the graphical construction, then $(\eta_t(y_1), \ldots, \eta_t(y_n)) = (\eta_0(X^{y_1}_t), \ldots, \eta_0(X^{y_n}_t))$.

Given a measurable function $f$, then for any initial state $\eta_0$,

\[
\mathbb{E}_{\eta_0}[f(\eta_t(y_1), \ldots, \eta_t(y_n))] = \mathbb{E}_{\eta_0}[f(\eta_0(X^{y_1}_t), \ldots, \eta_0(X^{y_n}_t))] = \sum_{z_1, \ldots, z_n} \rho^{y_1, \ldots, y_n}(z_1, \ldots, z_n) \mathbb{E}_{\eta_0}[f(\eta_0(z_1), \ldots, \eta_0(z_n))].
\]
How to obtain cumulants of $\xi$ II

$$\mathbb{E}_{\eta_0}[f(\eta_t(y_1), \ldots, \eta_t(y_n))] = \sum_{z_1, \ldots, z_n} p_{t}^{y_1, \ldots, y_n}(z_1, \ldots, z_n) \mathbb{E}_{\eta_0}[f(\eta_0(z_1), \ldots, \eta_0(z_n))].$$  

(1)
How to obtain cumulants of $\xi$ II

$$\mathbb{E}_{\eta_0}[f(\eta_t(y_1), \ldots, \eta_t(y_n))]$$

$$= \sum_{z_1, \ldots, z_n} \rho_t^{y_1, \ldots, y_n}(z_1, \ldots, z_n) \mathbb{E}_{\eta_0}[f(\eta_0(z_1), \ldots, \eta_0(z_n))].$$

(1)

Application of (1):

$$\mathbb{E}\left[\prod_{i=1}^{4} \xi(x_i, t_i)\right]$$
How to obtain cumulants of $\xi$ II

$$
\mathbb{E}_{\eta_0}[f(\eta_t(y_1), \ldots, \eta_t(y_n))]
= \sum_{z_1, \ldots, z_n} \rho^y_{t_1, \ldots, t_n}(z_1, \ldots, z_n) \mathbb{E}_{\eta_0}[f(\eta_0(z_1), \ldots, \eta_0(z_n))].
$$  \hfill (1)

Application of (1):

$$
\mathbb{E}\left[\prod_{i=1}^{4} \xi(x_i, t_i)\right] = \mathbb{E}\left[\prod_{i=1}^{3} \xi(x_i, t_i) \mathbb{E}_{\eta_3}[\xi(x_4, t_4)]\right]
$$
How to obtain cumulants of $\xi$ II

\[
\mathbb{E}_{\eta_0}[f(\eta_t(y_1), \ldots, \eta_t(y_n))] = \sum_{z_1, \ldots, z_n} p_t^{y_1, \ldots, y_n}(z_1, \ldots, z_n)\mathbb{E}_{\eta_0}[f(\eta_0(z_1), \ldots, \eta_0(z_n))]. \tag{1}
\]

Application of (1):

\[
\mathbb{E} \left[ \prod_{i=1}^{4} \xi(x_i, t_i) \right] = \mathbb{E} \left[ \prod_{i=1}^{3} \xi(x_i, t_i) \mathbb{E}_{\eta_3}[\xi(x_4, t_4)] \right] = \sum p_t^{x_4-t_3}(z^3) \mathbb{E} \left[ \prod_{i=1}^{3} \xi(x_i, t_i)\xi(z^3, t_3) \right].
\]
How to obtain cumulants of $\xi$ II

$$
\mathbb{E}_{\eta_0}[f(\eta_t(y_1), \ldots, \eta_t(y_n))] \\
= \sum_{z_1, \ldots, z_n} p_t^{y_1, \ldots, y_n}(z_1, \ldots, z_n) \mathbb{E}_{\eta_0}[f(\eta_0(z_1), \ldots, \eta_0(z_n))].
$$

(1)

Application of (1):

$$
\mathbb{E} \left[ \prod_{i=1}^{4} \xi(x_i, t_i) \right] = \mathbb{E} \left[ \prod_{i=1}^{3} \xi(x_i, t_i) \mathbb{E}_{\eta_3} [\xi(x_4, t_4)] \right] \\
= \sum p_{t_4-t_3}(z^3) \mathbb{E} \left[ \prod_{i=1}^{3} \xi(x_i, t_i) \xi(z^3, t_3) \right] \\
= \sum p_{t_4-t_3}(z^3) \mathbb{E} \left[ \prod_{i=1}^{2} \xi(x_i, t_i) \mathbb{E}_{\eta_2} [\xi(x_3, t_3) \xi(z^3, t_3)] \right]
$$
How to obtain cumulants of $\xi$ II

$$\mathbb{E}_{\eta_0}[f(\eta_t(y_1), \ldots, \eta_t(y_n))]$$

$$= \sum_{z_1, \ldots, z_n} p_t^{y_1, \ldots, y_n}(z_1, \ldots, z_n) \mathbb{E}_{\eta_0}[f(\eta_0(z_1), \ldots, \eta_0(z_n))]. \quad (1)$$

**Application of (1):**

$$\mathbb{E} \left[ \prod_{i=1}^{4} \xi(x_i, t_i) \right] = \mathbb{E} \left[ \prod_{i=1}^{3} \xi(x_i, t_i) \mathbb{E}_{\eta_3} \left[ \xi(x_4, t_4) \right] \right]$$

$$= \sum \rho_{t_4-t_3}^{x_4}(z^3) \mathbb{E} \left[ \prod_{i=1}^{3} \xi(x_i, t_i) \xi(z^3, t_3) \right]$$

$$= \sum \rho_{t_4-t_3}^{x_4}(z^3) \mathbb{E} \left[ \prod_{i=1}^{2} \xi(x_i, t_i) \mathbb{E}_{\eta_2} \left[ \xi(x_3, t_3) \xi(z^3, t_3) \right] \right]$$

$$= \sum \rho_{t_4-t_3}^{x_4}(z^3) \rho_{t_3-t_2}^{x_3, z^3}(z_1^2, z_2^2) \mathbb{E} \left[ \prod_{i=1}^{2} \xi(x_i, t_i) \xi(z_1^2, t_2) \xi(z_2^2, t_2) \right]$$
How to obtain cumulants of $\xi$ III

We obtain:

$$\mathbb{E} \left[ \prod_{i=1}^{4} \xi(x_i, t_i) \right]$$

$$= \sum \rho_{t_4-t_3}^{x_4} (z^3) \rho_{t_3-t_2}^{x_3, z^3} (z_1^2, z_2^2) \rho_{t_2-t_1}^{x_2, z_1^2, z_2^2} (z_1^1, z_2^1, z_3^1) \mathbb{E} \left[ \xi(x_1, t_1) \prod_{i=1}^{3} \xi(z_i^1, t_1) \right].$$
How to obtain cumulants of $\xi$ III

We obtain:

$$\mathbb{E} \left[ \prod_{i=1}^{4} \xi(x_i, t_i) \right]$$

$$= \sum p_{t_4-t_3}^{x_4} (z^3) p_{t_3-t_2}^{x_3,z_3^3} (z_2^2, z_2^2) p_{t_2-t_1}^{x_2,z_2^1,z_2^2} (z_1^1, z_1^1, z_1^1) \mathbb{E} \left[ \xi(x_1, t_1) \prod_{i=1}^{3} \xi(z_i^1, t_1) \right].$$

Since the initial configuration is $\nu_\rho = \bigotimes_{x \in \mathbb{Z}^d} \text{Ber}(\rho)$, only four terms contribute to the above sum:

1. $x_1 = z_1^1 = z_2^1 = z_3^1: p_{t_4-t_3}^{x_4} (x_3) p_{t_3-t_2}^{x_3} (x_2) p_{t_2-t_1}^{x_2} (x_1),$
How to obtain cumulants of $\xi$ III

We obtain:

$$\mathbb{E}\left[\prod_{i=1}^{4} \xi(x_i, t_i)\right]$$

$$= \sum \rho_{t_4-t_3}^{x_4}(z^3)\rho_{t_3-t_2}^{x_3, z^2}(z_1^2, z_2^2)\rho_{t_2-t_1}^{x_2, z_1^1, z_2^2}(z_1^1, z_2^1, z_3^1)\mathbb{E}\left[\xi(x_1, t_1) \prod_{i=1}^{3} \xi(z_i^1, t_1)\right].$$

Since the initial configuration is $\nu_\rho = \bigotimes_{x \in \mathbb{Z}^d} \text{Ber}(\rho)$, only four terms contribute to the above sum:

1. $x_1 = z_1^1 = z_2^1 = z_3^1$: $\rho_{t_4-t_3}^{x_4}(x_3)\rho_{t_3-t_2}^{x_3}(x_2)\rho_{t_2-t_1}^{x_2}(x_1)$,
2. $x_1 = z_1^1$ and $z_2^1 = z_3^1$: $\rho_{t_4-t_3}^{x_4}(x_3)\rho_{t_2-t_1}^{x_2}(x_1)$,
How to obtain cumulants of $\xi$ III

We obtain:

$$\mathbb{E} \left[ \prod_{i=1}^{4} \xi(x_i, t_i) \right]$$

$$= \sum p_{t_4-t_3}^{x_4} (z_3^3)p_{t_3-t_2}^{x_3,z_3^3}(z_2^2, z_2^2)p_{t_2-t_1}^{x_2,z_1^2,z_2^2}(z_1^1, z_1^1, z_1^3) \mathbb{E} \left[ \xi(x_1, t_1) \prod_{i=1}^{3} \xi(z_i^1, t_1) \right].$$

Since the initial configuration is $\nu_{\rho} = \bigotimes_{x \in \mathbb{Z}^d} \text{Ber}(\rho)$, only four terms contribute to the above sum:

1. $x_1 = z_1^1 = z_2^1 = z_3^1: p_{t_4-t_3}^{x_4} (x_3)p_{t_3-t_2}^{x_3}(x_2)p_{t_2-t_1}^{x_2}(x_1)$,
2. $x_1 = z_1^1$ and $z_1^1 = z_3^1: p_{t_4-t_3}^{x_4} (x_3)p_{t_2-t_1}^{x_2}(x_1)$,
3. $x_1 = z_2^1$ and $z_1^1 = z_3^1: \sum p_{t_4-t_3}^{x_4} (z)p_{t_3-t_2}^{x_3,z}(\bar{z}, x_2)p_{t_2-t_1}^{x_2}(x_1)$
How to obtain cumulants of $\xi$ III

We obtain:

$$
\mathbb{E} \left[ \prod_{i=1}^{4} \xi(x_i, t_i) \right]
$$

$$
= \sum p_{t_4-t_3}^{x_4} (z^3) p_{t_3-t_2}^{x_3,z^3} (z_1^2, z_2^2) p_{t_2-t_1}^{x_2,z_1^2,z_2^2} (z_1^1, z_1^1, z_1^3) \mathbb{E} \left[ \xi(x_1, t_1) \prod_{i=1}^{3} \xi(z_i^1, t_1) \right].
$$

Since the initial configuration is $\nu_{\rho} = \bigotimes_{x \in \mathbb{Z}^d} \text{Ber}(\rho)$, only four terms contribute to the above sum:

1. $x_1 = z_1^1 = z_1^2 = z_1^3$: $p_{t_4-t_3}^{x_4} (x_3) p_{t_3-t_2}^{x_3} (x_2) p_{t_2-t_1}^{x_2} (x_1)$,

2. $x_1 = z_1^1$ and $z_1^2 = z_1^3$: $p_{t_4-t_3}^{x_4} (x_3) p_{t_2-t_1}^{x_2} (x_1)$,

3. $x_1 = z_2^1$ and $z_1^1 = z_1^3$: $\sum p_{t_4-t_3}^{x_4} (z) p_{t_3-t_2}^{x_3,z} (\bar{z}, x_2) p_{t_2-t_1}^{\bar{z}} (x_1)$ and

4. $x_1 = z_3^1$ and $z_1^1 = z_2^1$: $\sum p_{t_4-t_3}^{x_4} (z) p_{t_3-t_2}^{x_3,z} (x_2, \bar{z}) p_{t_2-t_1}^{\bar{z}} (x_1)$. 
How to obtain cumulants of $\xi$

1. $p_{t_4-t_3}^{x_4} (x_3) p_{t_3-t_2}^{x_3} (x_2) p_{t_2-t_1}^{x_2} (x_1),$
2. $p_{t_4-t_3}^{x_4} (x_3) p_{t_2-t_1}^{x_2} (x_1),$
3. $\sum p_{t_4-t_3}^{x_4} (z) p_{t_3-t_2}^{x_3} (\bar{Z}, x_2) p_{t_2-t_1}^{\bar{Z}} (x_1)$
4. $\sum p_{t_4-t_3}^{x_4} (z) p_{t_3-t_2}^{x_3} (x_2, \bar{Z}) p_{t_2-t_1}^{\bar{Z}} (x_1).$
How to obtain cumulants of $\xi$

1. $p_{t_4-t_3}^{x_4}(x_3)p_{t_3-t_2}^{x_3}(x_2)p_{t_2-t_1}^{x_2}(x_1),$

2. $p_{t_4-t_3}^{x_4}(x_3)p_{t_2-t_1}^{x_2}(x_1),$

3. $\sum p_{t_4-t_3}^{x_4}(z)p_{t_3-t_2}^{x_3}(\bar{z}, x_2)p_{t_2-t_1}^{\bar{z}}(x_1)$

4. $\sum p_{t_4-t_3}^{x_4}(z)p_{t_3-t_2}^{x_3}(\bar{z}, x_2)p_{t_2-t_1}^{\bar{z}}(x_1).$

Since $\xi$ has mean zero,

$$\mathbb{E}_c(\xi(x_i, t_i), i \in \{1, \ldots, 4\})$$

$$= \mathbb{E}\left[ \prod_{i=1}^{4} \xi(x_i, t_i) \right] - \sum_{i<j} \mathbb{E}\left[ \prod_{k \in \{i,j\}} \xi(x_k, t_k) \right] \mathbb{E}\left[ \prod_{k \notin \{i,j\}} \xi(x_k, t_k) \right]$$
How to obtain cumulants of $\xi$ IV

1. $p_{t_4-t_3}^{x_4} (x_3) p_{t_3-t_2}^{x_3} (x_2) p_{t_2-t_1}^{x_2} (x_1)$,

2. $p_{t_4-t_3}^{x_4} (x_3) p_{t_2-t_1}^{x_2} (x_1)$,

3. $\sum p_{t_4-t_3}^{x_4} (z) p_{t_3-t_2}^{x_3,z} (\bar{z}, x_2) p_{t_2-t_1}^{\bar{z}} (x_1)$

4. $\sum p_{t_4-t_3}^{x_4} (z) p_{t_3-t_2}^{x_3,z} (x_2, \bar{z}) p_{t_2-t_1}^{\bar{z}} (x_1)$.

Since $\xi$ has mean zero,

$$E_c(\xi(x_i, t_i), i \in \{1, \ldots, 4\})$$

$$= E\left[ \prod_{i=1}^{4} \xi(x_i, t_i) \right] - \sum_{i<j} E\left[ \prod_{k \in \{i, j\}} \xi(x_k, t_k) \right] E\left[ \prod_{k \notin \{i,j\}} \xi(x_k, t_k) \right]$$

$\rightarrow$ The first term above survives. The second term perfectly cancels out. The third and fourth term would cancel out if $p_{t}^{x_1,x_2}(y_1, y_2) = p_{t}^{x_1}(y_1) p_{t}^{x_2}(y_2)$ for all $x_i, y_i$. 
factorising the transition probabilities I

**Goal:** Write $p^x_{1,x_2}(y_1, y_2) = p^x_{1}(y_1)p^x_{2}(y_2) + "stuff"$ and characterise the "stuff".
factorising the transition probabilities I

Goal: Write \( p_{t}^{x_1,x_2}(y_1, y_2) = p_{t}^{x_1}(y_1)p_{t}^{x_2}(y_2) + "stuff" \) and characterise the "stuff".

Let \( x, y, z \in \mathbb{Z}^d \) and define \( f_y(z) = \mathbb{1}\{y = z\} \) and denote the exclusion particle started at \( x \) by \( X^x \), then there is a martingale \( M_t^x(y) \) such that

\[
\mathbb{1}\{X_t^x = y\} = \mathbb{1}\{X_0^x = y\} + \int_0^t (Lf_y)(X_s^x) \, ds + M_t^x(y).
\]

Here, \( L \) is the generator of the exclusion process.
factorising the transition probabilities

Goal: Write $p^x_{t_1,x_2}(y_1, y_2) = p^x_{t_1}(y_1)p^x_{t_2}(y_2) + "stuff"$ and characterise the "stuff".

Let $x, y, z \in \mathbb{Z}^d$ and define $f_y(z) = \mathbb{1}\{y = z\}$ and denote the exclusion particle started at $x$ by $X^x$, then there is a martingale $M^x_t(y)$ such that

$$\mathbb{1}\{X^x_t = y\} = \mathbb{1}\{X^x_0 = y\} + \int_0^t (Lf_y)(X^x_s) \, ds + M^x_t(y).$$

Here, $L$ is the generator of the exclusion process. Solving this equation with Duhamel’s principle shows that

$$\mathbb{1}\{X^x_t = y\} = p^x_t(y) + \int_0^t \sum_z p^y_{t-s}(z) \, dM^x_s(z).$$
factorising the transition probabilities I

Goal: Write $p_t^{x_1, x_2}(y_1, y_2) = p_t^{x_1}(y_1)p_t^{x_2}(y_2) + "stuff"$ and characterise the "stuff".

Let $x, y, z \in \mathbb{Z}^d$ and define $f_y(z) = 1\{y = z\}$ and denote the exclusion particle started at $x$ by $X^x$, then there is a martingale $M_t^x(y)$ such that

$$1\{X^x_t = y\} = 1\{X^x_0 = y\} + \int_0^t (Lf_y)(X^x_s) \, ds + M_t^x(y).$$

Here, $L$ is the generator of the exclusion process. Solving this equation with Duhamel’s principle shows that

$$1\{X^x_t = y\} = p_t^x(y) + \int_0^t \sum_z p_t^{y}(z) \, dM_s^x(z).$$

Note that $M_{r,t}^x := \int_0^r \sum_z p_t^{y}(z) \, dM_s^x(z)$ is a zero mean martingale in $r$. 
We conclude that

\[ p_{t_1,t_2}(y_1, y_2) = E \left[ \prod_{i=1}^{2} \mathbb{1}\{X_{t}^{x_i} = y_i\} \right] = E \left[ \prod_{i=1}^{2} (p_{t_i}^{x_i}(y_i) + M_{t,t_i}^{x_i}) \right] = \prod_{i=1}^{2} p_{t_i}^{x_i}(y_i) + E \left[ \prod_{i=1}^{2} M_{t,t_i}^{x_i}(y_i) \right], \]
We conclude that

\[ p_{t}^{x_{1},x_{2}}(y_{1}, y_{2}) = E \left[ \prod_{i=1}^{2} 1\{X_{t}^{x_{i}} = y_{i}\} \right] = E \left[ \prod_{i=1}^{2} (p_{t}^{x_{i}}(y_{i}) + M_{t,t}^{x_{i}}) \right] = \prod_{i=1}^{2} p_{t}^{x_{i}}(y_{i}) + E \left[ \prod_{i=1}^{2} M_{t,t}^{x_{i}}(y_{i}) \right], \]

\[ \rightarrow \text{Thus } "\text{stuff}" \text{ is given by the expectation of the product of two martingales. We are able to handle that.} \]
We conclude that

\[
\rho_{t_1,t_2}^{x_1,x_2}(y_1, y_2) = E \left[ \prod_{i=1}^{2} \mathbb{1}\{X_t^{x_i} = y_i\} \right] = E \left[ \prod_{i=1}^{2} (\rho_t^{x_i}(y_i) + M_t^{x_i}) \right]
\]

\[
= \prod_{i=1}^{2} \rho_t^{x_i}(y_i) + E \left[ \prod_{i=1}^{2} M_t^{x_i}(y_i) \right],
\]

Thus "stuff" is given by the expectation of the product of two martingales. We are able to handle that.

Thus, the fourth cumulants consists of a term of the form

\[
\rho_{t_4-t_3}^{x_4}(x_3)\rho_{t_3-t_2}^{x_3}(x_2)\rho_{t_2-t_1}^{x_2}(x_1)
\]

and two terms of the form

\[
\sum \rho_{t_4-t_3}^{x_4}(z) "stuff" \rho_{t_2-t_1}^{x_2}(x_1).
\]