Alternating Sign Matrices and their various integrabilities

(PDF + Roger Behrend + Paul Zinn-Justin
+ E. Guitter + C. Kristjansen)

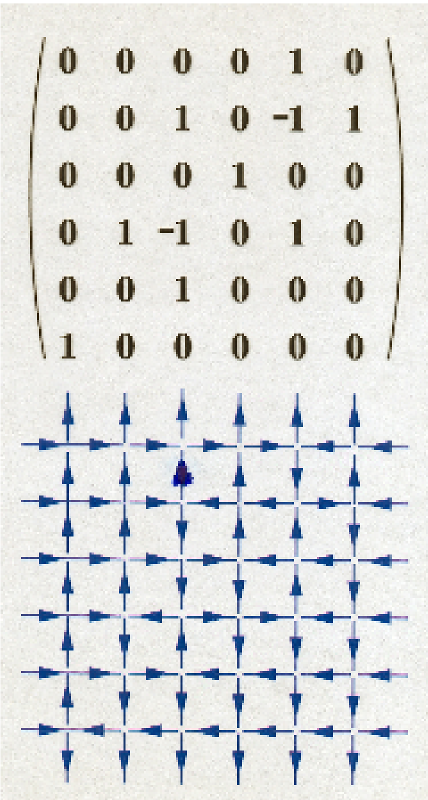
- Quantum Integrability
- Descending Plane Partitions
- Alternating Sign Matrices

identity between refined enumerations

$$\det(DPP) = \det(ASM)$$

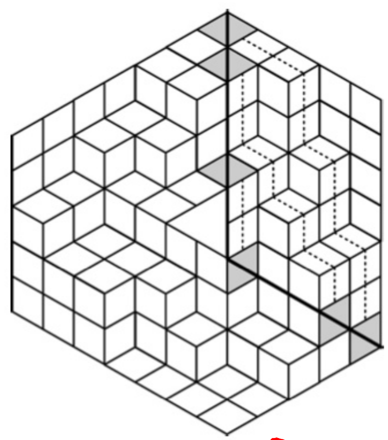
6V, ASM, , , and LOOP MODEL

$$A_n = \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}$$

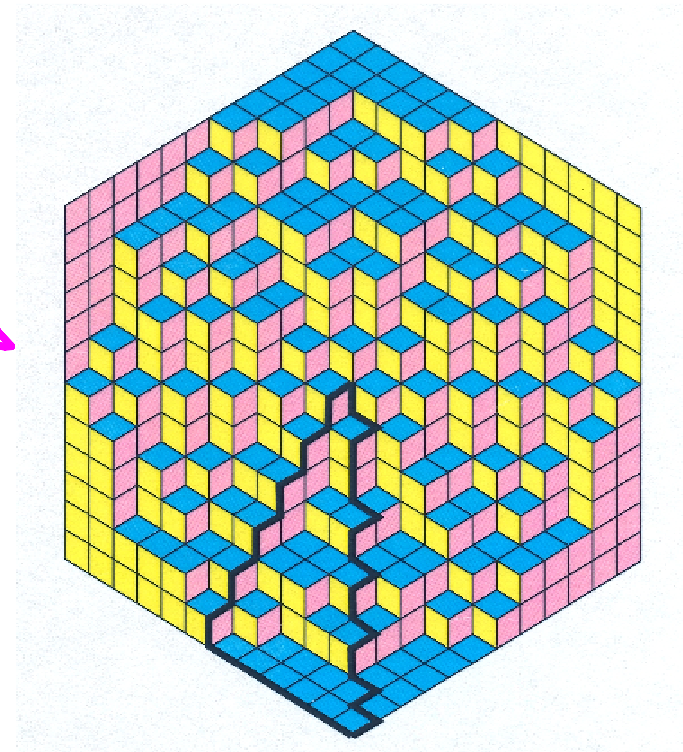


Alternating sign matrices

six vertex model

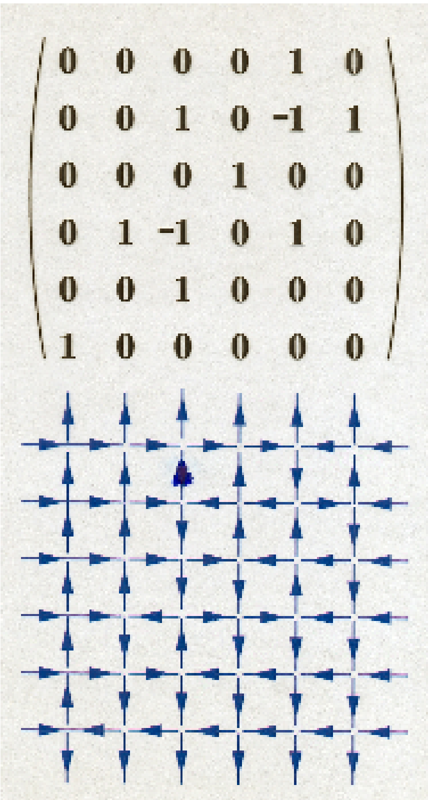


Descending Plane partitions



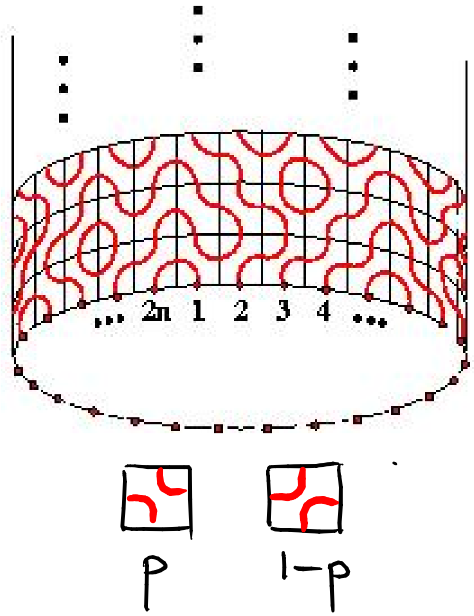
Totally Symmetric Self-Complementary Plane Partition

6V, ASM, , and



Alternating sign matrices (82)

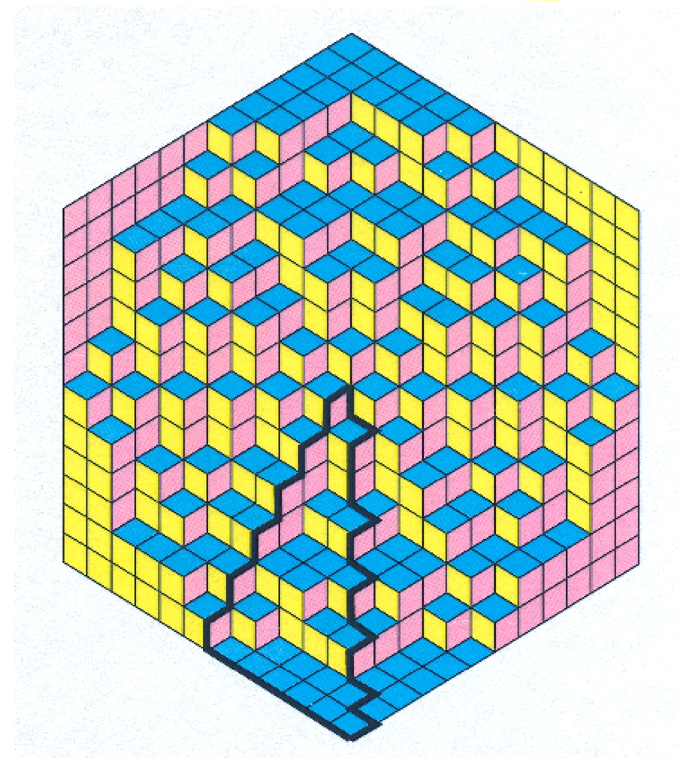
six vertex model



loop model

qKZ solution

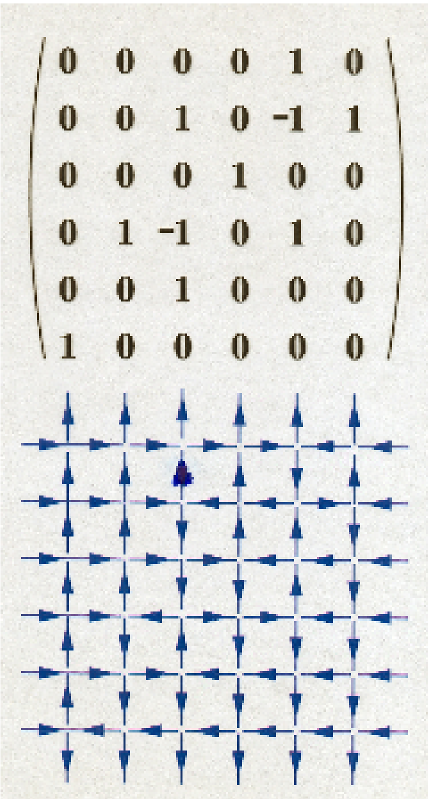
[Nienhuis de Gier
Razumov Stroganov
DF+Zinn-Justin]



Totally Symmetric Self-Complementary Plane Partition

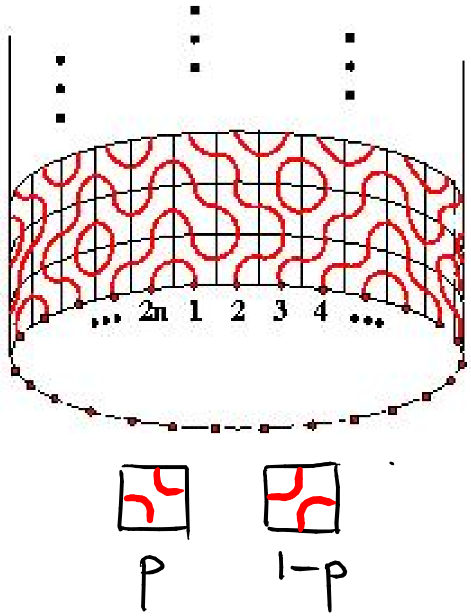
(70)

6V, ASM, , and



Alternating sign matrices (82)

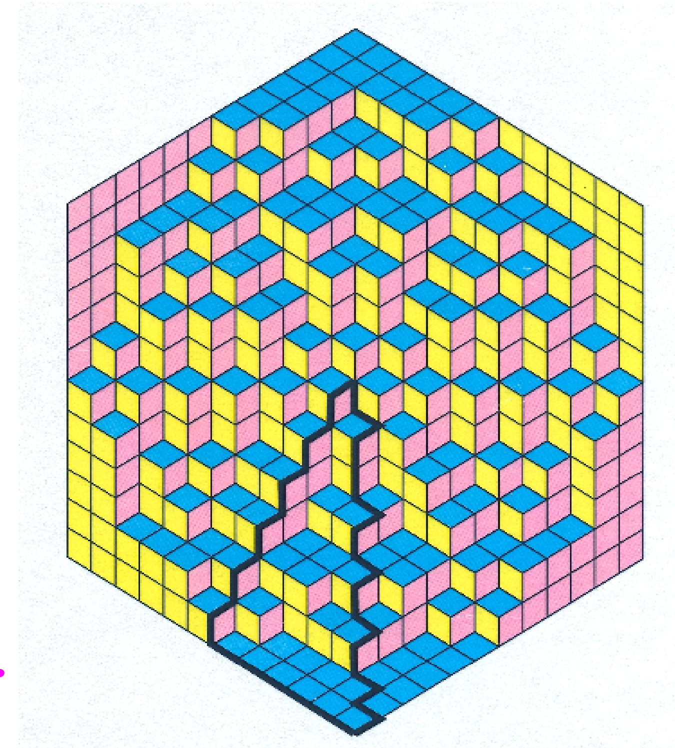
six vertex model



loop model

$$V = \{ M \in \mathcal{M}_{2n}(\mathbb{C}),\ M^2 = 0 \text{ and } M \text{ strictly upper triangular} \}$$

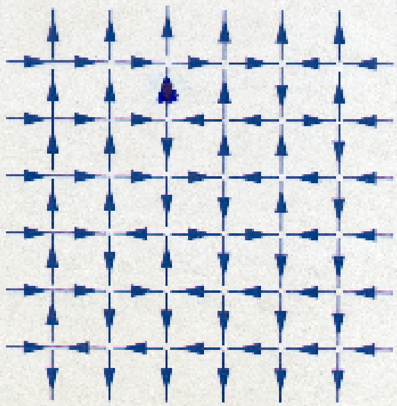
[DF-Zinn Justin 04-08]



Totally Symmetric Self-Complementary Plane Partition (70)

6V, ASM

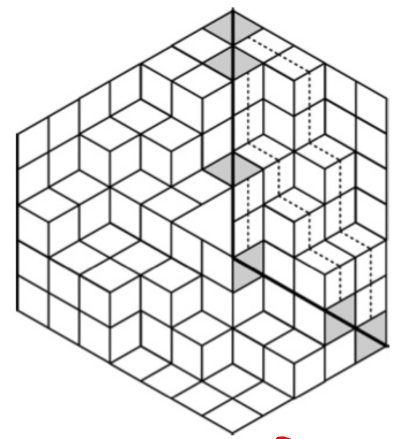
$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$



Alternating sign matrices (82)

six vertex model

DPP



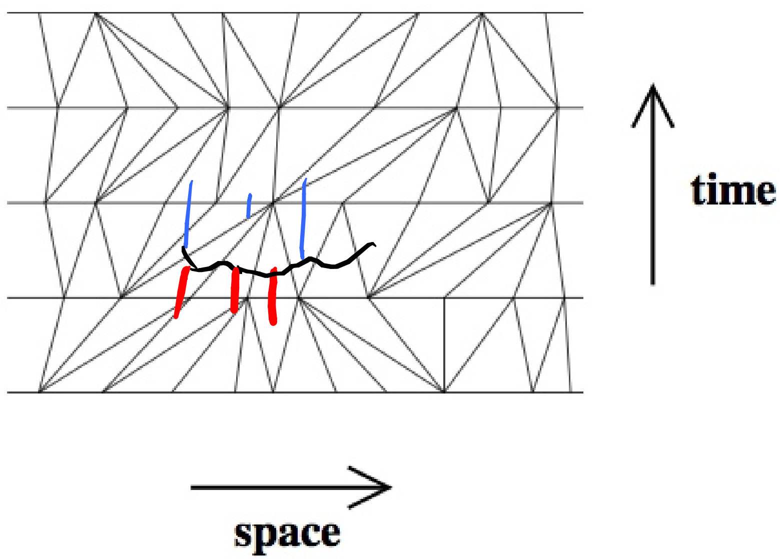
Descending Plane partitions

[DF Behrend Zinn Justin '11-12]

Refined Enumerations Coincide (Proof)

PREAMBLE : 1+1 Dimensional Lorentzian quantum gravity

(PDF + Emmanuel Guitter + Charlotte Kristjansen '99)



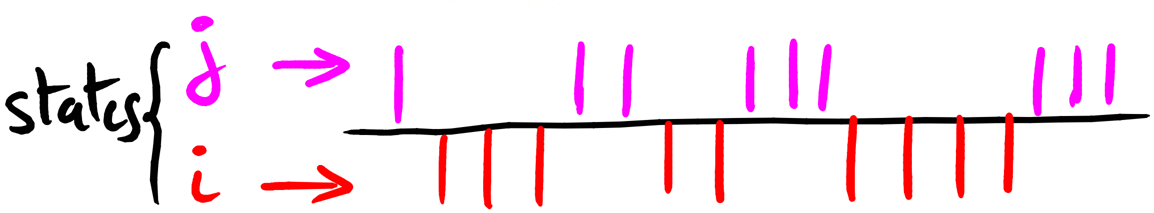
Triangulations that are

1. Random in space direction
2. Regular in time direction

$\Rightarrow$ TRANSFER MATRIX

$$\boxed{T_{ij} = \binom{i+j}{i}}$$

$(i,j \in \mathbb{Z}_+)$



Include $\begin{cases} \text{curvature weight } a \,/\, \perp\!\perp \text{ or } \top\!\top \\ \text{area weight } g \,/\, \perp \text{ or } \top \end{cases}$

Then

$$\boxed{T_{i,j}(g,a) = (ag)^{i+j} \sum_{k=0}^{\mathrm{Min}(i,j)} \binom{i}{k}\binom{j}{k} a^{-2k}}$$

Generating Function

$$\boxed{\sum_{i,j \geq 0} z^i w^j T_{i,j}(g,a) = \frac{1}{1 - ga(z+w) - g^2(1-a^2)zw}}$$

INFINITE MATRIX !

INTEGRABILITY

$$\boxed{\begin{gathered}[T(g,a), T(g',a')] = 0 \\ \iff \quad \varphi(g,a) = \varphi(g',a') \\ \varphi(g,a) = \frac{1-g^2(1-a^2)}{ag} \quad (= q+q^{-1})\end{gathered}}$$

- diagonalization of T
- correlation functions

DIAGONALIZATION of $T(g,a)$

$$\frac{1-g^2(1-a^2)}{ga} = q+q^{-1} \Rightarrow a = \frac{q}{1-q^2}\,\frac{1-\lambda}{\sqrt{\lambda}}\,;\; g = \sqrt{\lambda}\,\frac{1-q^2}{1-\lambda q^2}$$

$$\sum_{i,j\geq 0} T(g,a)_{ij}\, z^i w^j = \frac{1-\lambda q^2}{(1-qz)(1-qw)-\lambda(q-z)(q-w)}$$

$$= \sum_{m\geq 0} \underbrace{\sqrt{1-q^2}\,\frac{(q-z)^m}{(1-qz)^{m+1}}}_{\text{eigenvector}}\;\underbrace{\frac{1-\lambda q^2}{1-q^2}\,\lambda^m}_{\substack{\text{eigenvalue}\\ \Lambda_m}}\;\underbrace{\sqrt{1-q^2}\,\frac{(q-w)^m}{(1-qw)^{m+1}}}_{\text{eigenvector}}$$

eigenvector $\sum_{j\geq 0} z^j v^{(m)}_j$

eigenvalue Λ_m

eigenvector $\sum_{i\geq 0} w^i v^{(m)}_i$

$$\Longleftrightarrow\quad \boxed{T(g,a)\, v^{(m)} = \Lambda_m\, v^{(m)}}$$

END OF PREAMBLE

$\boxed{\text{DPP}}$ = Arrays of positive integers

$$\begin{array}{cccccc}
a_{11} & a_{12} & \cdots & & a_{1,\mu_1} & \\
 & a_{22} & a_{23} & \cdots & & a_{2,\mu_2} \\
 & & \ddots & & & \\
 & & & a_{r,r} & \cdots & a_{r,\mu_r}
\end{array}$$

1. $a \underset{\longrightarrow}{\geq} b$

2. $\begin{array}{c} a \\ \vee \\ b \end{array} \downarrow$

3. $\lambda_i = \mu_i - i + 1$ = # parts in row i

$\lambda_i < a_{i,i} \leq \lambda_{i-1}$

Vocabulary

- a_{ij} = part
- $a_{ij} \leq j-i$ = special part
- order n = $a_{ij} \leq n \ \forall i,j$.

OBSERVABLES

parts = n
special parts
non-special parts

$n=3$ 7 DPP's

$\emptyset \quad \boxed{2} \quad \boxed{3} \quad \boxed{3\ 3} \quad \boxed{3\ 2} \quad \boxed{3\ 1} \quad \begin{array}{|c|c|}\hline 3 & 3 \\ \hline & 2 \\ \hline \end{array}$

$\boxed{\text{DPP}}$ = Arrays of positive integers

$$\begin{array}{cccccc} a_{11} & a_{12} & \cdots & & \cdots & a_{1,\mu_1} \\ & a_{22} & a_{23} & \cdots & a_{2,\mu_2} & \\ & & \ddots & & \cdots & \\ & & a_{r,r} & \cdots & a_{r,\mu_r} & \end{array}$$

1. $a \geq b$ $\xrightarrow{}$
2. $\begin{array}{c} a \\ \vee \\ b \end{array}$ $\downarrow$
3. $\lambda_i = \mu_i - i + 1$ = # parts in row i

   $\lambda_i < a_{i,i} \leq \lambda_{i-1}$

Vocabulary

- a_{ij} = part
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OBSERVABLES

parts = n $\longrightarrow$ 3

special parts

non-special parts

n=3 : 7 DPP's

$\varnothing$	$\boxed{2}$	$\boxed{3}$	$\boxed{3\ 3}$	$\boxed{3\ 2}$	$\boxed{3\ 1}$	$\begin{array}{\|cc\|}\hline 3 & 3 \\ \hline & \multicolumn{1}{\|c\|}{2} \\ \hline \end{array}$
0	0	1	2	1	1	2

$\boxed{\text{DPP}}$ = Arrays of positive integers

$$\begin{array}{llll} a_{11} & a_{12} & \cdots & a_{1,\mu_1} \\ & a_{22} & a_{23} \cdots & a_{2,\mu_2} \\ & & \ddots & \\ & & a_{r,r} \cdots a_{r,\mu_r} & \end{array}$$

1. $a \geq b$ $\rightarrow$

2. $\downarrow$ $a > b$

3. $\lambda_i = \mu_i - i + 1$ = # parts in row i

$\lambda_i < a_{i,i} \leq \lambda_{i-1}$

Vocabulary

- a_{ij} = part
- $\boxed{a_{ij} \leq j - i = \text{special part}}$
- order n = $a_{ij} \leq n \ \forall i,j$.

OBSERVABLES

parts = n $\longrightarrow$ 3

special parts $\longrightarrow$

non-special parts

$n=3$: 7 DPP's

$\varnothing$	2	3	3 3	3 2	3	3 3 / 2
0	0	0	0	0	1	0

$\boxed{\text{ASM}}$

- $n \times n$ matrices with elements $0, \pm 1$
- ± 1 alternate along raws and columns
- raw and column sums $= 1$

Generalize Permutation matrices

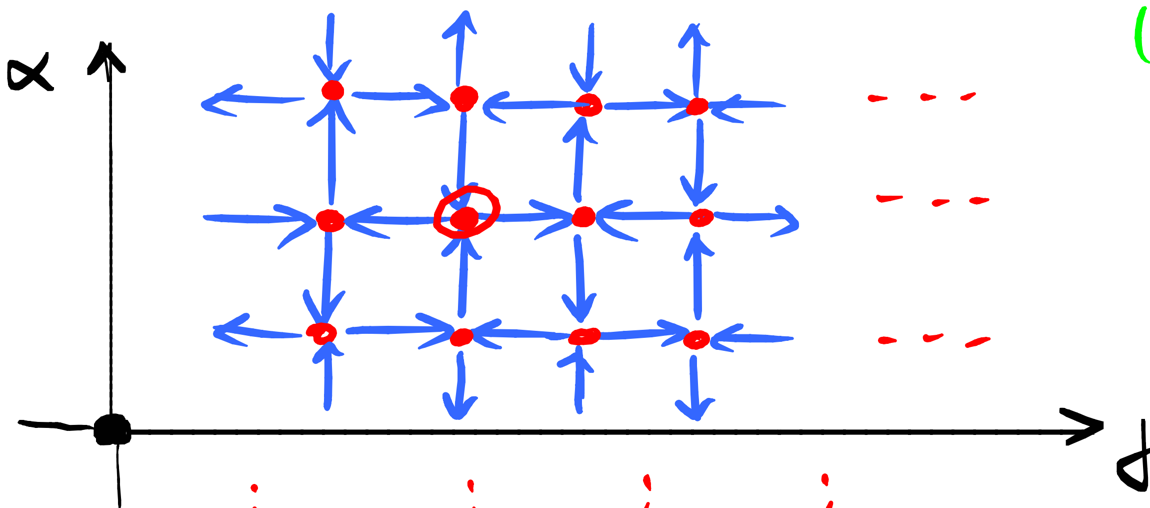
Arise from the Laurent phenomenon for the octahedron equation (Fomin-Zelevinsky '99)

$$\boxed{\begin{gathered} T_{i\,j\,k+1}\, T_{i\,j\,k-1} = T_{i\,j+1\,k}\, T_{i\,j-1\,k} + T_{i+1\,j\,k}\, T_{i-1\,j\,k} \\ T_{ijk}\left(\{T_{ij0}\,;\, T_{i'j'1}\}_{i,j,i',j' \in \mathbb{Z}^2}\right) = \text{Laurent Polynomial.} \end{gathered}}$$

THM The octahedron move is a mutation in an infinite rank Cluster Algebra

(DF Kedem 09)

Quiver:



Cluster:

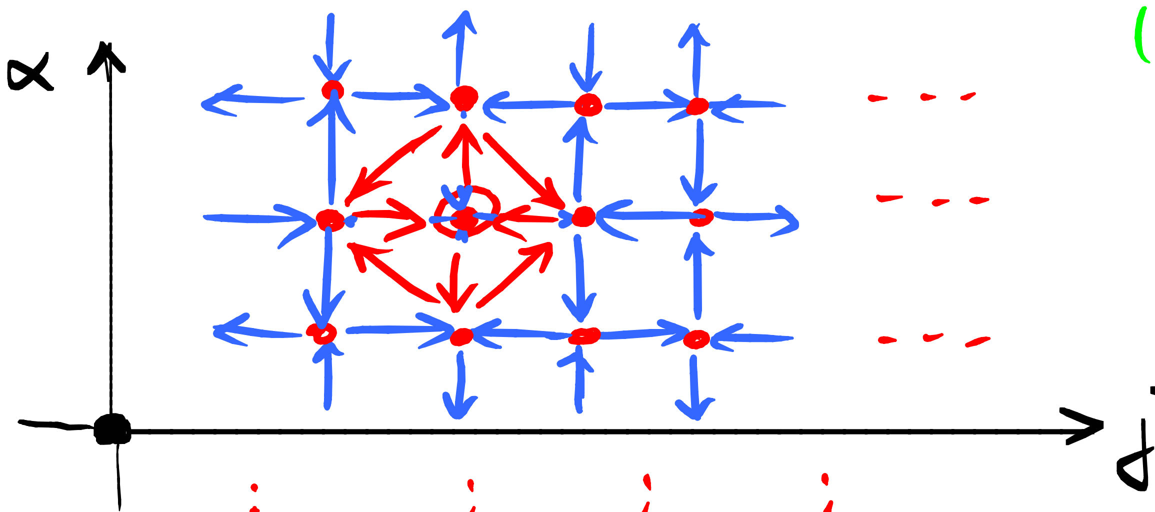
$$\begin{array}{cccccc} & \vdots & \vdots & \vdots & \vdots & \\ \cdots & T_{1,-1,0} & T_{1,0,1} & T_{1,1,0} & T_{1,2,1} & \cdots \\ \cdots & T_{0,-1,1} & T_{0,0,0} & T_{0,1,1} & T_{0,2,0} & \cdots \\ \cdots & T_{-1,-1,0} & T_{-1,0,1} & T_{-1,1,0} & T_{-1,2,1} & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \end{array}$$

THM The octahedron move is a mutation in an infinite rank Cluster Algebra

(DF Kedem 09)

Quiver:

mutation

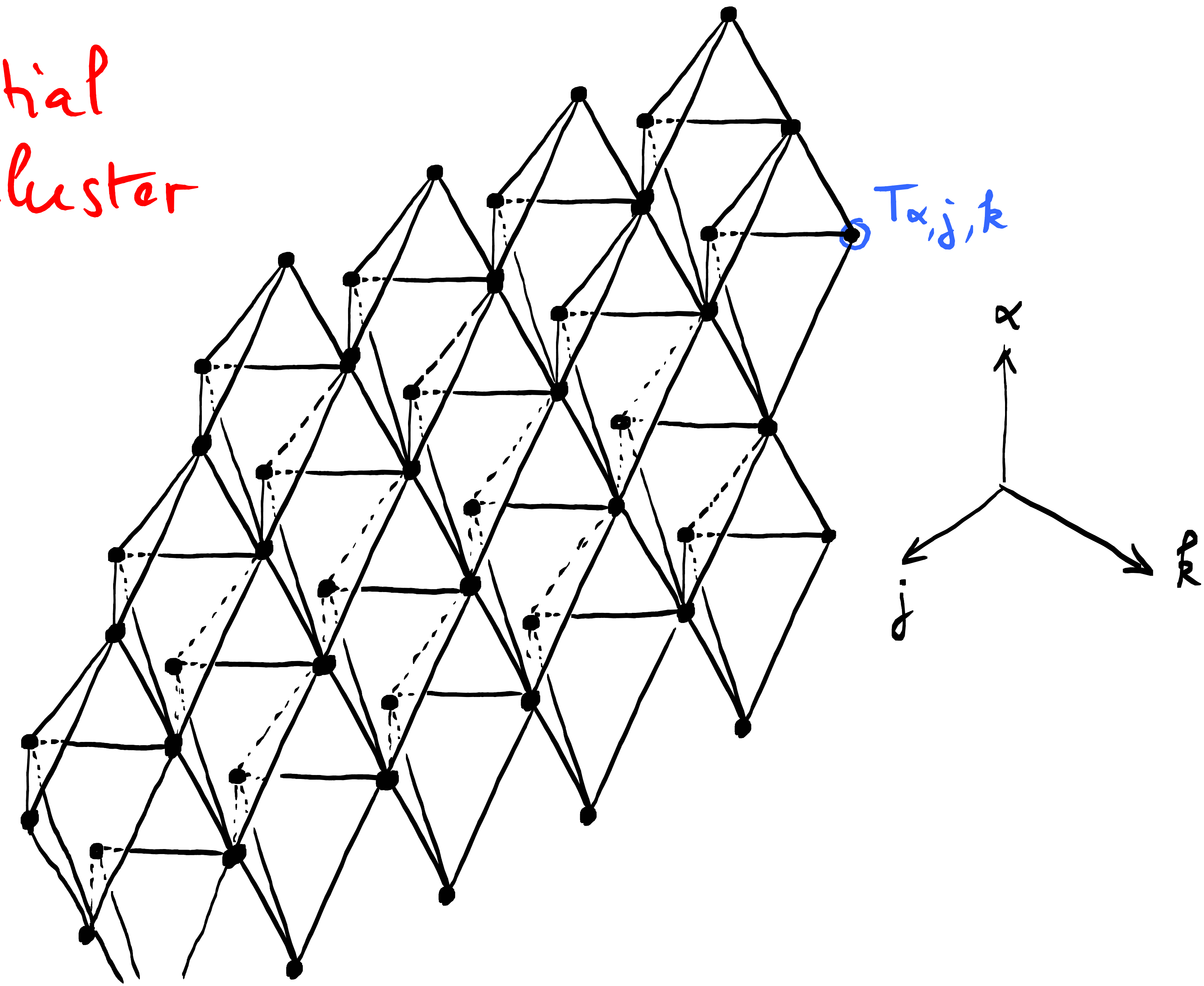


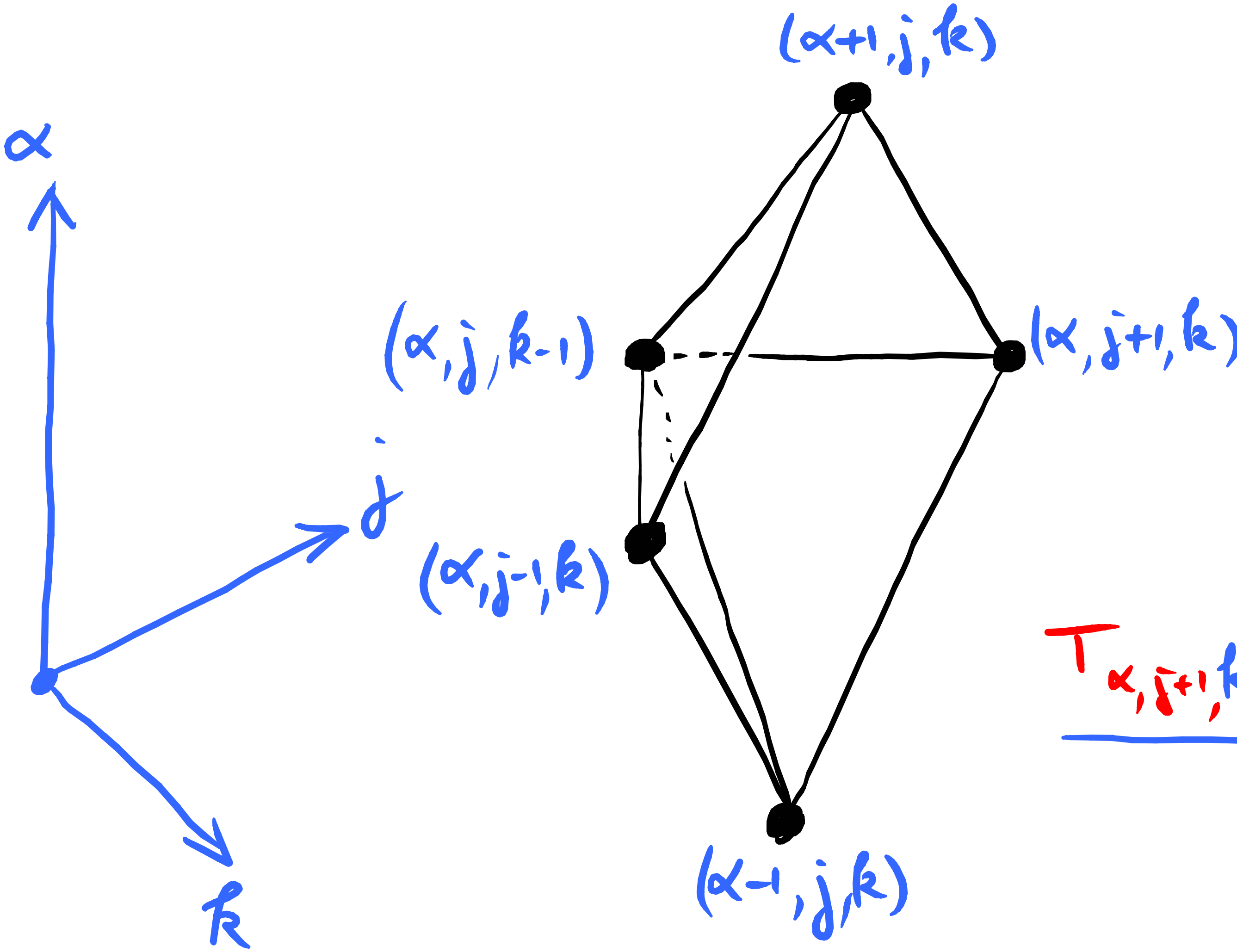
Cluster :

$$
\begin{array}{cccccc}
 & \vdots & \vdots & \vdots & \vdots & \\
\cdots & T_{1-10} & T_{101} & T_{110} & T_{121} & \cdots \\
\cdots & T_{0-11} & \bigcirc & T_{011} & T_{020} & \cdots \\
\cdots & T_{-1-10} & T_{-101} & T_{-110} & T_{-121} & \cdots \\
 & \vdots & \vdots & \vdots & \vdots &
\end{array}
$$

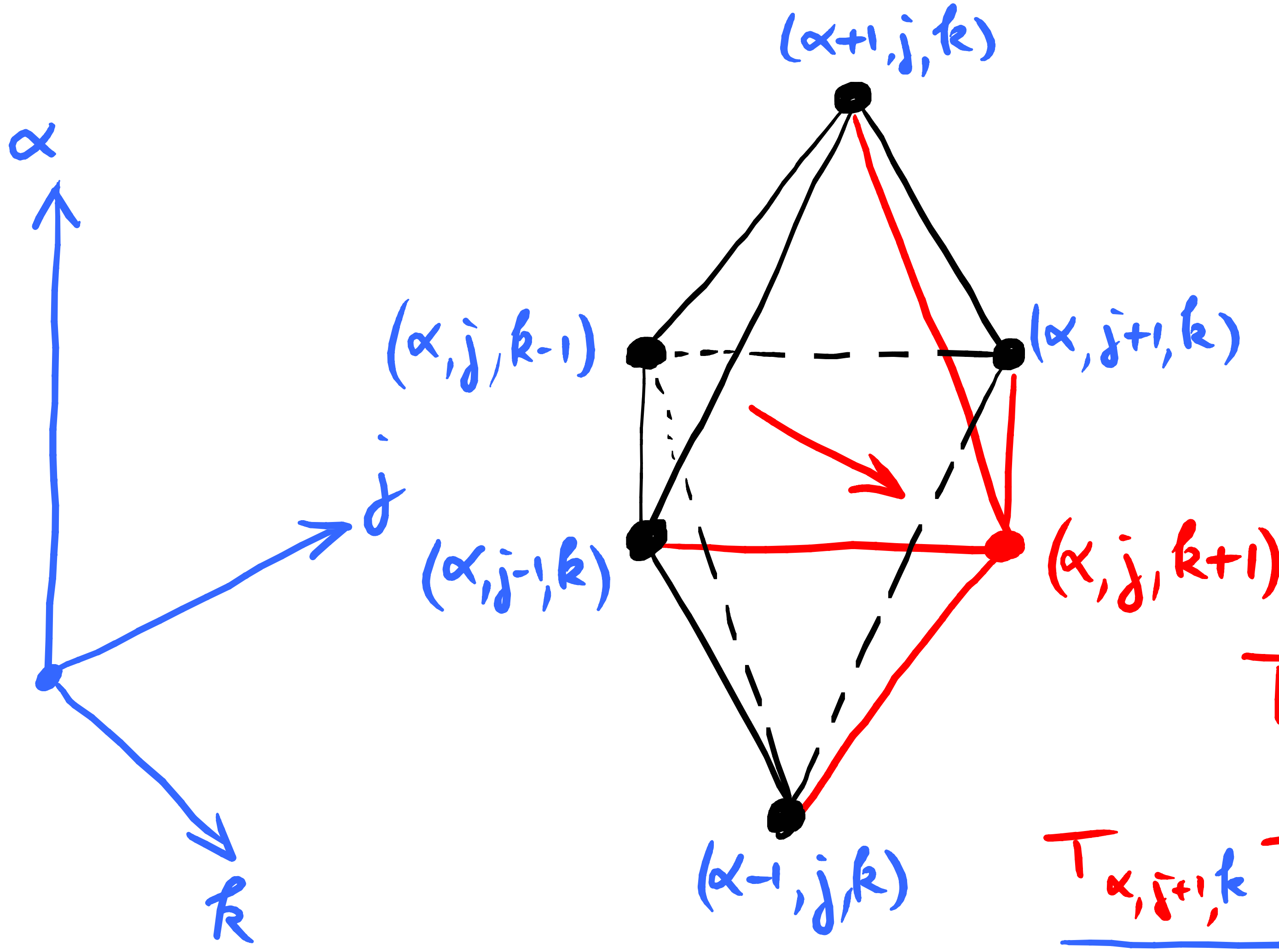
$$\frac{T_{0-11}T_{011}+T_{101}T_{-101}}{T_{000}}$$

initial cluster

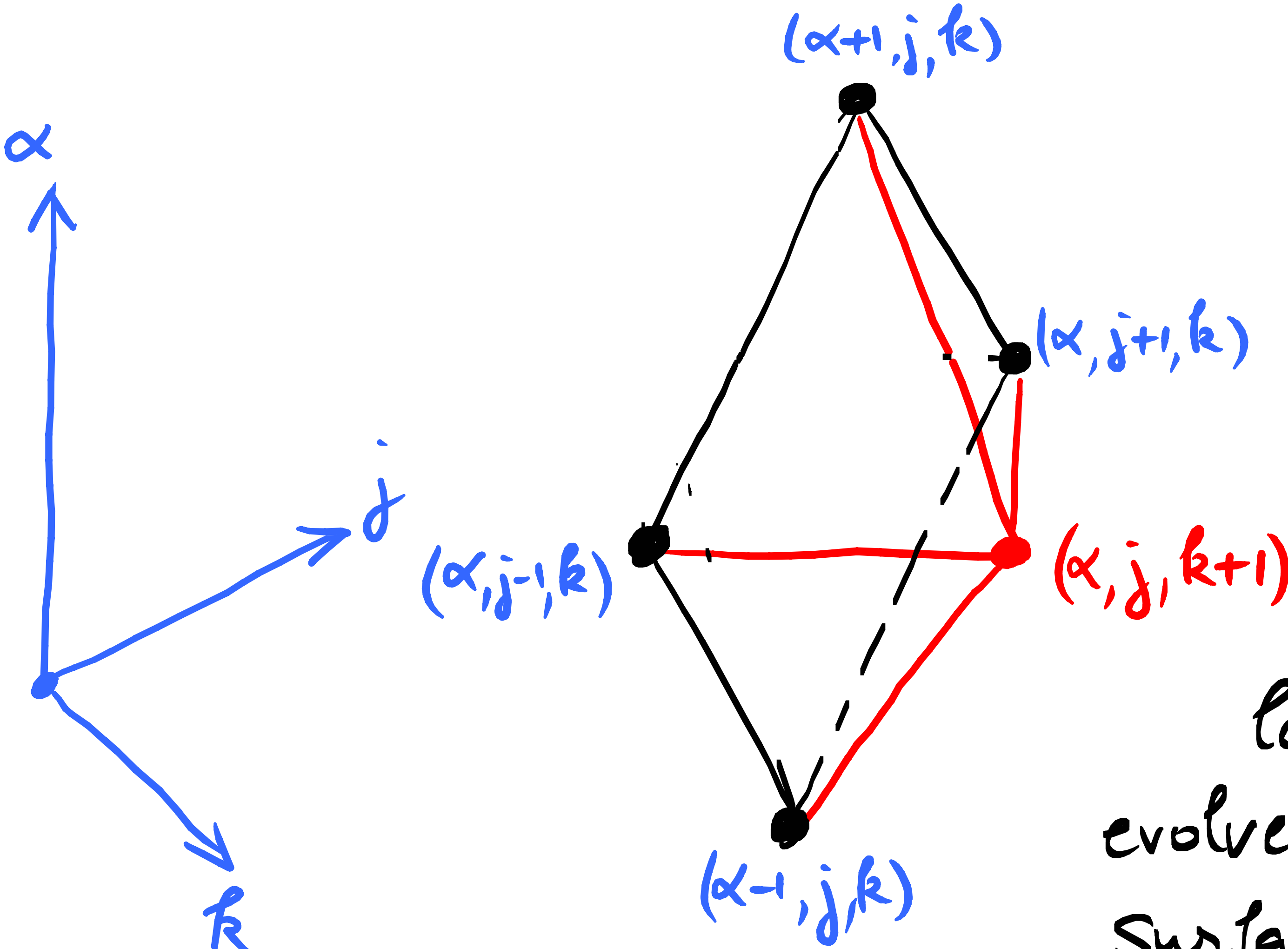




$$\frac{T_{\alpha,j+1,k}\,T_{\alpha,j-1,k} + T_{\alpha+1,j,k}\,T_{\alpha-1,j,k}}{T_{\alpha,j,k-1}}$$

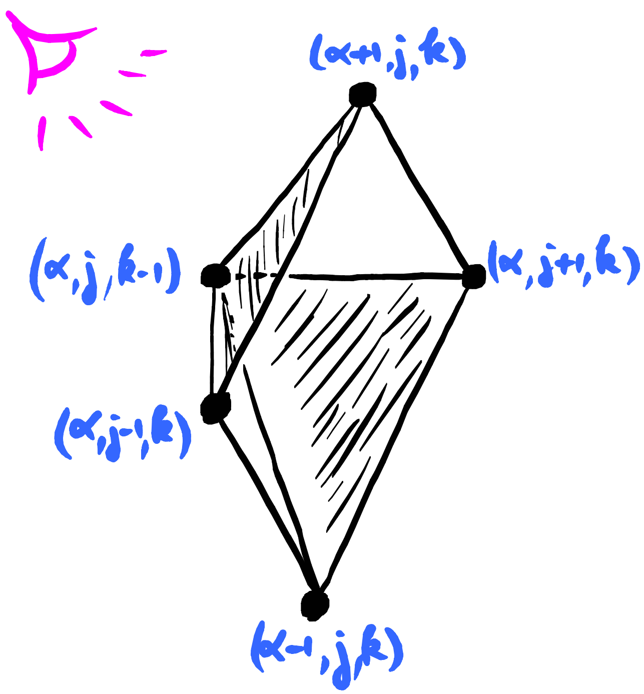


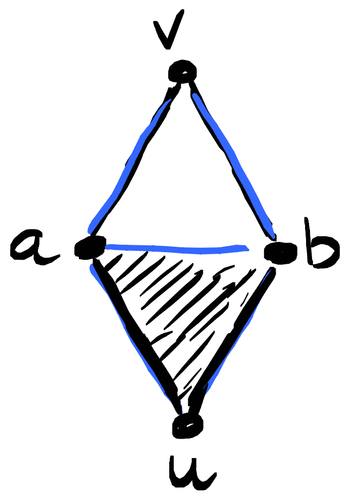
$$T_{\alpha,j,k+1} = \frac{T_{\alpha,j+1,k}\,T_{\alpha,j-1,k} + T_{\alpha+1,j,k}\,T_{\alpha-1,j,k}}{T_{\alpha,j,k-1}}$$



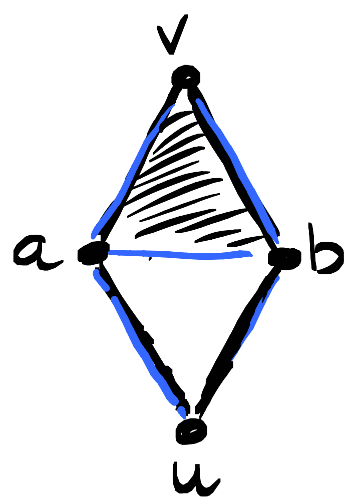
local move that evolves the stepped surface by "adding" an octahedron

DISCRETE INTEGRABILITY

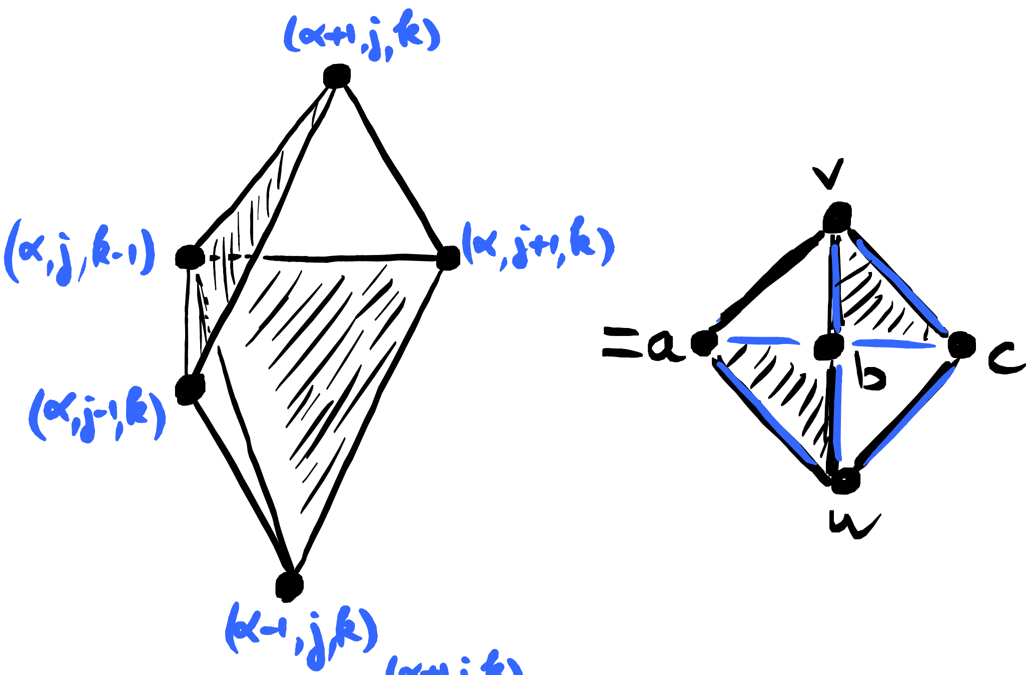




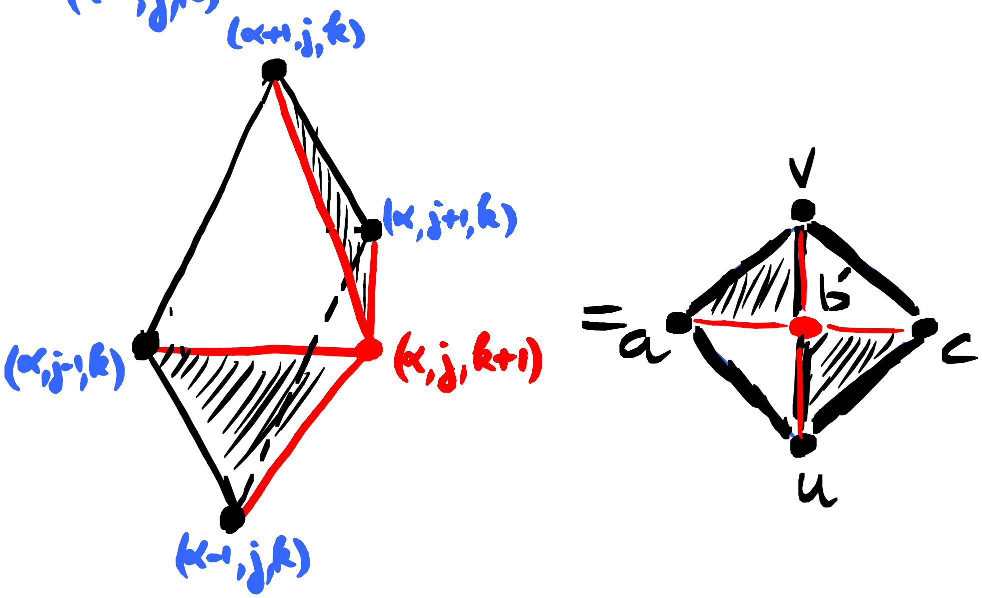
$$\rightarrow \quad D(u,a,b)=\begin{pmatrix} \frac{a}{b} & \frac{u}{b} \\ 0 & 1 \end{pmatrix}$$



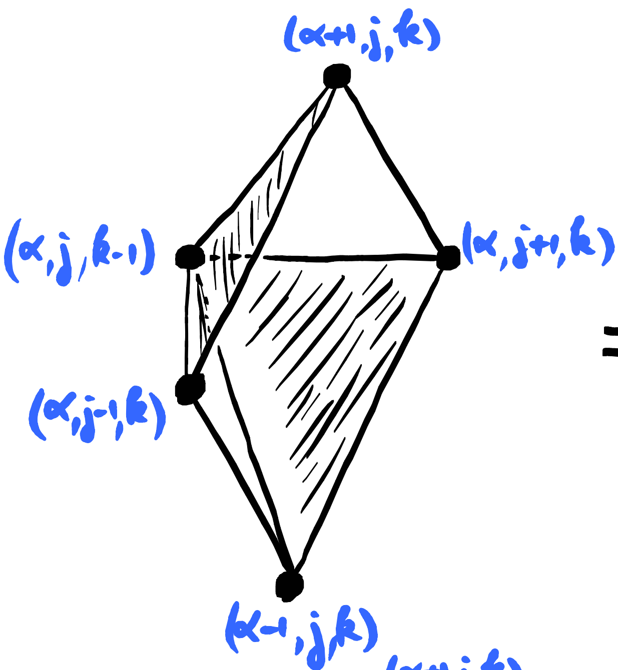
$$\rightarrow \quad U(a,b,v)=\begin{pmatrix} 1 & 0 \\ \frac{v}{b} & \frac{a}{b} \end{pmatrix}$$



$$= D(u,a,b)\,U(b,c,v)$$



$$= U(a,b',v)\,D(u,b',c)$$



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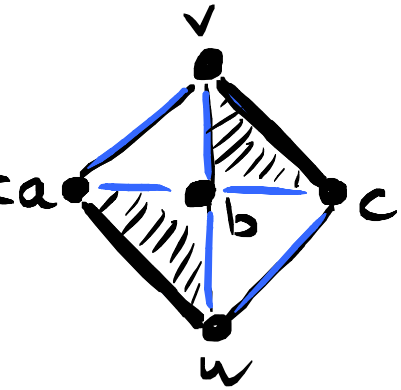
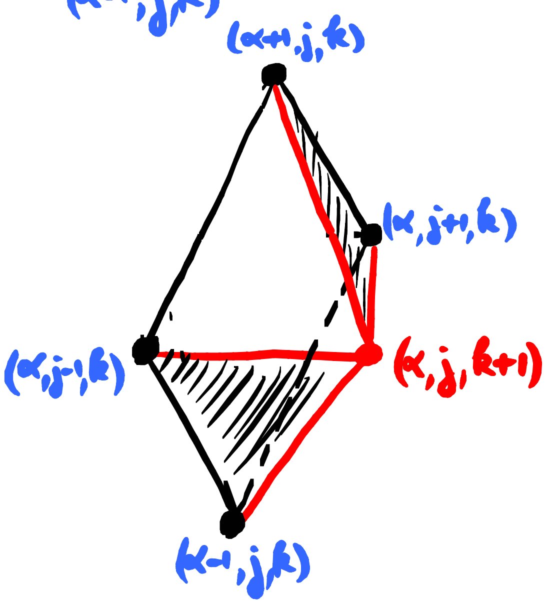


$$= D(u,a,b)\,U(b,c,v)$$

$$\Big\| \quad \Longleftrightarrow \quad \boxed{bb' = ac + uv}$$



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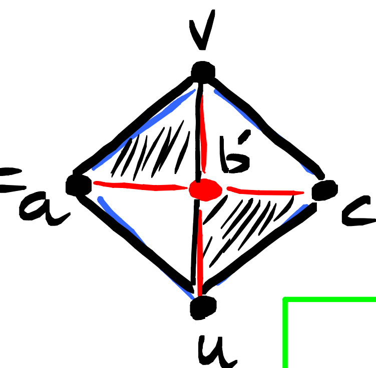
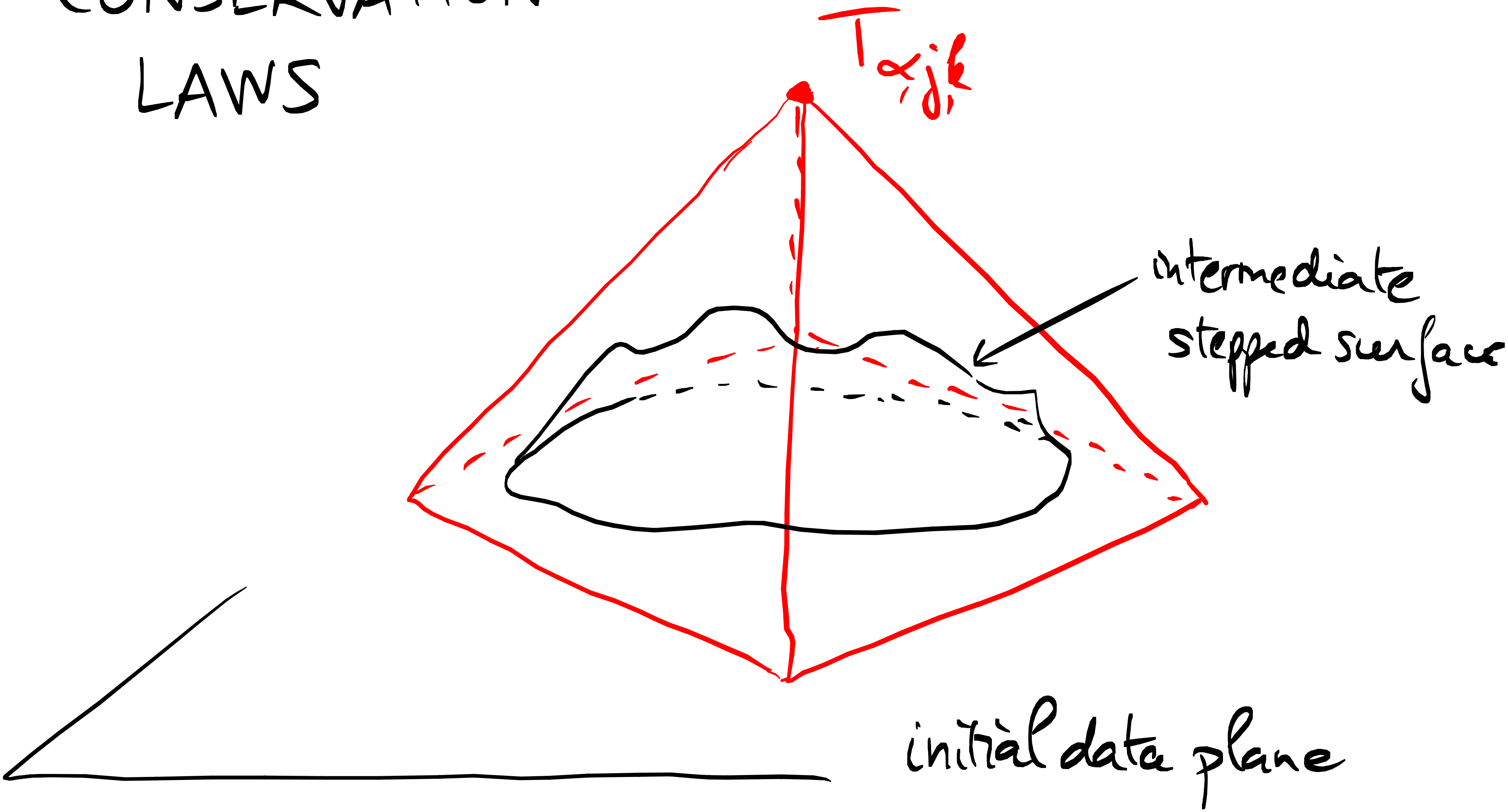


$$= U(a,b',v)\,D(u,b',c)$$

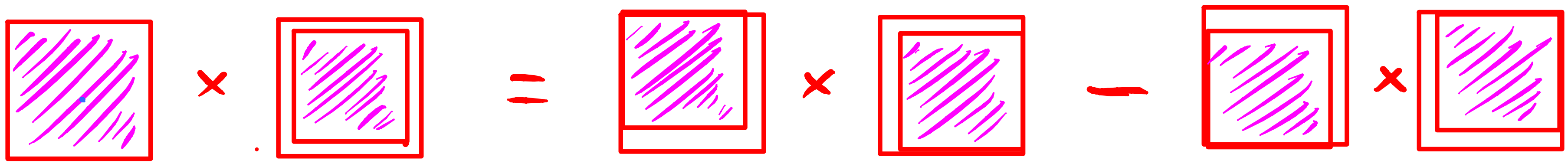
"Flat connection"
(related to Yang-Baxter eq)

CONSERVATION LAWS



Robbins-Rumsey (82): $\boxed{\lambda\text{-determinant}}$

use Desnanot-Jacobi relation for minors of $i+1 \times i+1$ matrix



size: $i+1$ $\quad i-1 \quad i \quad i \quad i \quad i$

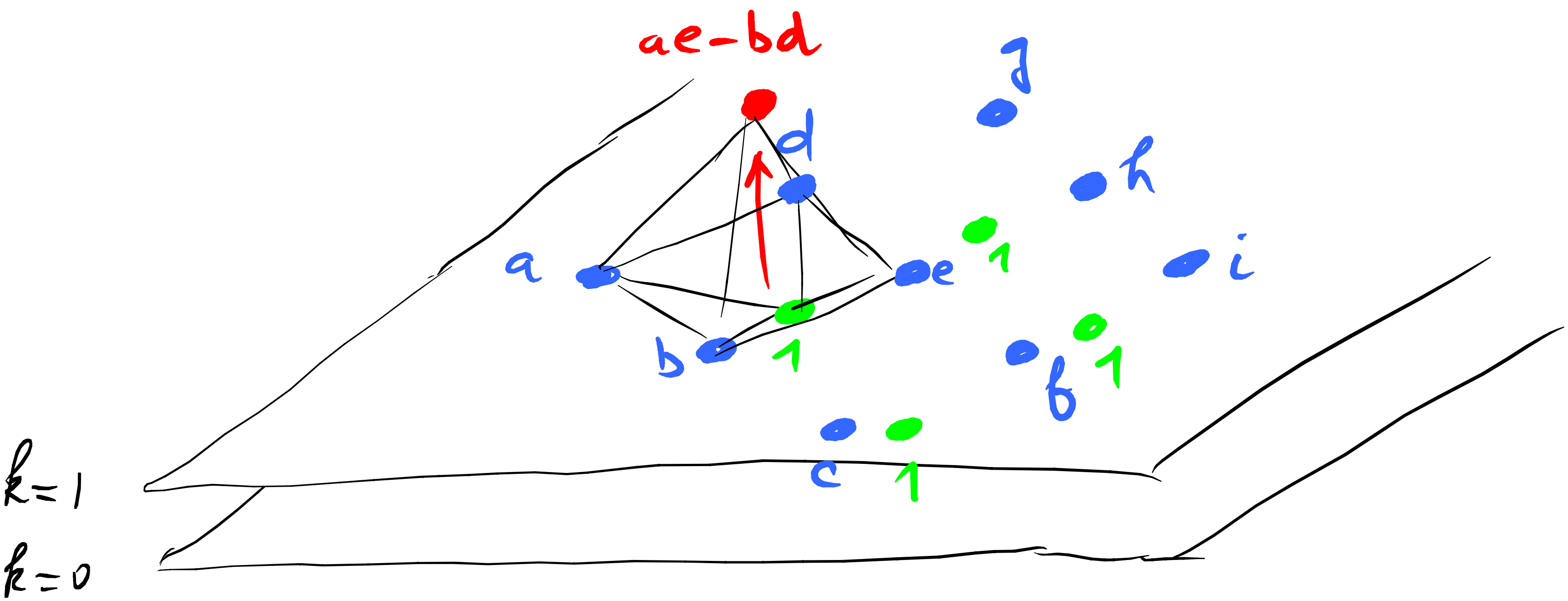
→ Lewis Carroll algorithm for computing determinant

$$T_{ij0} = 1 \;;\; T_{ij1} = a_{\frac{j-i}{2},\, \frac{i+j}{2}} \Rightarrow T_{0,k+1,k} = \det_{k \times k}(A)$$

$(\mu = 1,\ \lambda = -1)$

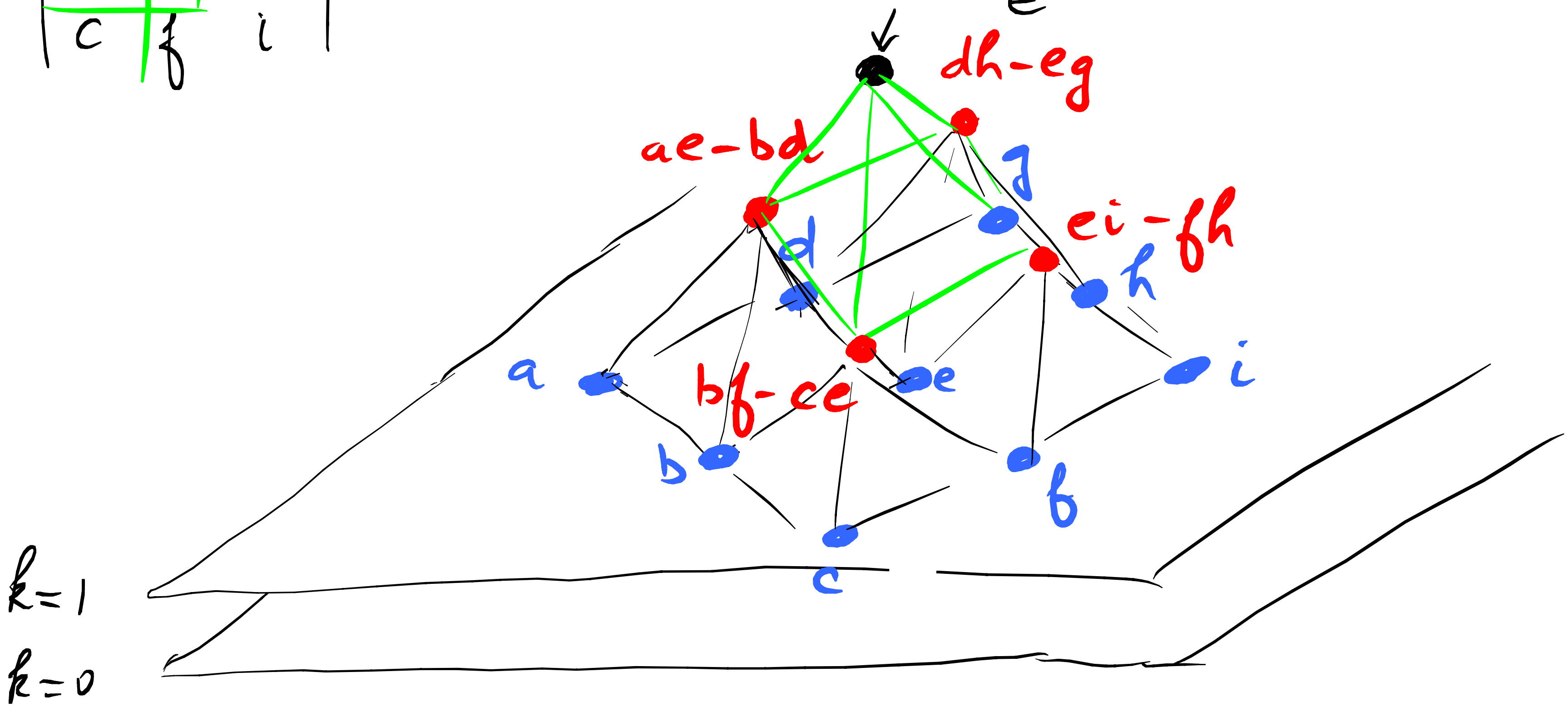
$$\boxed{\det_{k\times k}(A) = \sum_{\substack{\text{permutation}\\ \text{matrices } P_{ij}\\ k\times k}} (-1)^{\mathrm{inv}(P)} \prod_{ij} a_{ij}^{\,P_{ij}}}$$

Computing determinants is not more difficult than constructing pyramids



$$\begin{vmatrix} a & d \\ b & e \end{vmatrix} \times 1 = a \times e - b \times d$$

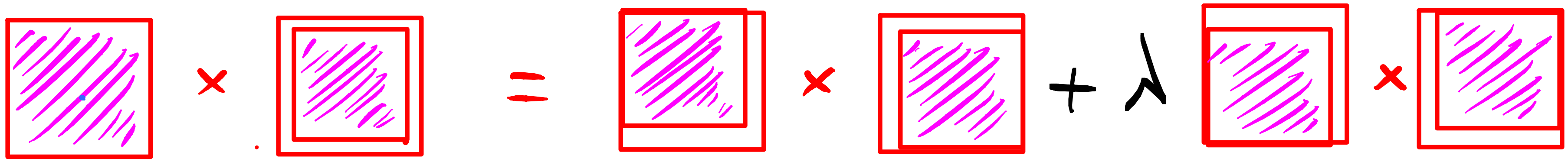
$$\begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix} = \frac{(ae-bd)(ei-fh)-(dh-eg)(bf-ce)}{e}$$



$$\begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix} \times |e| = \begin{vmatrix} a & d \\ b & e \end{vmatrix} \times \begin{vmatrix} e & h \\ f & i \end{vmatrix} - \begin{vmatrix} b & e \\ c & f \end{vmatrix} \times \begin{vmatrix} d & g \\ e & h \end{vmatrix}$$

Robbins-Rumsey (82): $\boxed{\lambda\text{-determinant}}$

use Desnanot-Jacobi relation for minors of $i+1 \times i+1$ matrix



size: $i+1$ $\quad i-1$ $\quad i$ $\quad i$ $\quad i$ $\quad i$

$\to$ Lewis Carroll algorithm for computing determinant

$T_{ij0} = 1$; $T_{ij1} = a_{\frac{j-i}{2},\frac{i+j}{2}} \Rightarrow T_{0,k+1,k} = |A|_\lambda$

$(\mu = 1, \ \lambda = \lambda)$

THM (MRR)

$$|A|_\lambda = \sum_{\substack{ASM_{k\times k} \\ B_{ij}}} \lambda^{Inv(B)} (1+\lambda^{-1})^{\#(-1)_B} \prod_{ij} a_{ij}^{B_{ij}}$$

THM (MRR)

$$|A|_\lambda = \sum_{\substack{ASM\ B_{ij} \\ k\times k}} \lambda^{Inv(B)}\left(1+\lambda^{-1}\right)^{\#(-1)_B} \prod_{ij} a_{ij}^{B_{ij}}$$

EXAMPLE:

n=3 $\quad$ 7 ASM's

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}\begin{pmatrix}0&1&0\\1&-1&1\\0&1&0\end{pmatrix}$$

OBSERVABLES FOR ASMs

- position of the 1 in the first row

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}\begin{pmatrix}0&1&0\\1&-1&1\\0&1&0\end{pmatrix}$$

$$0 \qquad 1 \qquad 0 \qquad 1 \qquad 2 \qquad 2 \qquad 1$$

OBSERVABLES FOR ASMs

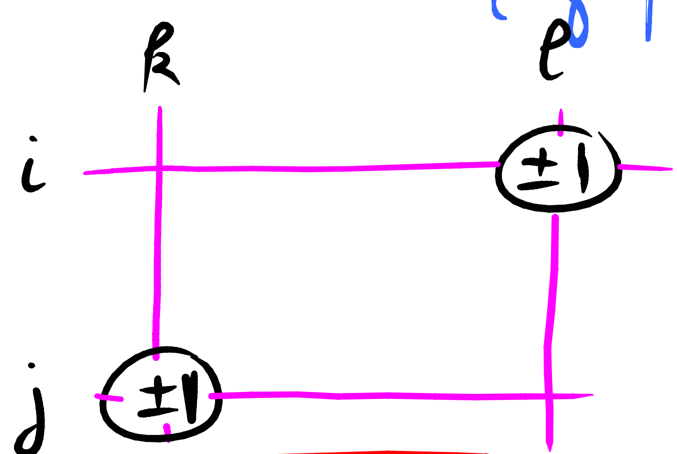
- position of the 1 in the first row
- $\#(-1)$ number of -1's

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}\begin{pmatrix}0&1&0\\1&-1&1\\0&1&0\end{pmatrix}$$

$$0 \qquad 0 \qquad 0 \qquad 0 \qquad 0 \qquad 0 \qquad 1$$

OBSERVABLES FOR ASMs

- position of the 1 in the first row
- $\#(-1)$ number of -1's
- inversion number (of permutations)



$$\boxed{\mathrm{inv}(B) = \sum_{i<j} \sum_{k<\ell} B_{i\ell} B_{jk}}$$

THE ASM–DPP CONJECTURE

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Alternating Sign Matrices and Descending Plane Partitions

W. H. MILLS, DAVID P. ROBBINS, AND HOWARD RUMSEY, JR.

Institute for Defense Analyses, Princeton, New Jersey 08540-3699

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Conjecture 3. Suppose that n, k, m, p are nonnegative integers, $1 \leqslant k \leqslant n$. Let $\mathscr{A}(n, k, m, p)$ be the set of alternating sign matrices such that

(i) the size of the matrix is $n \times n$,

(ii) the 1 in the top row occurs in position k,

(iii) the number of -1's in the matrix is m,

(iv) the number of inversions in the matrix is p.

ASM $(A(n))$
- • n = size
- • position top 1
- • # −1's
- • # inversions

On the other hand, let $\mathscr{D}(n, k, m, p)$ be the set of descending plane partitions such that

(I) no parts exceed n,

(II) there are exactly $k-1$ parts equal to n,

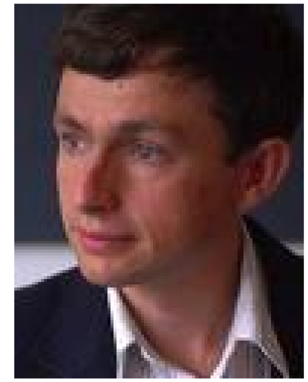
(III) there are exactly m special parts,

(IV) there are a total of p parts.

DPP $(D(n))$
- • n = order
- # parts = n
- # special parts
- # parts

Then $\mathscr{A}(n, k, m, p)$ and $\mathscr{D}(n, k, m, p)$ have the same cardinality.

Counting



- DPP : Andrews '79 → formula $D(n)$
- ASM : Zeilberger '96, Kuperberg '96

$A(n) = TSSCPP(n) = D(n)$

6 vertex model

↑

Izergin–Korepin integrable lattice model

Refinements ?

Computation of the refined numbers

Strategy: 1. reformulate in terms of known (and manageable objects)

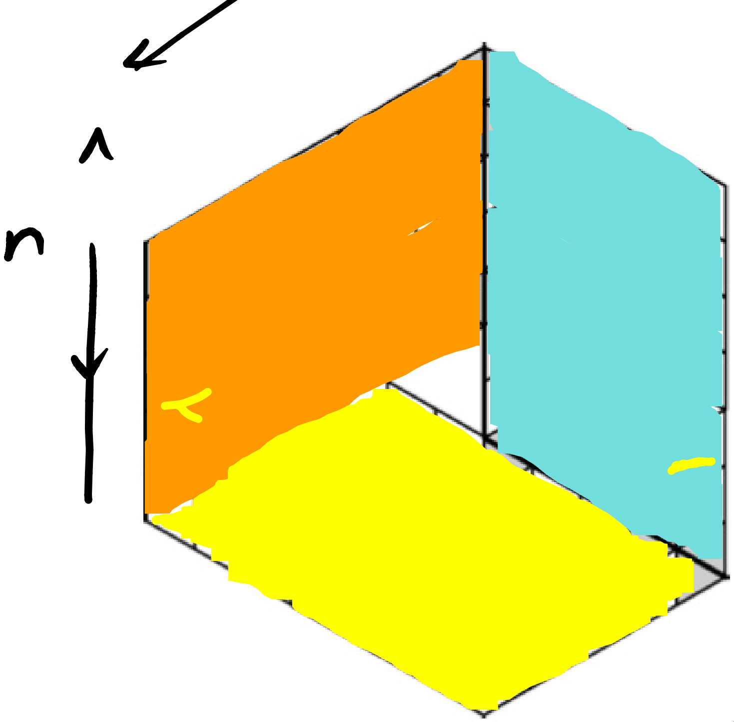
- DPP → lattice paths (lattice fermions)
- ASM → 6 vertex model (integrable lattice model)

2. write refined generating functions as determinants

3. Prove identity between determinants

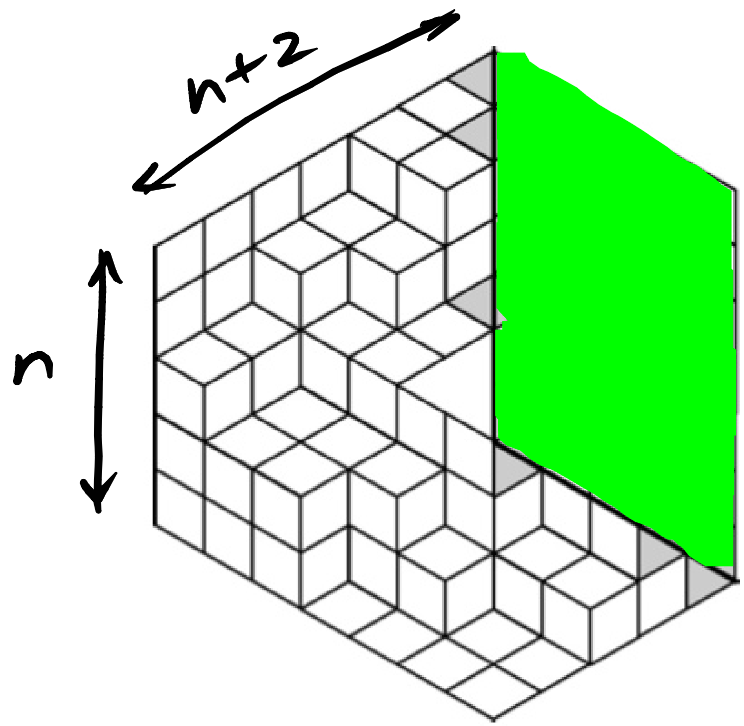
$\boxed{\text{DPP}}$ DPP as Lattice Paths

Lalonde '03
(Krattenthaler '06)

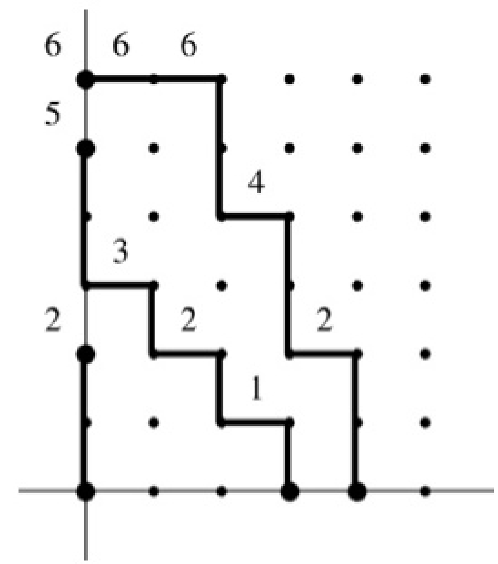


Cyclically symmetric Rhombus tilings of a Hexagon $(n, n+2, n, n+2, n, n+2)$ with Δ hole $_{2\times2\times2}$ $\longleftrightarrow$ Lattice Paths

DPP as Lattice Paths



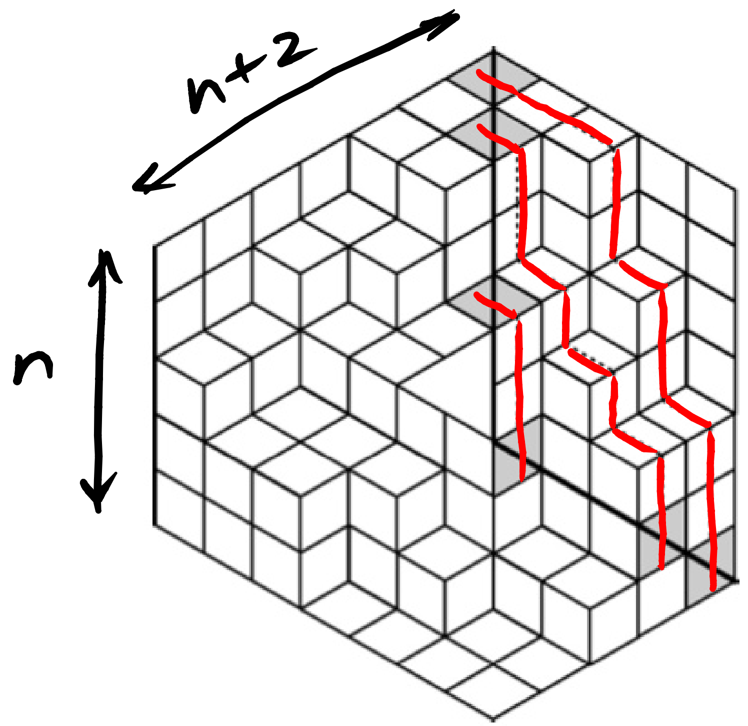
$\longleftrightarrow$

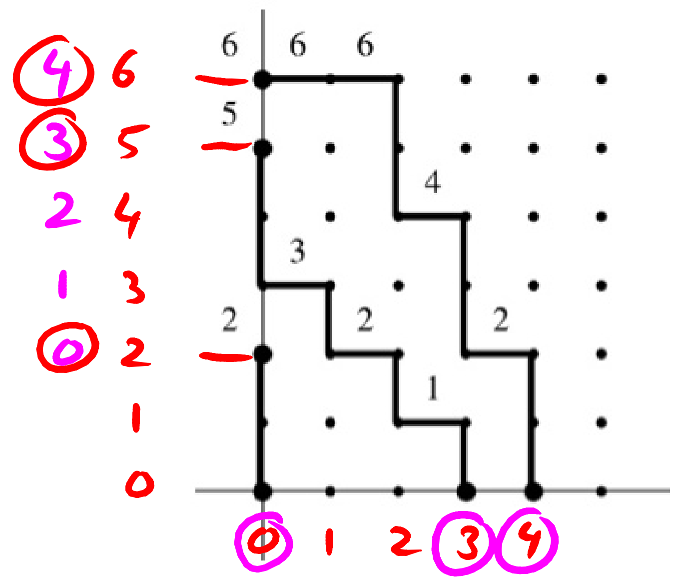


Rhombus Tilings of a Hexagon $(n, n+2, n, n+2, n, n+2)$ with $\underset{2\times 2\times 2}{\Delta}$ hole $\longleftrightarrow$ Lattice Paths

DPP as Lattice Paths

(Krattenthaler '06)

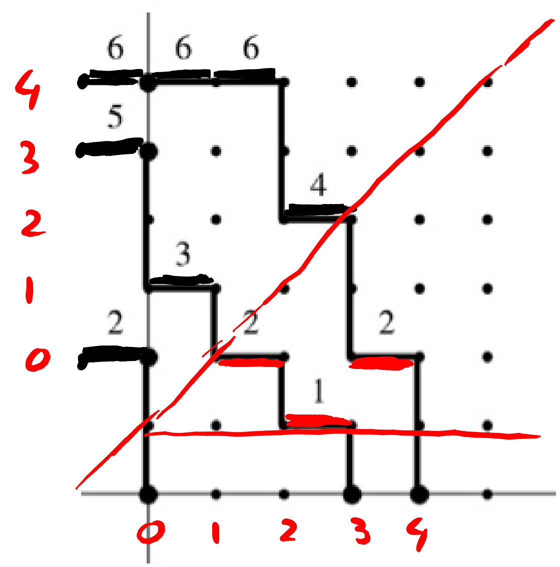




6	6	6	4	2
	5	3	2	1
		2		

Rhombus Tilings of a Hexagon $(n, n+2, n, n+2, n, n+2)$ with $\Delta_{2\times2\times2}$ hole $\leftrightarrow$ Lattice Paths

horizontal steps — = non-special parts



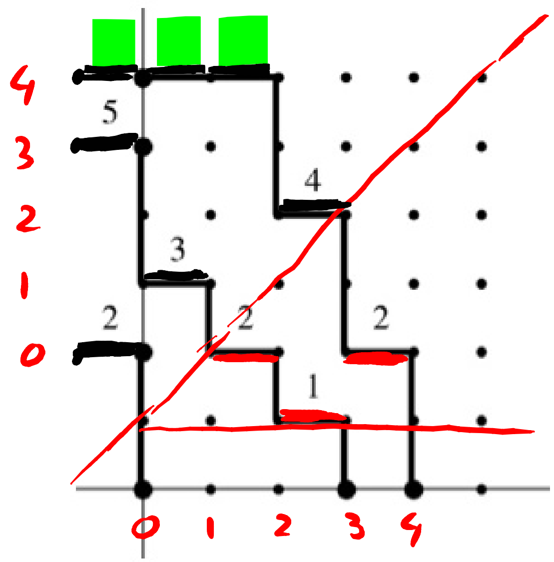
horizontal steps — = special parts

} horizontal steps here do not count

6	6	6	4	2
			2	1
		2		

6	6	6	4	2
	5	3	2	1
		2		

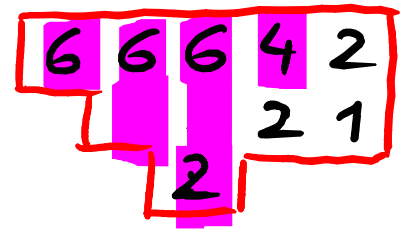
horizontal steps — = non-special parts

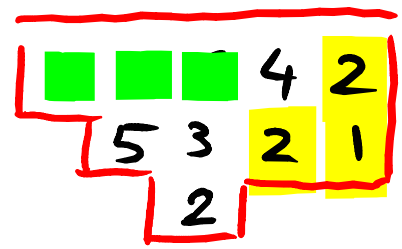


■ = parts equal to n

horizontal steps — = special parts

} horizontal steps here do not count



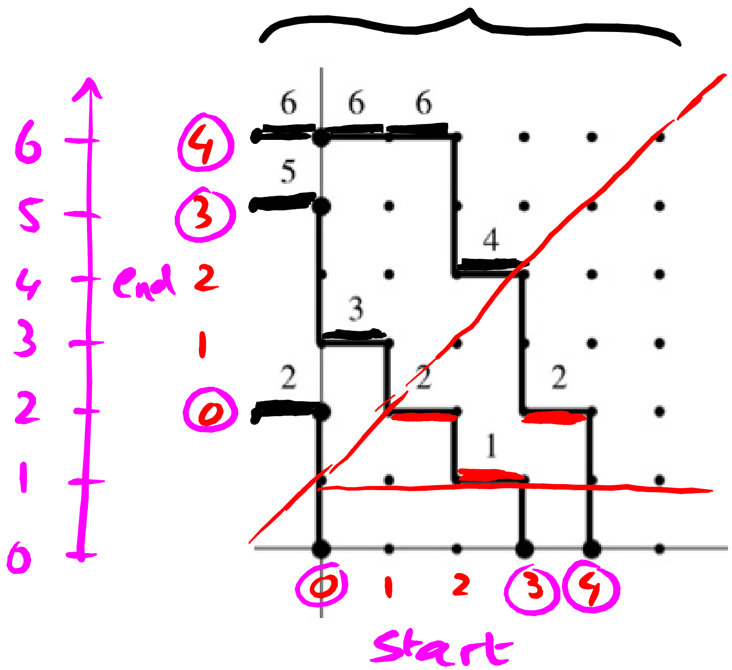


DPP : NON-INTERSECTING LATTICE PATHS

horizontal steps — = non-special parts

weight y

6	6	6	4	2
	5	3	2	1
		2		



horizontal steps — = special parts

weight x

6	6	6	4	2
	5	3	2	1
		2		

} horizontal steps here do not count

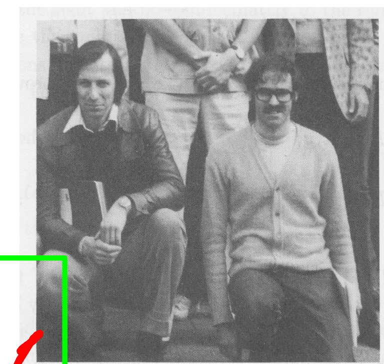
Pb = count these families of non-intersecting paths

• M_{ij} = Partition Function for 1 path : $(i,0) \to (0,j)$

$$M_{ij} = \sum_{\text{paths } (i,0)\to(0,j)} x^{\#(-)}\, y^{\#(\textbf{—})}$$

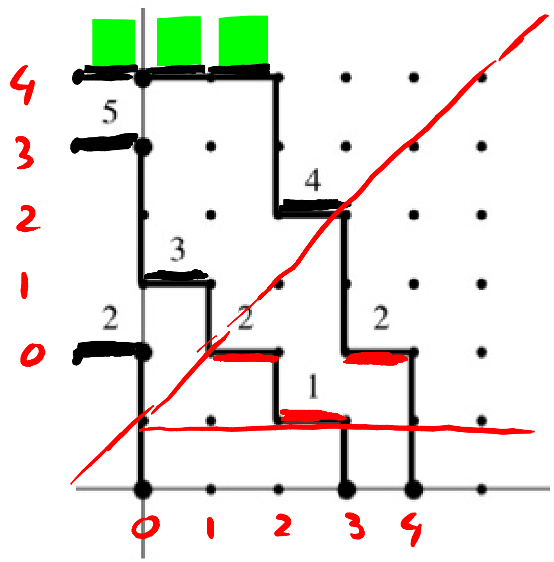
horizontal steps : lower wedge, upper wedge

• By Gessel–Viennot theorem



$\det\left(M^{i_1 \dots i_k}_{i_1 \dots i_k}\right)$ = Partition fctn for families of k non-intersecting paths starting at $(i_1,0) \cdots (i_k,0)$, ending at $(0,i_1) \cdots (0,i_k)$

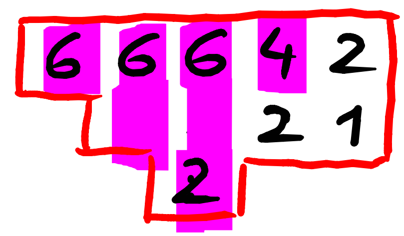
horizontal steps — = non-special parts

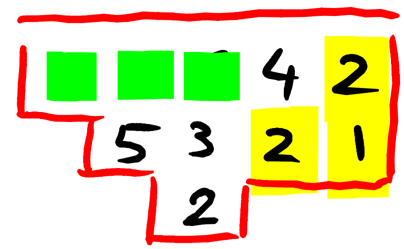


■ = parts equal to n

horizontal steps — = special parts

} horizontal steps here do not count

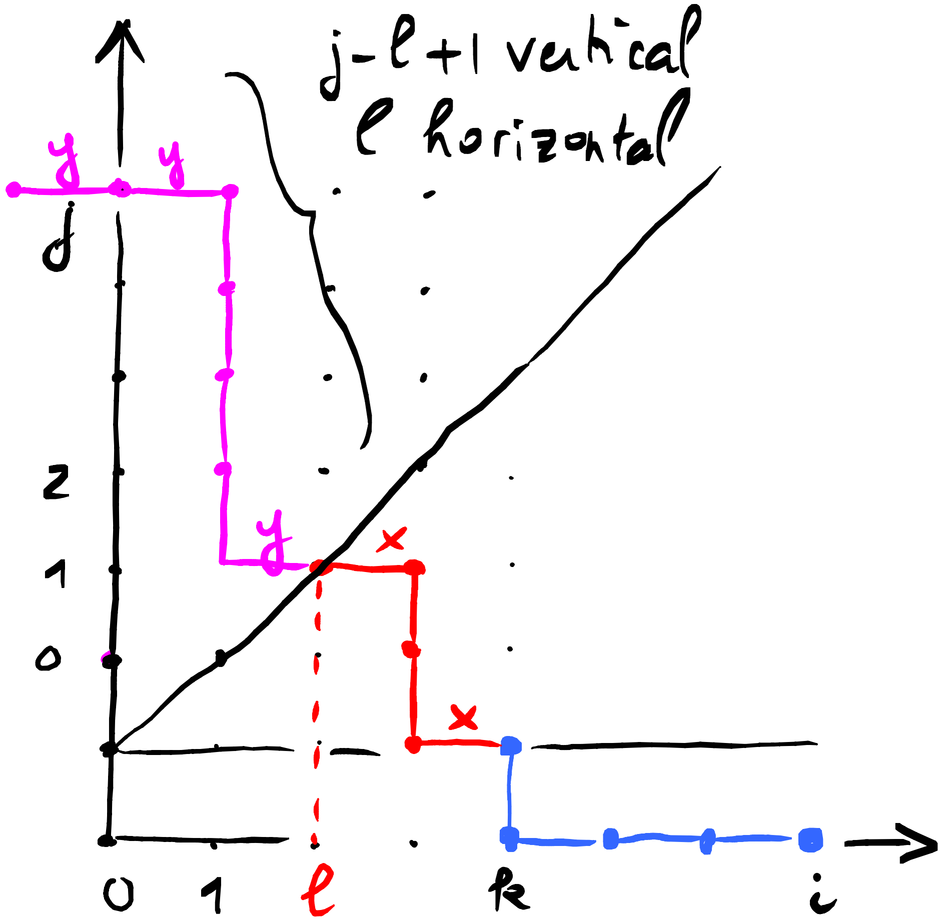




Lemma $$\det_{n\times n}(I+M) = \sum_{1\le i_1<\dots<i_k\le n} \det\left(M^{i_1\dots i_k}_{i_1\dots i_k}\right)$$

M's minor with rows $i_1 \dots i_k$, cols $i_1 \dots i_k$

Here: M_{ij} = Part. fctn (path $(i,0)\to(0,j)$)



$$Z^{(n)}_{DPP}(x, y) = \det(I + M)$$

per special part

per non-special part

$$M_{i,j} = \sum_{k=0}^{i} \sum_{\ell \geq 0} \binom{k}{\ell} x^{k-\ell} \binom{j+1}{\ell} y^{\ell+1}$$

Generating function:

$$f_{DPP}(z,w) = \sum_{i,j \geq 0} z^i w^j \, (I+M)_{i,j}$$

THM

$$\boxed{f_{DPP}(z,w) = \frac{1}{1-zw} + \frac{1}{1-z} \, \frac{yz}{1-xz-w-(y-x)zw}}$$

weights: x / special part $\quad y$ / non-special part

ASM

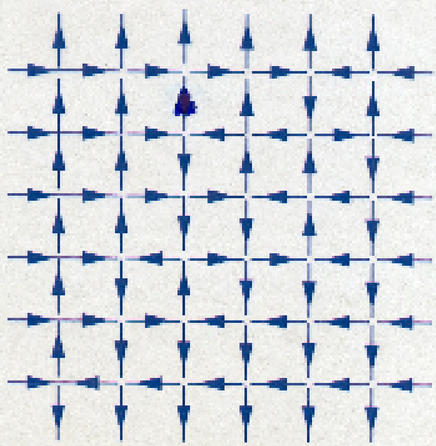
From ASM to 6 Vertex model with Domain Wall Boundary conditions (Kuperberg)

$n \times n$ ASM

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

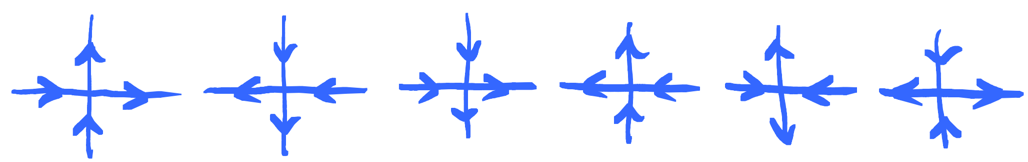
$\updownarrow$

6V + DWBC on $n \times n$ grid



Bijection:

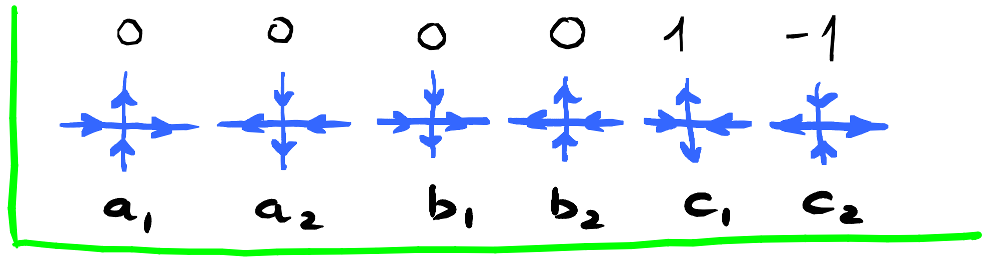
0 0 0 0 1 -1



$$\underbrace{a_1 \quad a_2}_{qz - q^{-1}w} \qquad \underbrace{b_1 \quad b_2}_{q^{-1}z - qw} \qquad \underbrace{c_1 \quad c_2}_{(q^2 - q^{-2})\sqrt{zw}}$$

(integrable weights)

Refinements :

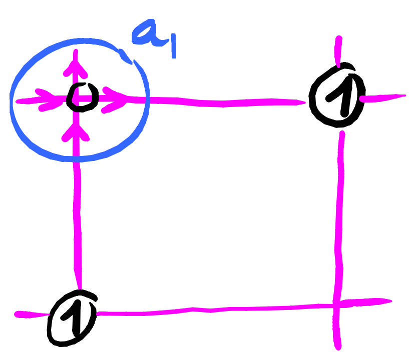


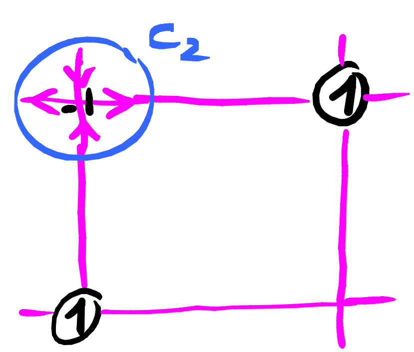
by symmetry :

$$\begin{cases} N_{a_1} = N_{a_2} = \frac{N_a}{2} & N_{b_1} = N_{b_2} = \frac{N_b}{2} \\ N_{c_1} = N_{c_2} + n & N_c = N_{c_1} + N_{c_2} \end{cases}$$

- $\boxed{\#(-1)} = N_{c_2} = \boxed{\frac{N_c - n}{2}}$
- $Inv(A) = N_{a_1} + N_{c_2}$

$$\Rightarrow \boxed{Inv(A) - \#(-1) = N_{a_1} = \frac{N_a}{2}}$$





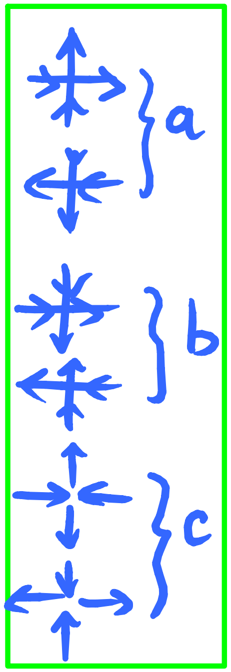
Partition function

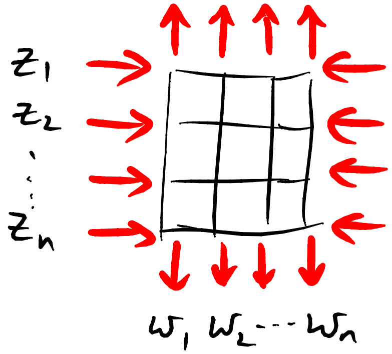
$$Z^{(n)}_{ASM}(x,y,z) = \sum_{\substack{\text{configs of } n\times n\\ \text{6V DWBC}}} x^{\frac{N_c-n}{2}}\, y^{\frac{N_a}{2}}\, \underset{\substack{\shortparallel \\ 1}}{z}^{N_{a'}}$$

usually, one considers $Z^{(n)}_{6V}(a,b,c) = \sum_{\substack{\text{configs}\\ \text{6V DWBC}}} a^{N_a}\, b^{N_b}\, c^{N_c} \times \underbrace{\left(\frac{a'}{a}\frac{b}{b'}\right)}_{\substack{\shortparallel \\ 1}}^{N_{a'}}$

$$Z^{(n)}_{6V}(a,b,c) = b^{n^2}\left(\frac{c}{b}\right)^n \sum \left(\frac{c}{b}\right)^{N_c-n} \left(\frac{a}{b}\right)^{N_a}$$

$$\Longleftrightarrow \boxed{x = \left(\frac{c}{b}\right)^2 \quad y = \left(\frac{a}{b}\right)^2}$$





Partition function of 6V + DWBC

$$Z_n = \frac{\displaystyle\sum_{\substack{\text{configs}\\ \text{on grid}}} \ \prod_{\substack{\text{vertices}\\ (i,j)}} \text{weights}(z_i, w_j)}{\displaystyle\prod_i c(z_i, w_i)}$$

THM

$$Z_n = \frac{\displaystyle\prod_{i,j} a(z_i,w_j)\, b(z_i,w_j)}{\displaystyle\prod_{i<j} (z_i - z_j)(w_i - w_j)} \det\left\{ \frac{1}{a(z_i,w_j)\, b(z_i,w_j)} \right\}_{1\le i,j\le n}$$

(Korepin – Izergin)

recursion relation + symmetries (from commutation of Transfer matrices).

Homogeneous limit:
$$\begin{cases} z_i \longrightarrow r & \forall i \\ w_j \longrightarrow r^{-1} & \forall j \end{cases}$$

$$a(z_i, w_j) \longrightarrow q r - q^{-1} r^{-1} = a(r, r^{-1})$$
$$b(z_i, w_j) \longrightarrow q^{-1} r - q r^{-1} = b(r, r^{-1})$$
$$c(z_i, w_j) \longrightarrow q^2 - q^{-2} = c(r, r^{-1})$$

$$\boxed{Z_n(q,r) = \frac{(ab)^{n^2}}{c^n} \det_{0\le i,j\le n-1}\left(\frac{\left(\frac{d}{du}\right)^i}{i!} \frac{\left(\frac{d}{dv}\right)^j}{j!} \left[\frac{c(u,v)}{a(u,v)\, b(u,v)} \right] \Bigg|_{\substack{u=r^{-1} \\ v=r^{-1}}} \right)}$$

by Taylor expansion around the limit

• Note: $$\frac{c(u^{-1},v)}{a(u^{-1},v)\, b(u^{-1},v)} = \frac{1}{uv - q^{-2}} - \frac{1}{uv - q^{2}}$$

Taylor-expand:

• Define: $$(A_+)_{i,j} = \frac{\left(\frac{d}{du}\right)^i}{i!} \frac{\left(\frac{d}{dv}\right)^j}{j!} \left.\frac{1}{uv-q^2}\right|_{\substack{u=r^{-1}\\ v=r^{-1}}} \qquad (A_-)_{ij} = \text{idem } (q \to q^{-1})$$

• Introduce upper triangular matrix $U(\alpha,\beta)$

$$U(\alpha,\beta)_{i,j} = \begin{cases} \binom{j}{i} \alpha^i \beta^j & \text{if } i \leq j \\ 0 & \text{if } i > j \end{cases} \qquad \begin{pmatrix} 1 & \beta & \beta^2 & \beta^3 & \cdots \\ 0 & \alpha\beta & 2\alpha\beta^2 & 3\alpha\beta^3 & \cdots \\ 0 & 0 & \alpha^2\beta^2 & 3\alpha^2\beta^3 & \\ 0 & 0 & 0 & \alpha^3\beta^3 & \\ \vdots & & & \ddots & \ddots \end{pmatrix}$$

$i,j \in \mathbb{Z}_+ \qquad \alpha,\beta \in \mathbb{C}^*$

THM $$A_+ = -\frac{1}{r^{-2}-q^2}\left[U^t(\alpha,\beta)\right]^{-1} U(\alpha',\beta')$$

with: $\alpha = \frac{1-q^2r^2}{r}$ $\beta = \frac{q^2-q^{-2}}{r^2-q^2}$ $\alpha' = -q^2r^2\beta$ $\beta' = -\frac{1}{\alpha}$

Proof: 1. generating function$_t$ $(U(\alpha\beta)) = \dfrac{1}{1-\beta w(1+\alpha z)} = Y$

2. $U^t(\alpha,\beta)^{-1} = U(-1/\beta, -1/\alpha)$; $gf = \dfrac{1}{1+\frac{1}{\alpha}z(1-\frac{1}{\beta}w)} = X$

3. generating function(A_+) $= \dfrac{1}{(r^{-1}+z)(r^{-1}+w)-q^2} = X*Y$

THM $$A_+ = -\frac{1}{r^{-2}-q^2}\left[U^t(\alpha,\beta)\right]^{-1} U(\alpha',\beta')$$

with: $\alpha = \frac{1-q^2r^2}{r}$ $\quad \beta = \frac{q^2-q^{-2}}{r^2-q^2}$ $\quad \alpha' = -q^2r^2\beta$ $\quad \beta' = -\frac{1}{\alpha}$

Proof: 1. generating $\text{function}_t(U(\alpha\beta)) = \frac{1}{1-\beta w(1+\alpha z)} = Y$

2. $U^t(\alpha,\beta)^{-1} = U(-1/\beta, -1/\alpha)$; g.f. $= \frac{1}{1+\frac{1}{\alpha}z(1-\frac{1}{\beta}w)} = X$

3. generating function$(A_+) = \frac{1}{(r^{-1}+z)(r^{-1}+w) - q^2} = X * Y$

Holds true for any finite truncation to $i,j \in [0, n-1]$

set $\quad V = U^t(\alpha, \beta) \qquad \bar{U} = U(\alpha', \beta')$

$\quad \bar{V} = V(q \to q^{-1}) \qquad \bar{\bar{U}} = U(q \to q^{-1})$

then:
$$\boxed{\begin{aligned} A_+ &= \frac{1}{q^2 - r^{-2}} \, V^{-1} U \\ A_- &= \frac{1}{q^{-2} - r^{-2}} \, \bar{V}^{-1} \bar{U} \end{aligned}}$$

$$\det(A_- - A_+) = \det(A_-) \det\left[I - \frac{q^{-2} - r^{-2}}{q^2 - r^{-2}} \bar{U}^{-1} \bar{V} V^{-1} U \right]$$

$$\propto \det\left[I - \frac{q^{-2} - r^{-2}}{q^2 - r^{-2}} (\bar{V} V^{-1})(U \bar{U}^{-1}) \right]$$

$U \bar{U}^{-1} = U(-1, 1)$

$\bar{V} V^{-1} = U^t(-y, x)$

(Proof by convolution of g.f.)

Collecting all prefactors, we get:

$$Z^{(n)}_{ASM}(x,y) = \det\big((1-v)I + vG\big)$$

$$v = \frac{r^{-2}-q^{-2}}{q^{2}-q^{-2}} \qquad G = U^t(-y,x)\,U(-1,1)$$

$$G_{ij} = \sum_{k\geq 0} \binom{i}{k} y^k \binom{j}{k} x^{i-k}$$

$\Rightarrow$ generating function of $(1-v)I + vG$ $= f_{ASM}(z,w)$

THM

$$\boxed{f_{ASM}(z,w) = \frac{1-v}{1-zw} + \frac{v}{1-zx-w-(y-x)zw}}$$

Final identity:

$$\boxed{Z^{(n)}_{DPP} = \det(I+M) = \det((1-v)I + vG) = Z^{(n)}_{ASM}}$$

Proof:

note that

$$(1-z)(1-(1-v)w)\, f_{DPP}(z,w) - \left(1-\tfrac{z}{1-v}\right)(1-w)\, f_{ASM}(z,w)$$

$$= \big(\underbrace{x\, v(1-v) - y(1-v) - v}_{=0}\big) \times \text{rational fraction}(z,w)$$

$$\left(x = \left(\frac{q^2-q^{-2}}{q^{-1}r - q r^{-1}}\right)^2,\quad y = \left(\frac{qr - q^{-1}r^{-1}}{q^{-1}r - qr^{-1}}\right)^2,\quad v = \left(\frac{r^{-2}-q^{-2}}{q^2-q^{-2}}\right),\ 1-v = \left(\frac{q^2-r^{-2}}{q^2-q^{-2}}\right)\right)$$

+ Remark: let $A = (A_{ij})_{i,j\geq 0}$ $F = \sum_{i,j\geq 0} A_{ij} z^i w^j$

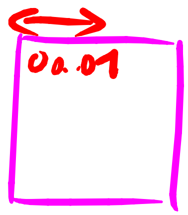
then $\boxed{(1-\lambda z)(1-\mu w)\, F(z,w)}$ is the generating function for $(I-\lambda S)\, A\, (I-\mu S^t)$ where

$S_{i,j} = \delta_{i,j+1}$ "shift" matrix strictly lower triangular $\Rightarrow$ the determinant is unchanged.

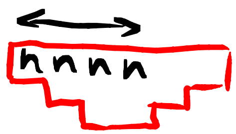
Proof completed for $z=1$

REFINEMENTS

$\#a_1$'s

- ASM 

$Z^{\#0\text{'s left of } 1 \text{ in 1st row (ASM)}}$

- DPP 

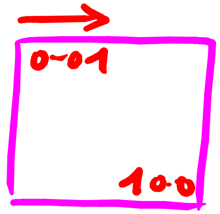
$Z^{\#\text{parts} = n}$

modifies the last column of the matrix only
→ change the g.f accordingly, same idea of proof

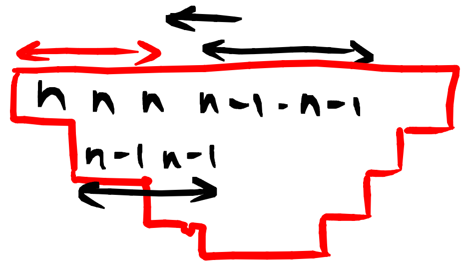
$$\boxed{Z_{ASM}(x, y, z) = Z_{DPP}(x, y, z)}$$

(MRR)

DOUBLE REFINEMENT

- ASM 

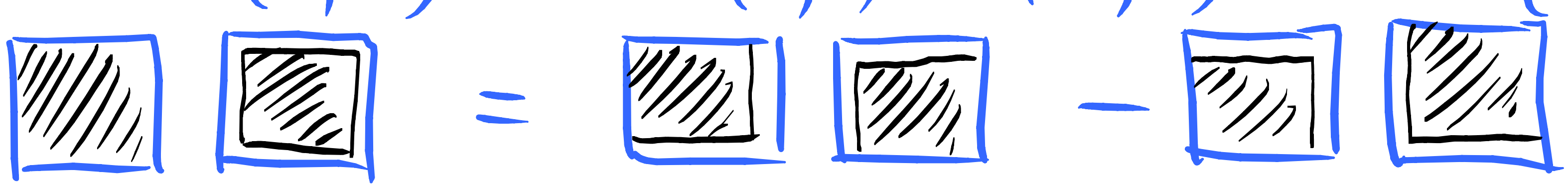
$z^{\#0\text{'s left of top }1}\, w^{\#0\text{'s right of bottom }1}$

- DPP 

$z^{\#(\text{parts}=n)}\, w^{\#(\text{parts}=n-1)}$
$\times\, w^{\#(\text{rows of length } n-1)}$

THM
$$Z_{ASM}(x, y, z, w) = Z_{DPP}(x, y, z, w)$$

Proof: use Desnanot-Jacobi Lewis-Carroll to relate $Z(z,w)$ to $Z(z,1)$ $Z(w,1)$ and $Z(1,1)$



For both $Z = Z_{ASM}$ and $Z = Z_{DPP}$:

$$\boxed{\begin{aligned}(z-w)\,Z_N(z,w)\;Z_{N-1}(1,1)\\ =:(z-1)\,w\,Z_N(z,1)\;Z_{N-1}(1,w)\\ -(w-1)\,z\;Z_N(1,w)\;Z_{N-1}(z,1)\end{aligned}}$$

(z, w affects only 2 columns of the matrix).

CONCLUSION

MRR PROVED by method of generating fctns for the matrix of which we take the det

something special about the class of gen.-fctns

→ Bijection ASM – DPP ? (– TSSCPP ?)

fermions? (– O(n) ?)

→ More refinements ?

other spectral parameters ?

$\rightarrow$ Generalizations: DPP with symmetries

$\leftrightarrow$ ASM with symmetries [in progress]

$\rightarrow$ q-deformation: $|D| = \sum a_{ij}$ for a DPP

$$\boxed{\sum_{D \in DPP(n)} q^{|D|} = \prod_{j=0}^{n-1} \frac{(3j+1)!_q}{(n+j)!_q}}$$

$[j]_q = \frac{1-q^j}{1-q}$

$j!_q = [1]_q [2]_q \cdots [j]_q$

pf = q-enumeration of ASMs?

$\rightarrow$ Razumov-Stroganov for DPP?

REMARK : integrability of 1+1D Lorentzian gravity v/s integrability of the 6V model

Critical varieties

- $\varphi(g,a) = \dfrac{1-g^2(1-a^2)}{ga}$
- $2\Delta(a,b,c) = \dfrac{a^2+b^2-c^2}{ab} = \dfrac{1+y-x}{\sqrt{x}}$

REMARK : integrability of 1+1D Lorentzian gravity v/s integrability of the 6V model

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not same!

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$x = g^2 \qquad y = g^2a^2$

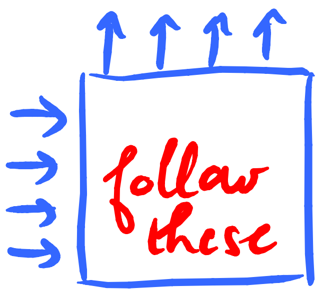
$\begin{cases} x \leftrightarrow y \ ? \\ a \leftrightarrow c \ ? \end{cases}$

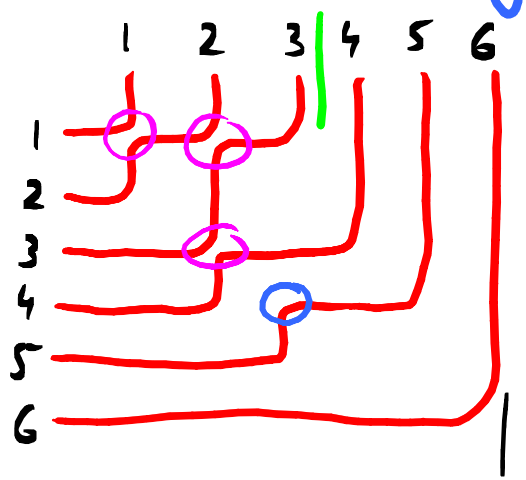
IT'S A SMALL WORLD!

FREE V/S NON-FREE FERMIONS

- DPP, TSSCPP → non-intersecting paths
  = free fermions

- 6V, ASM → osculating paths





O → y interaction
O → x
| → z
| → w

The determinant identity disentangles the interaction → make it more precise?