On a scaling limit of the parabolic Anderson model with exclusion interaction

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Joint work in progress with Martin Hairer

Setup

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$$\partial_t u = \Delta u + F(u, \xi),$$

where F is non-linear in u, affin in ξ and ξ is an irregular input.

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- Solution: Use regularity structures to renormalise the equation.

Directions within regularity structures: White noise

- Look at convergence of the sequence of equations

$$\partial_t u_\varepsilon = \Delta u_\varepsilon + F(u_\varepsilon, \xi_\varepsilon), \qquad \xi_\varepsilon \text{ mollified white noise.}$$

- Recall: $\xi \in \mathcal{S}'(\mathbb{R}^d)$ is called white noise if ξ is Gaussian and

$$\mathbb{E}[\xi(x)\xi(y)] = \delta_x(y).$$

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- Example: Generalized KPZ (Bruned, Hairer, Zambotti)

$$\partial_t u = \Delta u + f(u)(\partial_x u)^2 + k(u)\partial_x u + h(u) + g(u)\xi$$

$\longrightarrow$ 120 terms need to be controlled $\Rightarrow$ not doable by hands.

Directions within regularity structures: non gaussian approximation

- Look at the convergence of the sequence of equations

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where ξ_ε is a non gaussian, smooth, strongly mixing field that approximates white noise.

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Examples:

- KPZ (Shen/Hairer)

$$\partial_t u_\varepsilon = \Delta u_\varepsilon + (\partial_x u_\varepsilon)^2 + \xi_\varepsilon$$

- "generalised" Φ^4_3 equation (Shen/Xu)

$$\partial_t u_\varepsilon = \Delta u_\varepsilon + V(u_\varepsilon) + \xi_\varepsilon,$$

for some suitable polynomial V.

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$\longrightarrow$ The theory of regularity structures needs to be adapted to the discrete setting.

- Successfully dealt with by Hairer/Matetski who studied

$$\partial_t u_\varepsilon(x,t) = \Delta u_\varepsilon(x,t) - u_\varepsilon(x,t)^3 + \xi_\varepsilon(x,t),$$

  where $x \in (\mathbb{Z}/\varepsilon\mathbb{Z})^3$, $t \geq 0$, and ξ_ε is a gaussian approximation of white noise.

The PAM equation

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = \Delta^{\mathrm{d}} u(x,t) + \xi(x,t) u(x,t), \quad x \in \mathbb{Z}^d, t \geq 0 \\ u(x,0) = u_0(x). \end{cases}$$

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$$u(x,t) = E_x\left[\exp\left\{\int_0^t \xi(X(s), t-s)\, ds\right\} u_0(X(t))\right],$$

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Goal: Investigate the above equation under an appropriate space-time scaling.

The simple symmetric exclusion process

- The simple symmetric exclusion process is a Markov process on $\{0,1\}^{\mathbb{Z}^d}$ whose generator acts on local function $f:\{0,1\}^{\mathbb{Z}^d}\to\mathbb{R}$ via
$$(Lf)(\eta)=\sum_{||u-v||=1}\eta(u)[1-\eta(v)][f(\eta^{u,v})-f(\eta)],$$
where
$$\eta^{u,v}(z)=\begin{cases}\eta(z) & \text{if } z\neq u,v,\\ \eta(u) & \text{if } z=v,\\ \eta(v) & \text{if } z=u.\end{cases}$$

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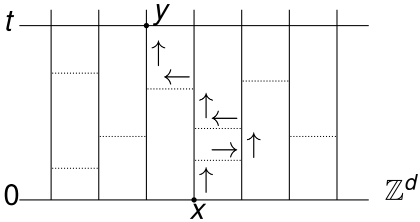
- $\xi = \eta - \rho$.

Graphical representation of the exclusion process



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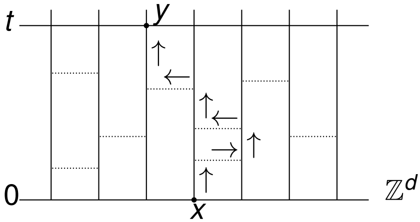

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$$\eta_t(y) = \eta_0(X_t^y).$$

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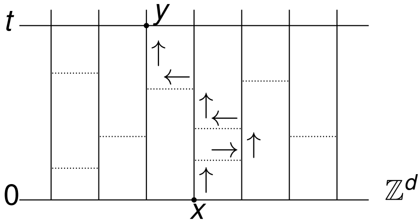

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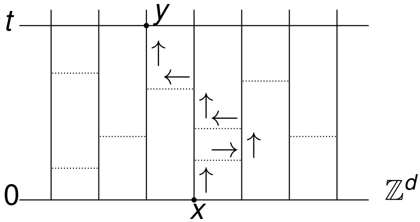

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$\Longrightarrow$

$$\mathbb{E}[\xi(x,0)\xi(y,t)] = \mathbb{E}[(\eta_0(x) - \rho)(\eta_0(X_t^y) - \rho)]$$

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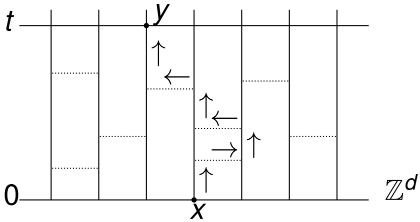

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$\Longrightarrow$

$$\begin{aligned}\mathbb{E}[\xi(x,0)\xi(y,t)] &= \mathbb{E}[(\eta_0(x)-\rho)(\eta_0(X_t^y)-\rho)]\\ &= \sum p_t(y,z)\mathbb{E}[(\eta_0(x)-\rho)(\eta_0(z)-\rho)]\\ &= p_t(y,x)\rho(1-\rho).\end{aligned}$$

Fluctuations of the exclusion process

With the help of the Markov property we may even deduce that

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Reasonable (and correct) guess:

- $2^{Nd/2}\xi(x2^N, t2^{2N})$ converges to a Gaussian process Φ such that $\mathbb{E}[\Phi(x,s)\Phi(y,t)] = \rho(1-\rho)(4\pi(t-s))^{-d/2}e^{-||x-y||^2/(t-s)}$.

Need for renormalisation

Scaling time by 2^{2N}, space by 2^N and ξ by $2^{Nd/2}$ we obtain

$$u(2^N x, 2^{2N} t) \approx E_{2^N x}\left[\exp\left\{2^{Nd/2}\int_0^t \xi(2^N X(2^{2N}s), 2^{2N}s)\,ds\right\}\right]$$
$$= 1 + 2^{Nd/2} E_{2^N}\left[\int_0^t \xi(2^N X(2^{2N}s), 2^{2N}s)\,ds\right]$$
$$+ 2^{Nd-1} E_{2^N x}\left[\int_{0\leq s_1, s_2 \leq t} \prod_{i=1}^{2} \xi(2^N X(2^{2N}s_i), 2^{2N}s_i) ds_i\right] + \cdots$$

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Taking expectation with respect to ξ of the last term we get approximately

$$E_{2^N x}\left[\int_{0\le s_1, s_2\le t} \frac{1}{|s_1 - s_2|^{d/2}} e^{-||X(2^N s_1) - X(2^N s_2)||^2/(s_1 - s_2)}\, ds_1\, ds_2\right]$$

$\longrightarrow$ problematic as soon as $d \geq 2$.

The result

- Let $\mathbb{T}_N^d = [0, 2^N)^d \cap \mathbb{Z}^d$ and define $\xi_N(x, t) = 2^{Nd/2}\xi(2^N x, 2^{2N} t)$ for $x \in \mathbb{T}_N^d 2^{-N}$. Let u_N be the solution to

$$\frac{\partial u_N}{\partial t}(x, t) = 2^{2N}\Delta^{\mathrm{d}} u_N(x, t) + [\xi_N(x, t) - C_N]u_N(x, t)$$

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Main result. Let $d \in \{2,3\}$. There is a sequence of constants C_N tending to infinity and $T > 0$ such that w_N converges in distribution in $\mathcal{C}^{\alpha,\alpha/2}(\mathbb{T}^d \times [0,T])$ with $\alpha < 2 - d/2$. The limit w formally satisfies

$$\frac{\partial w}{\partial t}(x,t) = \Delta w(x,t) + (\Phi(x,t) - \infty)w(x,t).$$

Introduction to cumulants I

- Given an index set $\mathcal{A}$ and a collection of random variables $\{X_a\}_{a\in\mathcal{A}}$. We write for $B \subseteq \mathcal{A}$

$$X_B = \{X_a : \ a \in B\} \quad \text{and} \quad X^B = \prod_{a\in B} X_a.$$

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Definition

Fix a finite subset $B \subseteq \mathcal{A}$. We define the cumulant $E_c(X_B)$ via

$$E_c(X_B) = \sum_{\pi\in\mathcal{P}(B)} (|\pi| - 1)!(-1)^{|\pi|-1} \prod_{\bar{B}\in\pi} E(X^{\bar{B}}).$$

Introduction to cumulants II

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- $E_c[X_{\{1,2\}}] = E[X_1 X_2] - EX_1 EX_2$.

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Remark:

- Cumulants are a mean to measure joint interaction of all random variables involved.
- If the X_i's are gaussian, then $E_c(X_B) = 0$ unless $|B| \leq 2$.

Introduction to Wick products

Definition

The Wick product $:X_A:$ is recursively defined via $:X_\emptyset: \; = 1$ and

$$X^A = \sum_{B \subseteq A} :X_B: \sum_{\pi \in \mathcal{P}(A \setminus B)} \prod_{\bar{B} \in \pi} E_c(X_{\bar{B}}).$$

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Remark: Expectations of Wick products can be expressed in terms of cumulants.

Application to the PAM equation

Recall that

$$2^{Nd-1}\mathbb{E}\left(E_{2^N}\left[\int_{0\le s_1,s_2\le t}\prod_{i=1}^{2}\xi(2^N X(2^{2N}s_i),2^{2N}s_i)ds_i\right]\right)$$

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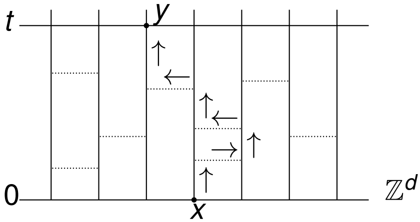
Idea: Replace $\prod_{i=1}^{2}\xi(2^N X(2^{2N}s_i),2^{2N}s_i)$ by its Wick product.
$\longrightarrow$ need to control cumulants of ξ.

How to obtain cumulants of ξ I



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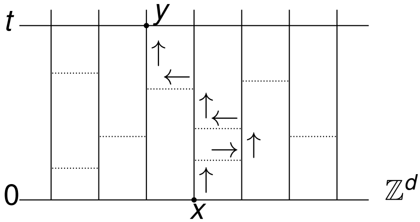

- Let $X^{y_1}, \dots, X^{y_n}$ be a collection of random walks that jump according to the exponential clocks from the graphical construction, then $(\eta_t(y_1), \dots, \eta_t(y_n)) = (\eta_0(X_t^{y_1}), \dots, \eta_0(X_t^{y_n}))$.

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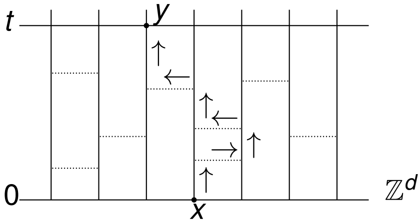

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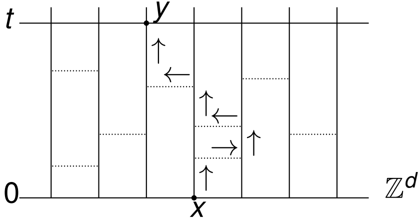

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$$\begin{aligned} &\mathbb{E}_{\eta_0}[f(\eta_t(y_1), \dots, \eta_t(y_n))] \\ &\quad = \mathbb{E}_{\eta_0}[f(\eta_0(X_t^{y_1}), \dots, \eta_0(X_t^{y_n})] \\ &\quad = \sum_{z_1, \dots, z_n} p_t^{y_1, \dots, y_n}(z_1, \dots, z_n) \mathbb{E}_{\eta_0}[f(\eta_0(z_1), \dots, \eta_0(z_n))]. \end{aligned}$$

How to obtain cumulants of ξ II

$$\begin{aligned}&\mathbb{E}_{\eta_0}[f(\eta_t(y_1),\dots,\eta_t(y_n))]\\&\quad=\sum_{z_1,\dots,z_n} p_t^{y_1,\dots,y_n}(z_1,\dots,z_n)\mathbb{E}_{\eta_0}[f(\eta_0(z_1),\dots,\eta_0(z_n))].\end{aligned}\tag{1}$$

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Application of (1)**:**

$$\mathbb{E}\Big[\prod_{i=1}^{4}\xi(x_i,t_i)\Big]$$

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$$\mathbb{E}\Big[\prod_{i=1}^{4}\xi(x_i,t_i)\Big] = \mathbb{E}\Big[\prod_{i=1}^{3}\xi(x_i,t_i)\mathbb{E}_{\eta_{t_3}}[\xi(x_4,t_4)]\Big]$$

How to obtain cumulants of ξ II

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&= \sum p_{t_4-t_3}^{x_4}(z^3)\mathbb{E}\Big[\prod_{i=1}^{3}\xi(x_i,t_i)\xi(z^3,t_3)\Big] \\
&= \sum p_{t_4-t_3}^{x_4}(z^3)\mathbb{E}\Big[\prod_{i=1}^{2}\xi(x_i,t_i)\mathbb{E}_{\eta_{t_2}}[\xi(x_3,t_3)\xi(z^3,t_3)]\Big]
\end{aligned}$$

How to obtain cumulants of ξ II

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$$= \sum p_{t_4-t_3}^{x_4}(z^3)\mathbb{E}\Big[\prod_{i=1}^2 \xi(x_i,t_i)\mathbb{E}_{\eta_{t_2}}[\xi(x_3,t_3)\xi(z^3,t_3)]\Big]$$

$$= \sum p_{t_4-t_3}^{x_4}(z^3)p_{t_3-t_2}^{x_3,z^3}(z_1^2,z_2^2)\mathbb{E}\Big[\prod_{i=1}^2 \xi(x_i,t_i)\xi(z_1^2,t_2)\xi(z_2^2,t_2)\Big]$$

How to obtain cumulants of ξ III

We obtain:

$$\mathbb{E}\Big[\prod_{i=1}^{4}\xi(x_i,t_i)\Big]$$

$$=\sum p_{t_4-t_3}^{x_4}(z^3)p_{t_3-t_2}^{x_3,z^3}(z_1^2,z_2^2)p_{t_2-t_1}^{x_2,z_1^2,z_2^2}(z_1^1,z_2^1,z_3^1)\mathbb{E}\Big[\xi(x_1,t_1)\prod_{i=1}^{3}\xi(z_i^1,t_1)\Big].$$

How to obtain cumulants of ξ III

We obtain:

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Since the initial configuration is $\nu_\rho=\bigotimes_{x\in\mathbb{Z}^d}\mathrm{Ber}(\rho)$, only four terms contribute to the above sum:

1. $x_1=z_1^1=z_2^1=z_3^1$: $p_{t_4-t_3}^{x_4}(x_3)p_{t_3-t_2}^{x_3}(x_2)p_{t_2-t_1}^{x_2}(x_1)$,

How to obtain cumulants of ξ III

We obtain:

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How to obtain cumulants of ξ III

We obtain:

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3. $x_1=z^1_2$ and $z^1_1=z^1_3$: $\sum p^{x_4}_{t_4-t_3}(z)p^{x_3,z}_{t_3-t_2}(\bar{z},x_2)p^{\bar{z}}_{t_2-t_1}(x_1)$

How to obtain cumulants of ξ III

We obtain:

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2. $x_1=z_1^1$ and $z_2^1=z_3^1$: $p_{t_4-t_3}^{x_4}(x_3)p_{t_2-t_1}^{x_2}(x_1)$,
3. $x_1=z_2^1$ and $z_1^1=z_3^1$: $\sum p_{t_4-t_3}^{x_4}(z)p_{t_3-t_2}^{x_3,z}(\bar z,x_2)p_{t_2-t_1}^{\bar z}(x_1)$ and
4. $x_1=z_3^1$ and $z_1^1=z_2^1$: $\sum p_{t_4-t_3}^{x_4}(z)p_{t_3-t_2}^{x_3,z}(x_2,\bar z)p_{t_2-t_1}^{\bar z}(x_1)$.

How to obtain cumulants of ξ IV

1. $p^{x_4}_{t_4-t_3}(x_3)p^{x_3}_{t_3-t_2}(x_2)p^{x_2}_{t_2-t_1}(x_1)$,
2. $p^{x_4}_{t_4-t_3}(x_3)p^{x_2}_{t_2-t_1}(x_1)$,
3. $\sum p^{x_4}_{t_4-t_3}(z)p^{x_3,z}_{t_3-t_2}(\bar{z},x_2)p^{\bar{z}}_{t_2-t_1}(x_1)$
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Since ξ has mean zero,

$$\begin{aligned}&\mathbb{E}_c(\xi(x_i,t_i),\ i\in\{1,\dots,4\})\\ &=\mathbb{E}\Big[\prod_{i=1}^{4}\xi(x_i,t_i)\Big]-\sum_{i<j}\mathbb{E}\Big[\prod_{k\in\{i,j\}}\xi(x_k,t_k)\Big]E\Big[\prod_{k\notin\{i,j\}}\xi(x_k,t_k)\Big]\end{aligned}$$

How to obtain cumulants of ξ IV

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$\longrightarrow$ The first term above survives. The second term perfectly cancels out. The third and fourth term would cancel out if $p^{x_1,x_2}_t(y_1,y_2)=p^{x_1}_t(y_1)p^{x_2}_t(y_2)$ for all x_i,y_i.

factorising the transition probabilities I

Goal: Write $p_t^{x_1,x_2}(y_1,y_2) = p_t^{x_1}(y_1)p_t^{x_2}(y_2)$ + "stuff" and characterise the "stuff".

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Let $x, y, z \in \mathbb{Z}^d$ and define $f_y(z) = 1\!\!1\{y = z\}$ and denote the exclusion particle started at x by X^x, then there is a martingale $M_t^x(y)$ such that

$$1\!\!1\{X_t^x = y\} = 1\!\!1\{X_0^x = y\} + \int_0^t (Lf_y)(X_s^x)\, ds + M_t^x(y).$$

Here, L is the generator of the exclusion process.

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Note that $M_{r,t}^x := \int_0^r \sum_z p_{t-s}^y(z)\,dM_s^x(z)$ is a zero mean martingale in r.

factorising the transition probabilities II

We conclude that

$$p_t^{x_1,x_2}(y_1,y_2) = E\Big[\prod_{i=1}^{2} \mathbb{1}\{X_t^{x_i} = y_i\}\Big] = E\Big[\prod_{i=1}^{2}(p_t^{x_i}(y_i) + M_{t,t}^{x_i})\Big]$$

$$= \prod_{i=1}^{2} p_t^{x_i}(y_i) + E\Big[\prod_{i=1}^{2} M_{t,t}^{x_i}(y_i)\Big],$$

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$\longrightarrow$ Thus "stuff" is given by the expectation of the product of two martingales. We are able to handle that.

factorising the transition probabilities II

We conclude that

$$p_t^{x_1,x_2}(y_1,y_2) = E\Big[\prod_{i=1}^{2} 1\!\!1\{X_t^{x_i} = y_i\}\Big] = E\Big[\prod_{i=1}^{2}(p_t^{x_i}(y_i) + M_{t,t}^{x_i})\Big]$$

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$\longrightarrow$ Thus "stuff" is given by the expectation of the product of two martingales. We are able to handle that.

Thus, the fourth cumulants consists of a term of the form $p^{x_4}_{t_4-t_3}(x_3)p^{x_3}_{t_3-t_2}(x_2)p^{x_2}_{t_2-t_1}(x_1)$ and two terms of the form $\sum p^{x_4}_{t_4-t_3}(z)$"stuff"$p^{\bar{z}}_{t_2-t_1}(x_1)$.