

RANDOM SURFACES

[P. Di Francesco, E. Guitten, J. Bouttier, R. Kedem]

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1. 2D Quantum Gravity & Matrix models

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1. 2D Quantum Gravity & Matrix models
maps / integrability

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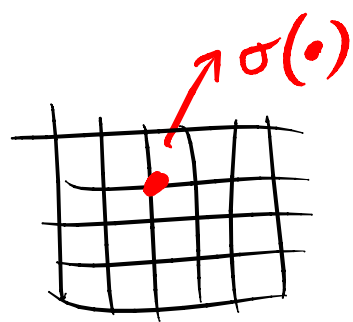
1. 2D Quantum Gravity & Matrix models
maps / integrability
2. Geodesic distance in planar maps

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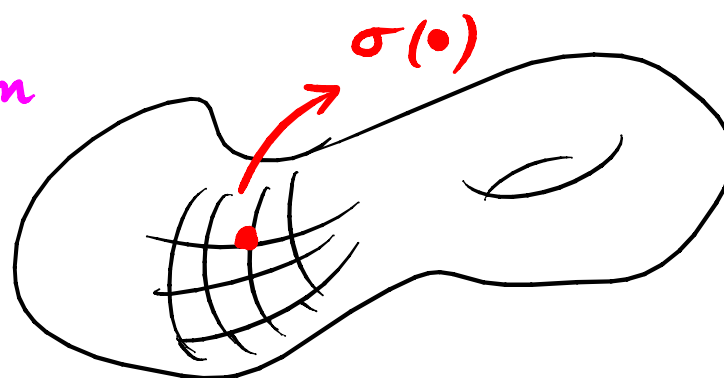
1. 2D Quantum Gravity & Matrix models
maps / integrability
2. Geodesic distance in planar maps
discrete integrability

PLANAR MAPS & GEODESICS



2D lattice model

2D quantum
Gravity
→
(Euclidian)
[90's]



2D random tessellation

- Matrix models, exact solutions
- Integrability (KP _{KdV} tau-functions)
- Correlations at fixed geodesic distance?

MAPS: 2D embedding of a connected graph with notions of:

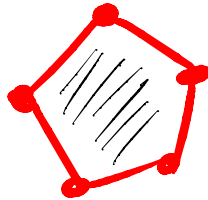
• vertex



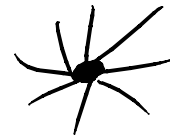
• edge



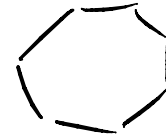
• face



• degree

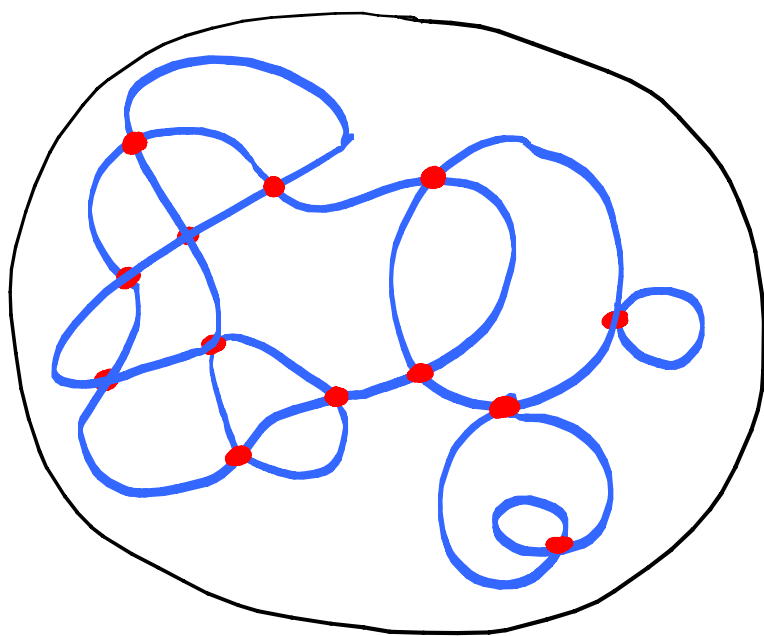


||
valence

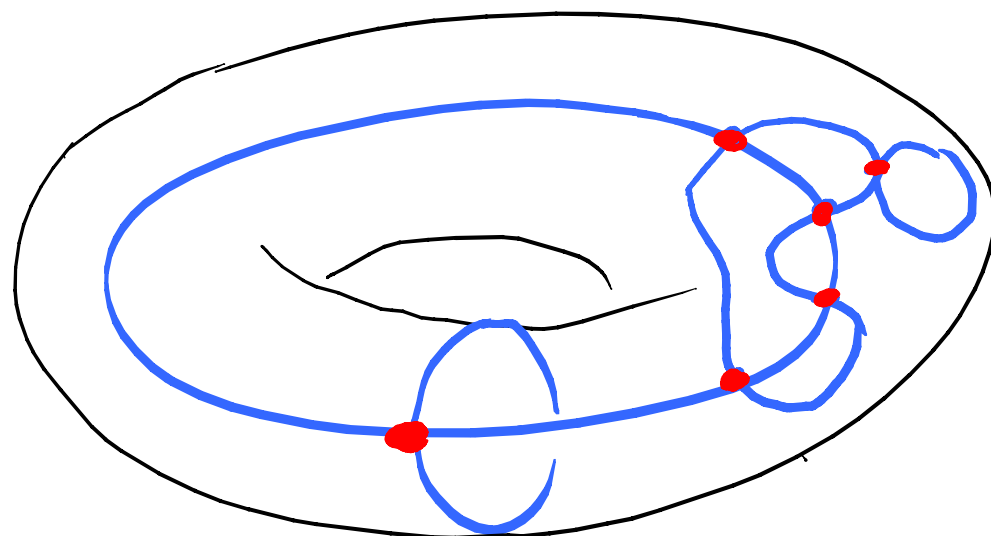


usually, rooting at an edge or a vertex

Examples of 4-valent maps



GENUS 0



GENUS 1

MATRIX MODELS

- statistical ensemble $\{M, N \times N \text{ Hermitian matrices}\}$
equipped w/ measure:

$$d\mu(M) = \frac{1}{Z_0} \underbrace{dM}_{\text{Haar measure}} e^{-N \text{Tr} \left(\frac{M^2}{2} - \sum_{k \geq 1} g_{2k} \frac{M^{2k}}{2k} \right)}$$

$\prod_{i < j} d(\text{Re } M_{ij}) d(\text{Im } M_{ij})$

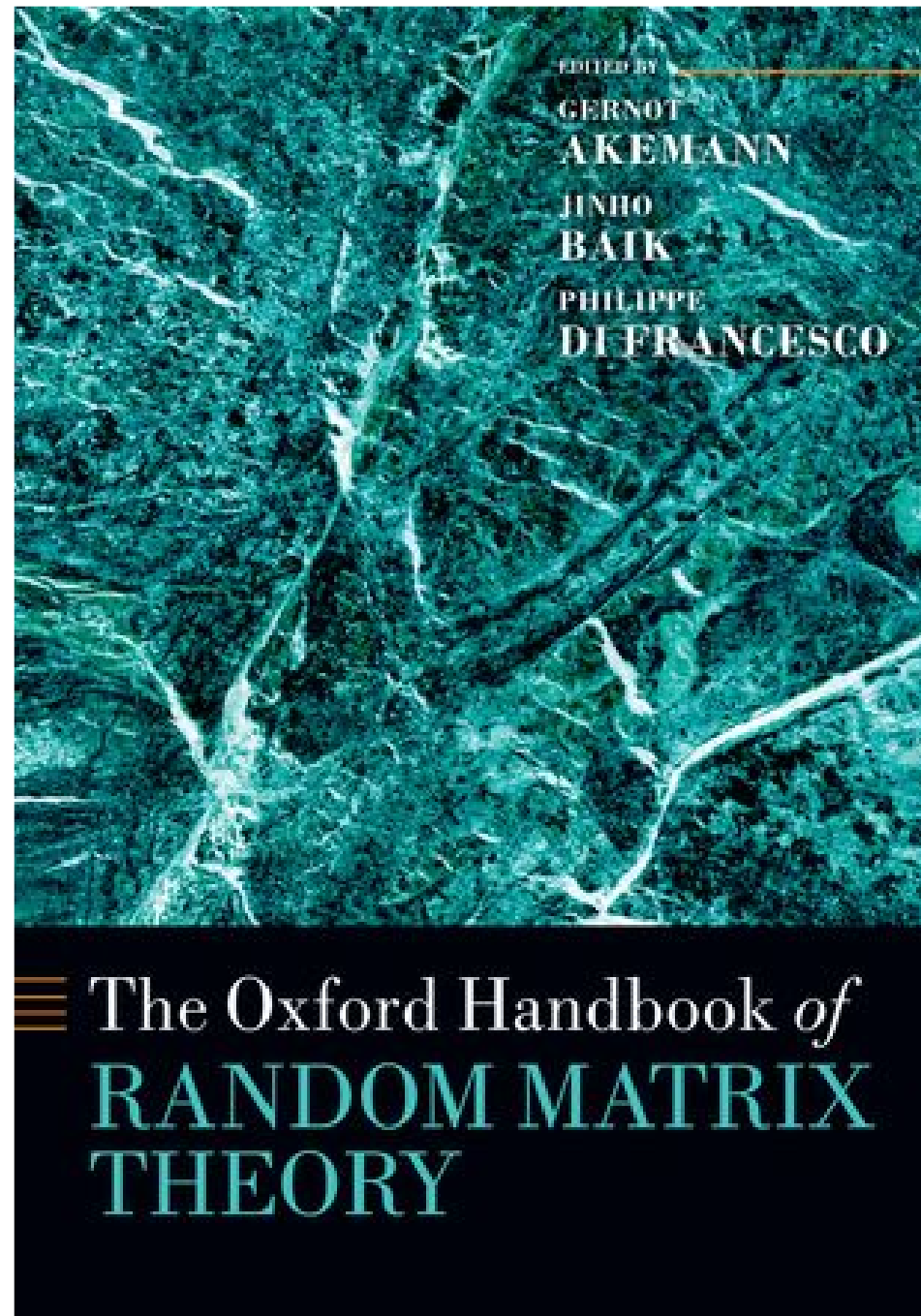
- Normalization / Partition function
 $Z_0 = \int dM e^{-N \text{Tr} \left(\frac{M^2}{2} \right)}$ (Gaussian)

$$Z(N; \{g_{2k}\}) = \int d\mu(M)$$

THM $\text{Log } Z(N; \{g_{2k}\}) =$ generating function
 for connected maps $\mathcal{M}$ of even valences with weight
 g_{2k} per $2k$ -vertex and $N^{X(\mathcal{M})}$ where
 $X(\mathcal{M}) =$ Euler characteristic of the map, namely
 $2 - 2 \times \text{genus of the underlying Riemann surface}$

$$\text{Log } Z(N; \{g_{2k}\}) = \sum_{h \geq 0} N^{2-2h} \sum_{\substack{\text{connected} \\ \text{maps } \mathcal{M} \\ \text{genus } h}} \prod_k g_{2k}^{V_{2k}(\mathcal{M})} \frac{1}{|\text{Aut } \mathcal{M}|}$$

$\downarrow$
 # vertices of degree $2k$ in $\mathcal{M}$



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42 Chapters

919 pages

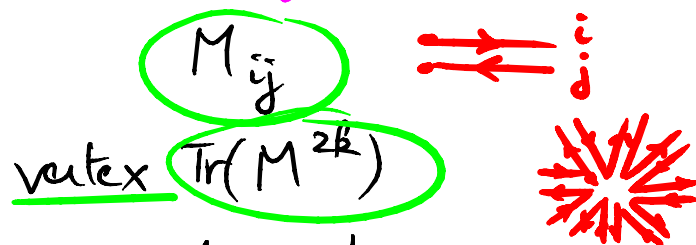
or

THM $\text{Log } Z(N; \{g_{2k}\}) =$ generating function
 for connected maps $\mathcal{M}$ of even valences with weight
 g_{2k} per $2k$ -vertex and $N^{X(\mathcal{M})}$ where
 $X(\mathcal{M}) =$ Euler characteristic of the map, namely
 $2 - 2 \times \text{genus of the underlying Riemann surface}$

Proof = "combinatorial lego"

1. expand $Z = \left\langle \sum_{n_k} \prod_k \frac{N g_{2k}}{k!} \frac{\text{Tr } M^{2k}}{2k} \right\rangle^{n_k}$
 $\text{d}\mu(M) = dM e^{-N \text{Tr}(\frac{M^2}{2})}$
 Gaussian average

2. Represent:

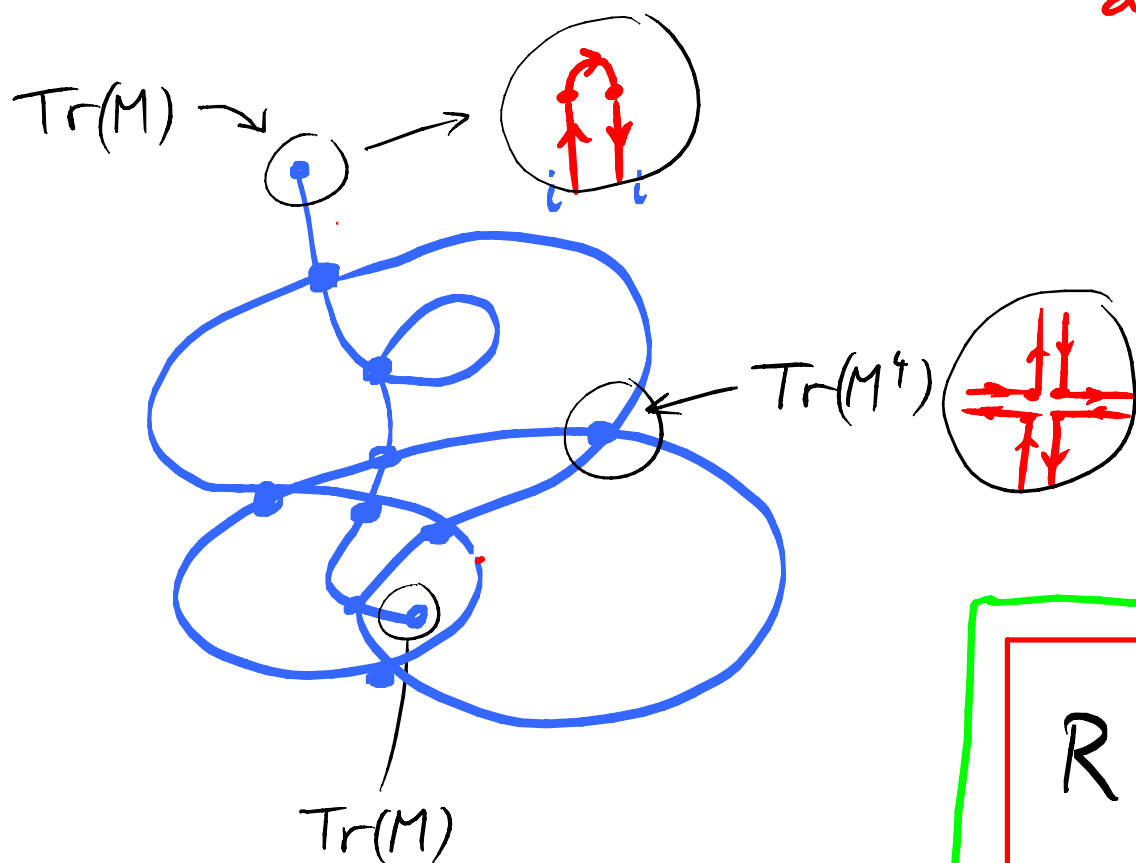


3. Wick THM pairings of M_{ij} 's via propagators $\langle M_{ij} M_{kl} \rangle = \frac{\delta_{il} \delta_{jk}}{N}$
 interpreted as edges $M_{ij} \rightleftarrows M_{kl}$

Ex: the 2-leg planar (genus 0) map gen. func.

$$R(\{g_{2k}\}) = \lim_{N \rightarrow \infty} \int [\text{Tr}(M)]^2 d\mu(M)$$

$$d\mu(M) = d\mu_0(M) e^{+N \text{Tr}(\sum_k g_{2k} \frac{M^{2k}}{2k})}$$



THM R is the unique solution with formal power series $\exp^n$ in the g 's to the algebraic eqn:

$$R = 1 + \sum_{k \geq 1} \binom{2k-1}{k} g_{2k} R^k$$

Results :

- compute matrix integrals : orthogonal polynomial technique
- Integrability : $Z = \text{tau-function of the KdV hierarchy (Miwa vars)}$
 $= g_{2h}/2h$ $\rightarrow$ genus expansion

Missing :

- an explanation for too simple answers
- intrinsic geometry

COMPUTING THE MATRIX INTEGRAL

0. Problem = compute $Z = \int dM e^{-N \text{Tr} V(M)}$

1. Go to eigenvalues $M = U m U^+$; $m = \text{diag}(m_i)$
 $U \in U(N)/U(1)^N$; $m \in \mathbb{R}^N$.

$$dM = dU \Delta(m)^2 ; \quad \Delta(m) = \prod_{i < j} (m_i - m_j) \text{ Vandermonde}$$

$$Z = \int d^N m \Delta(m)^2 e^{-N \text{Tr} V(m)}$$

2. Use orthogonal polynomials

$$d\mu(x) = dx e^{-NV(x)} \text{ on } \mathbb{R}$$

• $\Delta(m) = \det(p_{i-1}(m_j))_{1 \leq i, j \leq N}$

$\uparrow$ $p_m = \text{any monic polynomial of degree } m.$

COMPUTING THE MATRIX INTEGRAL

- Pick p_m orthogonal wrt: $d\mu(x) = dx e^{-NV(x)}$
on $\mathbb{R}$

namely $(p_m, p_n) = \int_{\mathbb{R}} d\mu(x) p_m(x) p_n(x) = h_n \delta_{m,n}$
 $\uparrow$ norms

- Then: $Z = \sum_{\substack{\sigma, \tau \\ \in S_N}} \text{sgn}(\sigma\tau) \int d\mu(m_1) \dots d\mu(m_N) \prod_{i=1}^N p_{\sigma(i)-1}(m_i) p_{\tau(i)-1}(m_i)$
 $= \sum_{\sigma, \tau \in S_N} \text{sgn}(\sigma\tau) \prod_{i=1}^N (p_{\sigma(i)-1}, p_{\tau(i)-1})$

$$Z = N! \prod_{i=1}^N h_{i-1}$$

$\uparrow_{\sigma=\tau} \uparrow$ by orthogonality

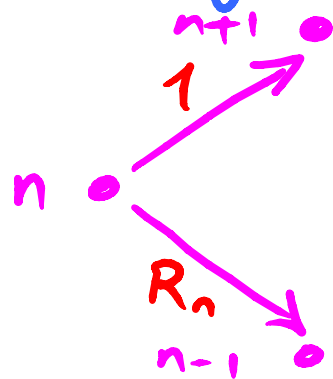
COMPUTING THE MATRIX INTEGRAL

3. Recursion relations for h_i :

Q : operator $x \cdot$ on $p_n(x)$: $x p_n(x) = (Q p_n)(x)$

P : operator $\frac{d}{dx}$ on $p_n(x)$: $\frac{d}{dx} p_n(x) = (P p_n)(x)$

- Then: $x p_n(x) = p_{n+1}(x) + R_n p_{n-1}(x)$ $R_n = \frac{h_n}{h_{n-1}}$
(use self duality of Q : $(Q p_n, p_m) = (p_n, Q p_m)$ and parity)

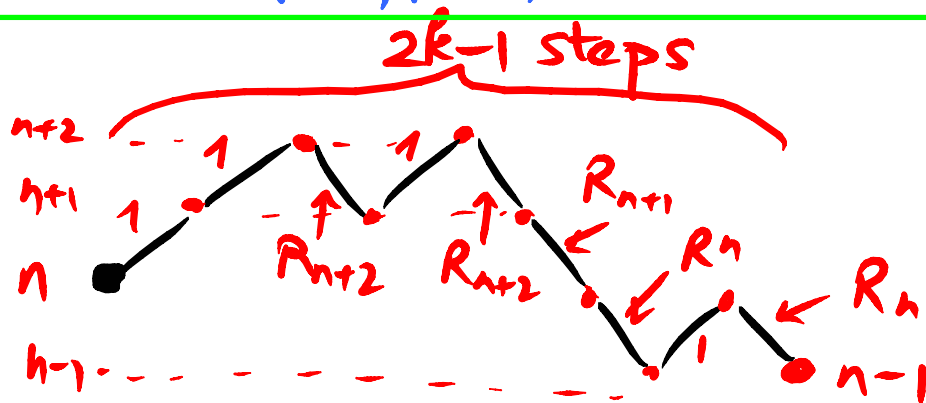


$\Rightarrow$ powers of Q described by paths.

COMPUTING THE MATRIX INTEGRAL

- $$\begin{aligned}
 (P_n, P_{n-1}) &= n h_{n-1} = \int P_n' P_{n-1} e^{-NV(x)} dx \\
 &\stackrel{\text{by parts}}{=} - \int P_n P_{n-1}' e^{-NV(x)} + N \int V'(x) P_n P_{n-1} e^{-NV(x)} dx \\
 &\quad \text{by orthogonality} \\
 &= N(V'(Q) P_n, P_{n-1})
 \end{aligned}$$

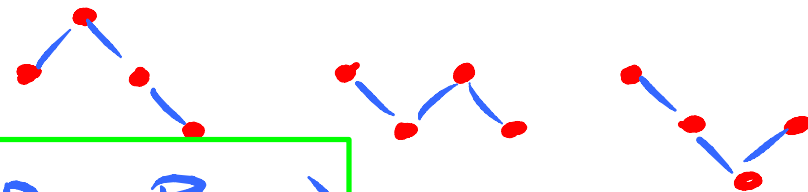
$$\Leftrightarrow \frac{n}{N} = \frac{(V'(Q) P_n, P_{n-1})}{(P_{n-1}, P_{n-1})} = R_n - \sum_{k \geq 2} g_{2k} \underbrace{\left(\frac{Q^{2k-1} P_n, P_{n-1}}{(P_{n-1}, P_{n-1})} \right)}_{\text{Polynomial of } R_n, R_{n \pm 1}, \dots, R_{n \pm (k-1)} \text{ of degree } k.}$$



Polynomial of $R_n, R_{n \pm 1}, \dots, R_{n \pm (k-1)}$ of degree k .

Examples

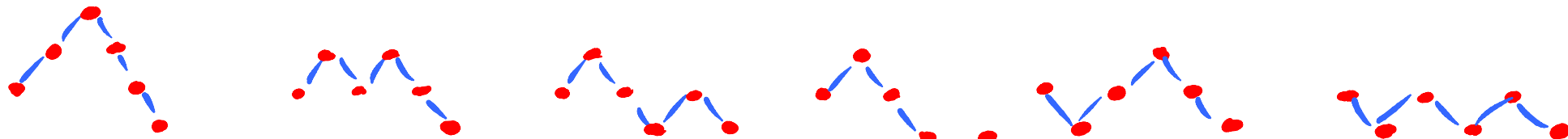
$k=2$ (4 valent) : $\frac{(Q^3 p_n, p_{n-1})}{(p_{n-1}, p_{n-1})} = R_{n+1}R_n + R_n^2 + R_n R_{n-1}$



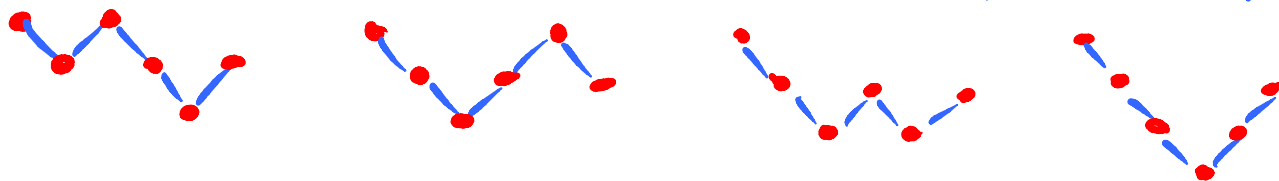
$$\frac{n}{N} = R_n - g R_n (R_{n+1} + R_n + R_{n-1})$$

$k=3$ (6 valent) : $\frac{(Q^5 p_n, p_{n-1})}{(p_{n-1}, p_{n-1})} =$

$$R_{n+2}R_{n+1}R_n + R_n^2 R_{n-1} + R_n R_{n-1}^2 + R_{n+1}R_n R_{n-1} + R_n^2 R_{n+1} + R_n^3$$



$$+ R_n^2 R_{n-1} + R_n^2 R_{n-1} + R_n R_{n-1}^2 + R_n R_{n-1} R_{n-2}$$



$\binom{5}{3} = 10$ terms

COMPUTING R

$$R = \int (\text{Tr} M)^2 d\mu(M) = \frac{1}{Z} \int d^N m \Delta(m)^2 \left(\underbrace{\sum_i m_i^2}_{N \text{ terms}} + \underbrace{\sum_{i \neq j} m_i m_j}_{N(N-1) \text{ terms}} \right) e^{-N \text{Tr} V(m)}$$

$$\begin{aligned} \bullet \int m_1^2 d\mu(M) &= \frac{1}{Z} \sum_{\sigma, \tau \in S_N} \text{sgn}(\sigma\tau) (m_1^2 P_{\sigma(1)-1}^{(m,1)}, P_{\tau(1)-1}^{(m,1)}) \prod_{i=2}^N (P_{\sigma(i)-1}, P_{\tau(i)-1}) \\ &= \frac{1}{Z} \sum_{i=0}^{N-1} \frac{(Q^2 P_i, P_i)}{(P_i, P_i)} = \underbrace{\sum_0^{N-1} (R_{i+1} + R_i)}_{\substack{\sigma=\tau \\ \text{by orthogonality}}} \end{aligned}$$

$$\begin{aligned} \bullet \int m_1 m_2 d\mu(M) &= \frac{1}{Z} \sum_{\sigma, \tau} \text{sgn}(\sigma\tau) (m_1 P_{\sigma(1)-1}^{(m,1)}, P_{\tau(1)-1}^{(m,1)}) (m_2 P_{\sigma(2)-1}^{(m,2)}, P_{\tau(2)-1}^{(m,2)}) \prod_{i=3}^N (P_{\sigma(i)-1}, P_{\tau(i)-1}) \\ &\Rightarrow \underbrace{\sigma(2) = \sigma(1) \pm 1}_{\text{"}} \quad \underbrace{\sigma(1)}_{\text{"}} \quad \underbrace{\sigma=\tau \text{ on } [3, N]}_{\text{"}} \\ &= -2 \sum_0^{N-1} R_i \end{aligned}$$

Total = $R_N \xrightarrow{N \rightarrow \infty} R$ and theorem follows.

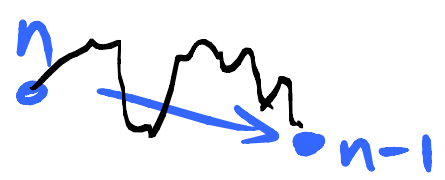
COMPUTING R

$$R_N \rightarrow R = 1 + \sum_{k \geq 2} g_{2k} R^k \binom{2k-1}{k}$$

($\frac{n}{N}$ at $n=N$)

↓

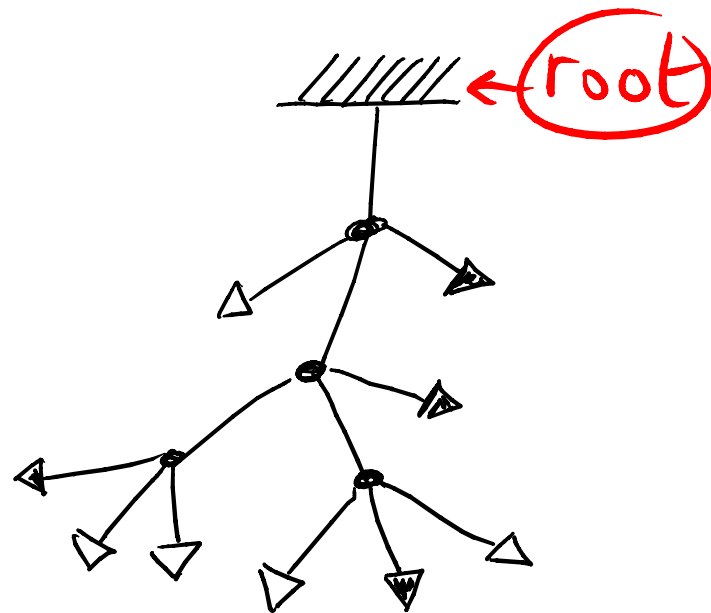
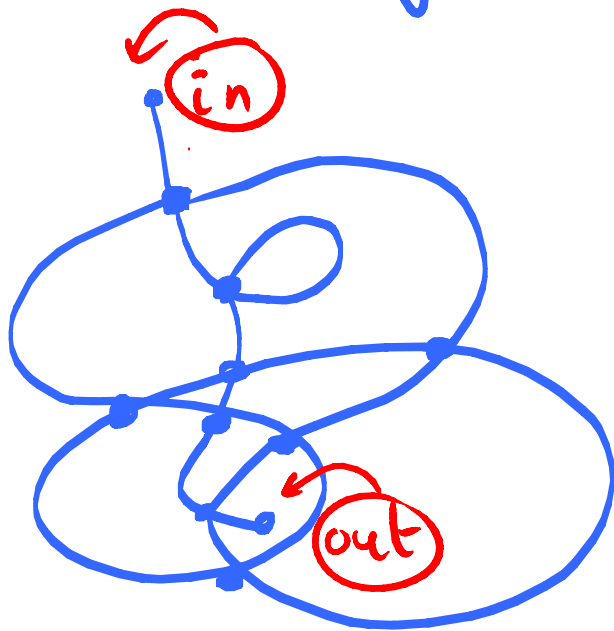
paths of $2k-1$ steps
from n



w/ weight R / descent.

A COMBINATORIAL APPROACH

$\left\{ \begin{array}{l} \text{tetraivalent planar maps} \\ \text{with 2 legs} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{rooted 4-valent blossom} \\ \text{trees with } 1 \downarrow \text{ at each vertex} \end{array} \right\}$



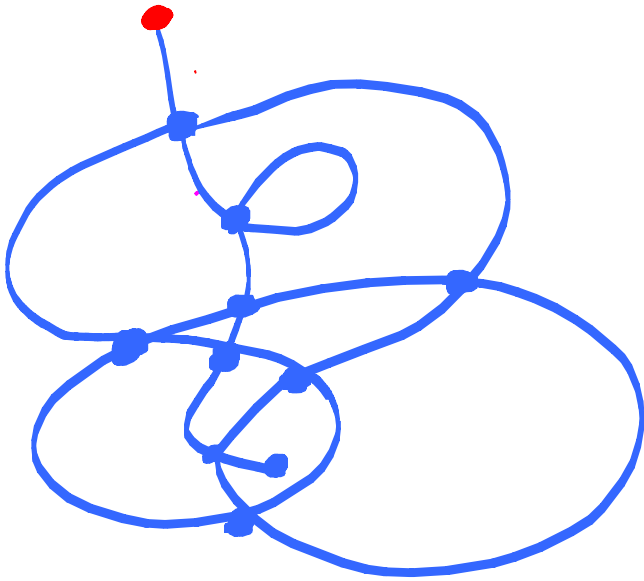
- Explains algebraic eqn (TREES).
- Allows to track distance between legs

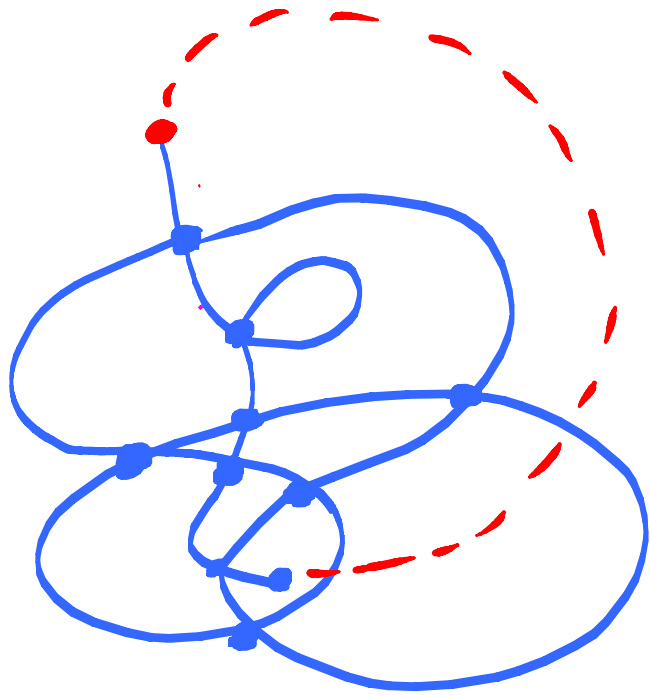
- Model for discrete random surfaces



Planar map

- tetravalent
- 2 legs



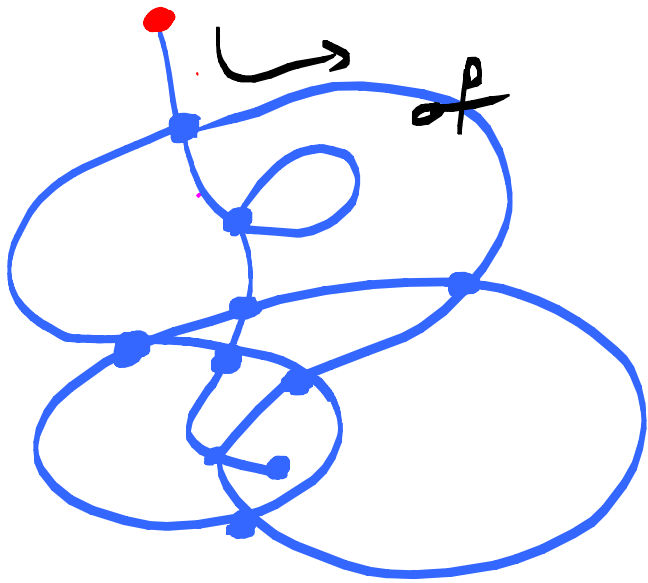


- intrinsic geometry

Planar map

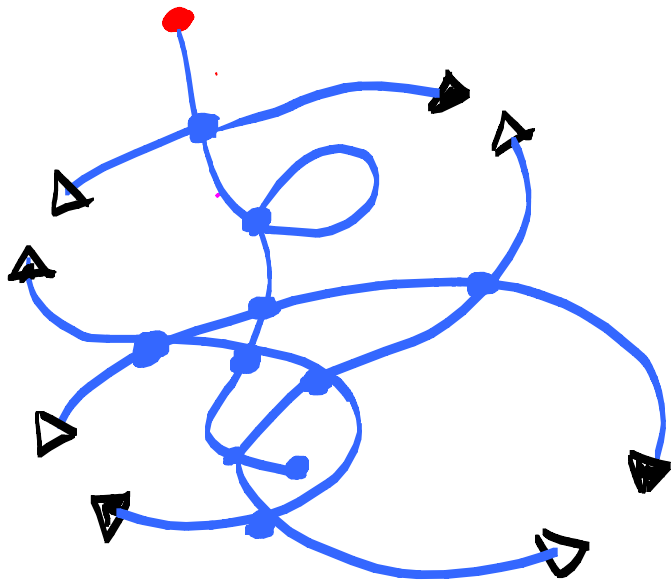
- tetravalent
- 2 legs
- geodesic distance between the legs = 2 here

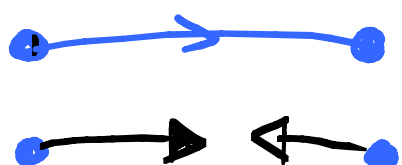
MAP - TREE BIJECTION



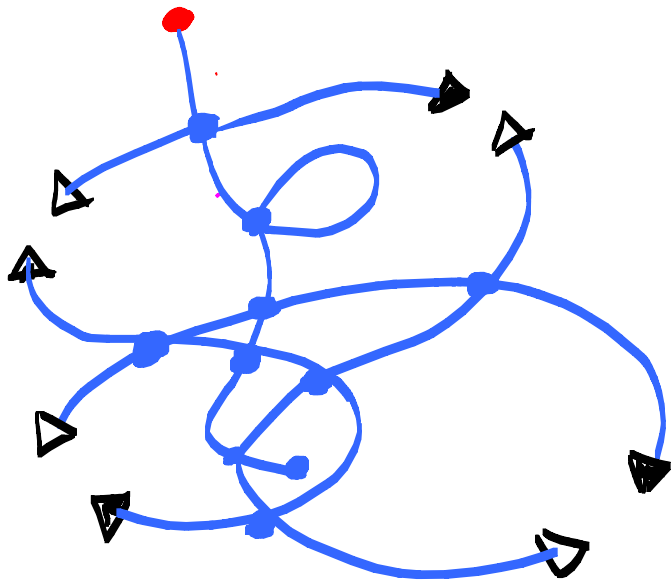
- cutting algorithm
- ① 1. walk along border of external face
2. for each edge, cut if it doesn't disconnect the graph
3. replace with
-

MAP - TREE BIJECTION



- cutting algorithm
- ① 1. walk along border of external face
2. for each edge, cut if it doesn't disconnect the graph
3. replace with
- 

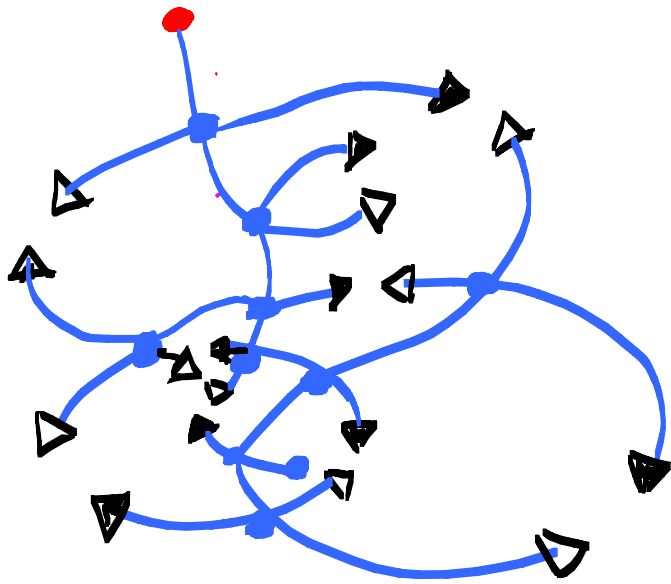
MAP-TREE BIJECTION



• cutting algorithm

ⓑ repeat ⓐ with this
new planar map

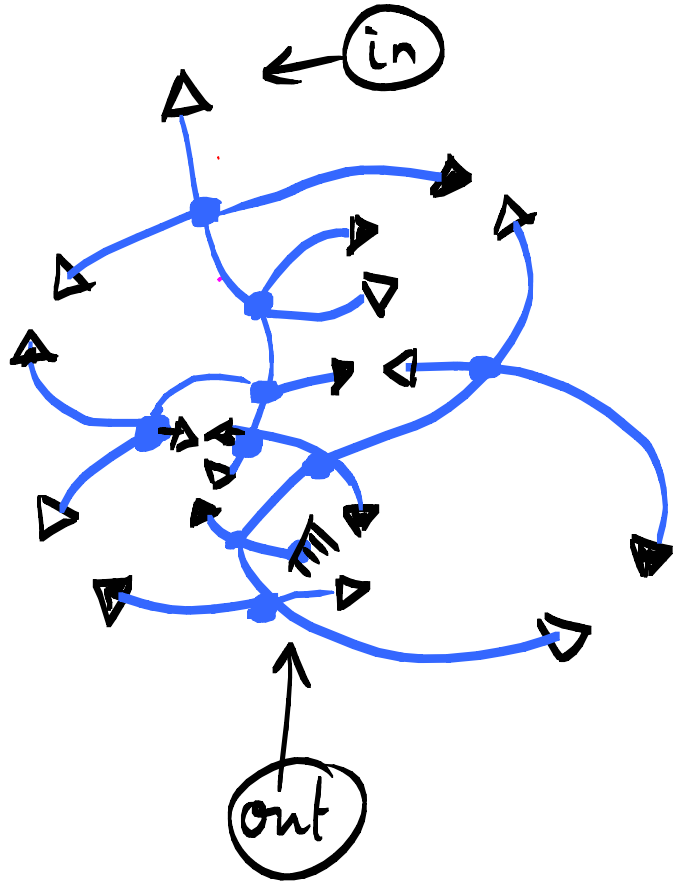
MAP - TREE BIJECTION



- cutting algorithm

Ⓐ repeat Ⓐ with this new planar map etc. until we have a tree (only 1 face).

MAP - TREE BIJECTION



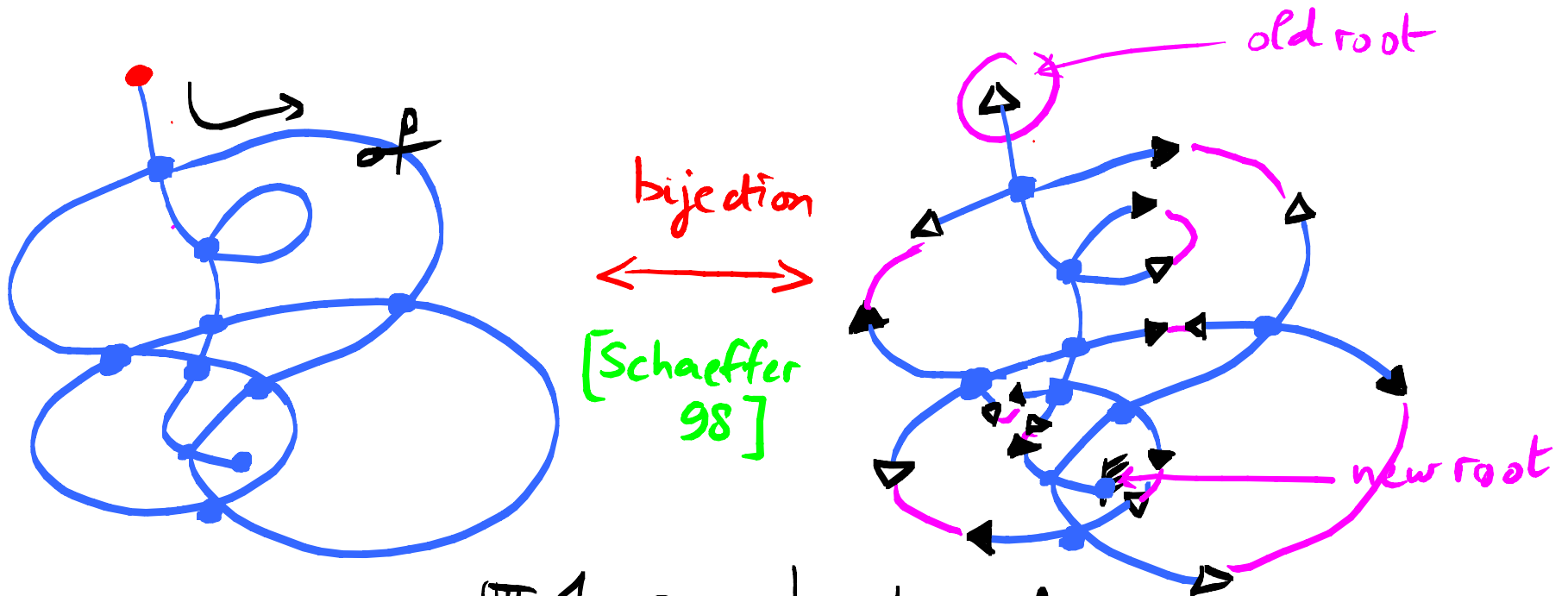
Change the
 (out) vertex into a root
 (in) vertex into a $\downarrow$

$\rightarrow$ Blossom-tree = tree with
 $\downarrow$ and $\downarrow$ leaves and

(i) exactly 1 $\downarrow$ at each inner vertex

(ii) $\#(\downarrow) = \#(\downarrow) + 1$

MAP - TREE BIJECTION



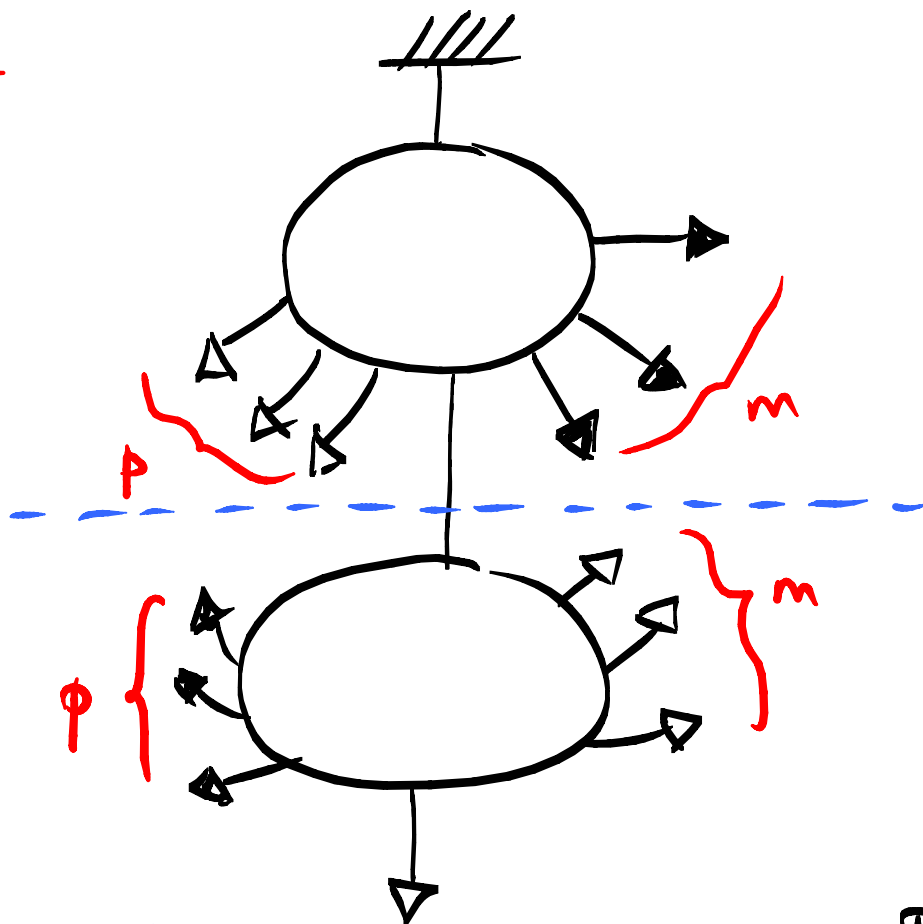
1. one $\downarrow$ at each inner vertex

2. Total "charge" = $\# \downarrow - \# \downarrow = +1$

(THM)

Equivalently: each descendant subtree has total charge = +1

Proof:



descendant
subtree

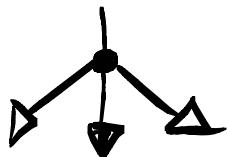
(+ parity of # legs) (extra)
as valence = 4

• we have necessarily

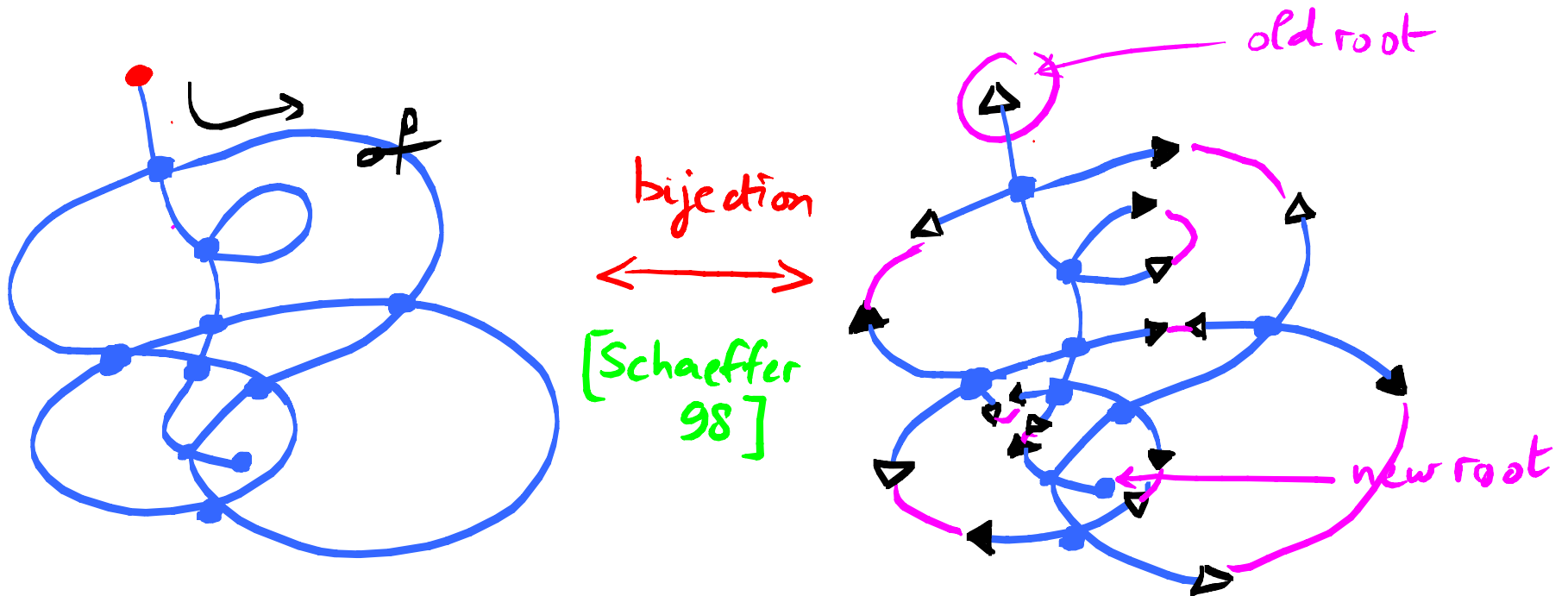
$m = p$ or $p+1$
but $m+p$ even
 $\Rightarrow m = p$

• change (descendant)
 $= m - p + 1 = +1$

• Proof of 1-2 by erosion

iterate( $\rightarrow$ )

MAP - TREE BIJECTION



R = generating function for rooted blossom trees

$$R = 1 + 3g + R^2$$

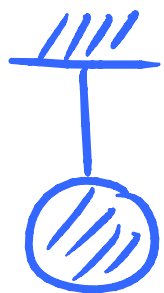
The diagram illustrates the generating function equation $R = 1 + 3g + R^2$ for rooted blossom trees. The equation is written in black ink. Below the equation, the terms are represented by diagrams: a rooted blossom tree (a circle with a vertical line and a horizontal line), a single edge with a triangle at the end, and three examples of a root vertex connected to two other vertices (each with a circle and a vertical line). The entire diagram is enclosed in a green box.

Generalization to arbitrary even valences $g_2, g_4, \dots$

• same cutting algorithm

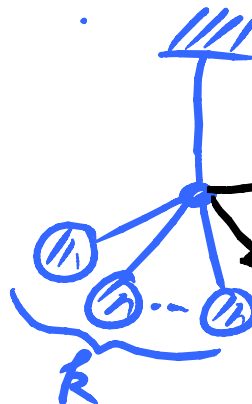
$\{\text{two-leg planar maps}\} \leftrightarrow \left\{ \begin{array}{l} \text{rooted planar blossom} \\ \text{trees with exactly} \\ k-1 \downarrow \text{ at each} \\ 2k\text{-valent vertex} \end{array} \right\}$

$$R = 1 + \sum_{k \geq 2} g_{2k} \binom{2k-1}{k} R^k$$



+

$\sum_{k \geq 2}$



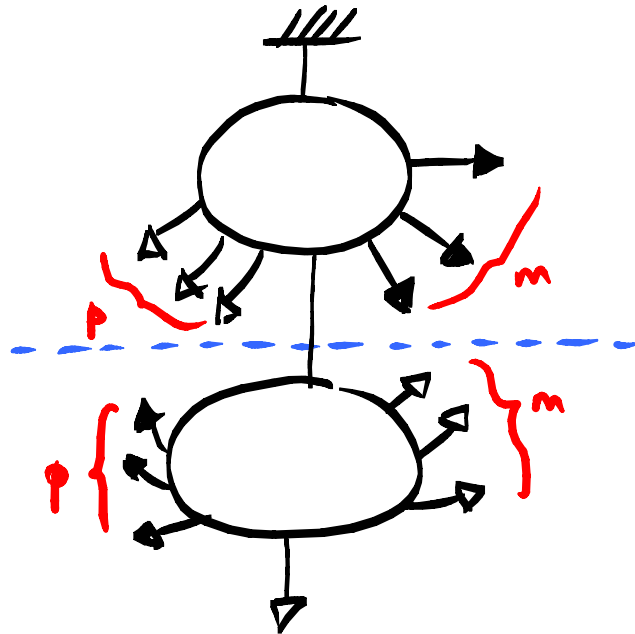
$\} k-1$

$\} \binom{2k-1}{k} \text{ configs.}$

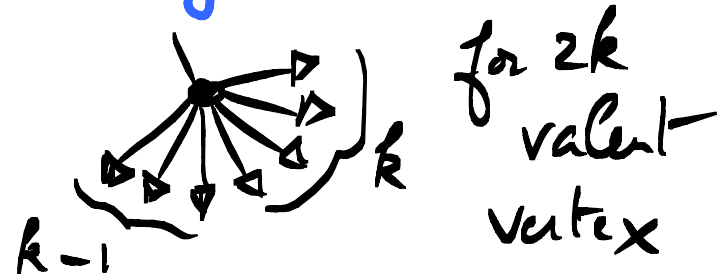
Proof: same as before

THM Equivalent characterization = each descendant subtree has total charge +1 and inner vertices have even valences 4, 6, 8...

Same argument



+ erosion argument



CRITICAL POINTS

Pure gravity:

$$R = 1 + 3g R^2 \Rightarrow R = \frac{1 - \sqrt{1 - 12g}}{6g}$$

$$g_c = \frac{1}{12} = \text{critical point}$$

$$\begin{aligned} \# \text{ maps with } n \text{ vertices} &= \frac{3^n}{n+1} \binom{2n}{n} \\ &\sim \frac{1}{\sqrt{\pi}} \frac{12^n}{n^{3/2}} \end{aligned}$$

$$\# \text{ unmarked maps} \sim \frac{12^n}{n^{3/2}} = \frac{\mu^n}{n^{3-\gamma_{st}}}$$

$$\gamma_{st} = -\frac{1}{2} \quad \text{string susceptibility.} \quad R_c - R \sim (g_c - g)^{-\gamma_{st}}$$

CRITICAL POINTS

General maps with even valence:

$$g_4 = g \quad ; \quad g_{2k} = \gamma_{2k} g^{k-1} \quad ; \quad \mathcal{F} = g^R$$

$$\Rightarrow g = \varphi(\mathcal{F}) = \mathcal{F} - \sum_2^d \gamma_{2k} \mathcal{F}^k$$

• Critical point: if γ_{2k} generic g_c such that $\varphi'(g_c) = 0$

$$\gamma_{2k} = -\frac{1}{2} \text{ as } g_c - g \sim \frac{(g_c - \mathcal{F})^2}{2} \varphi''(g_c) \quad \left\{ \begin{array}{l} g_c = \varphi(g_c) \end{array} \right.$$

• Multicritical point: γ_{2k} fine-tuned $g_c = \varphi(g_c)$

$$\varphi'(g_c) = \varphi''(g_c) = \dots = \varphi^{(m)}(g_c) = 0$$

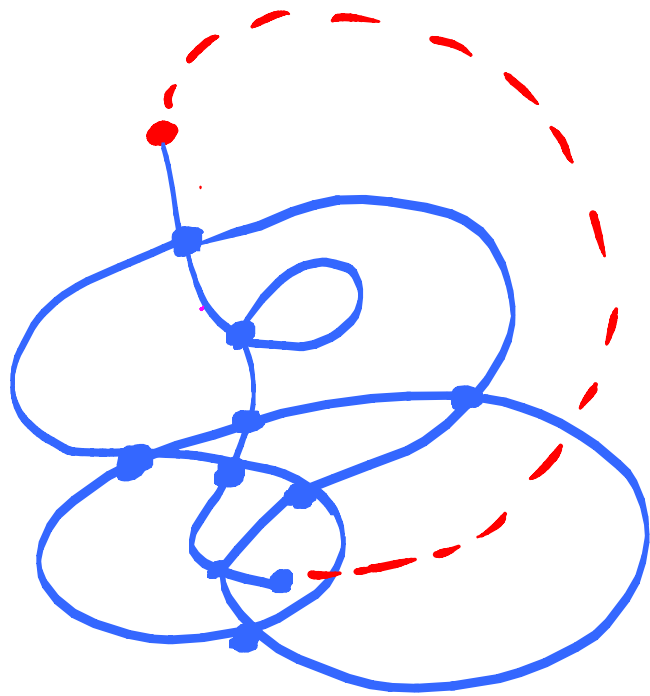
then $\gamma_{2k} = -\frac{1}{m+1}$ as

$$g_c - g \sim \frac{(g_c - \mathcal{F})^{m+1}}{(m+1)!} \varphi^{(m+1)}(g_c)$$

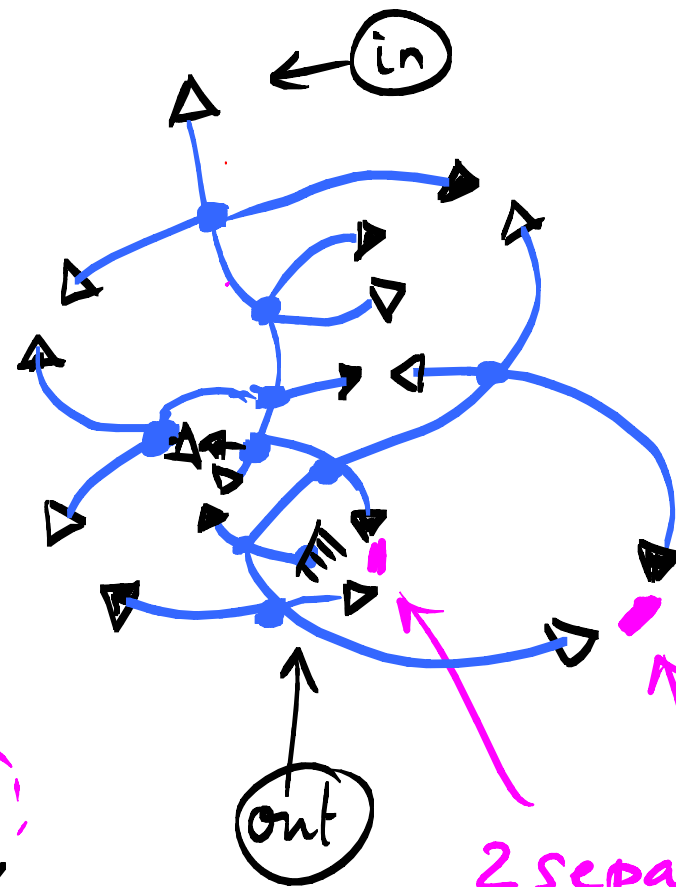
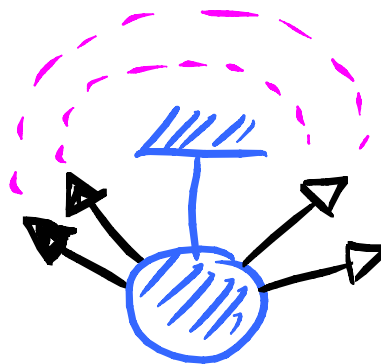
$\# \text{ maps}_{n \text{ vertices}} \sim \frac{g_c^{-n}}{n^{3-\gamma_{2k}}}$

another universality class!

GEODESIC DISTANCE:



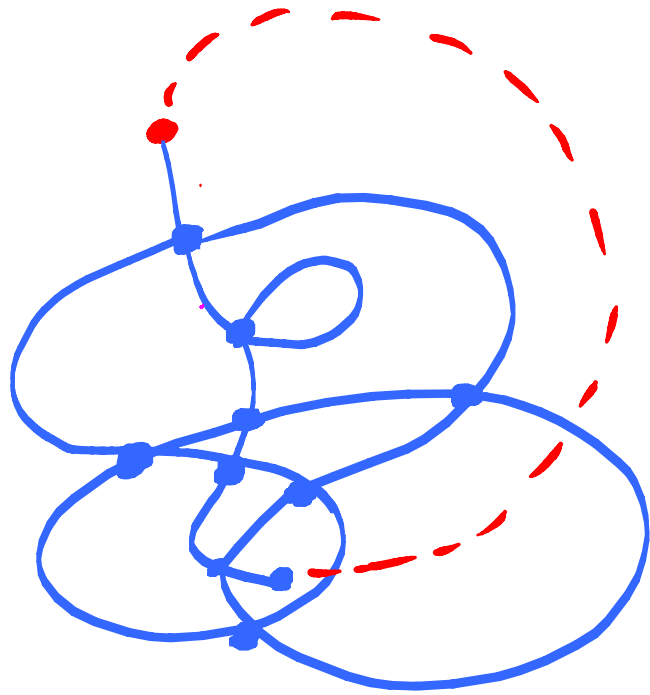
$d = 2$



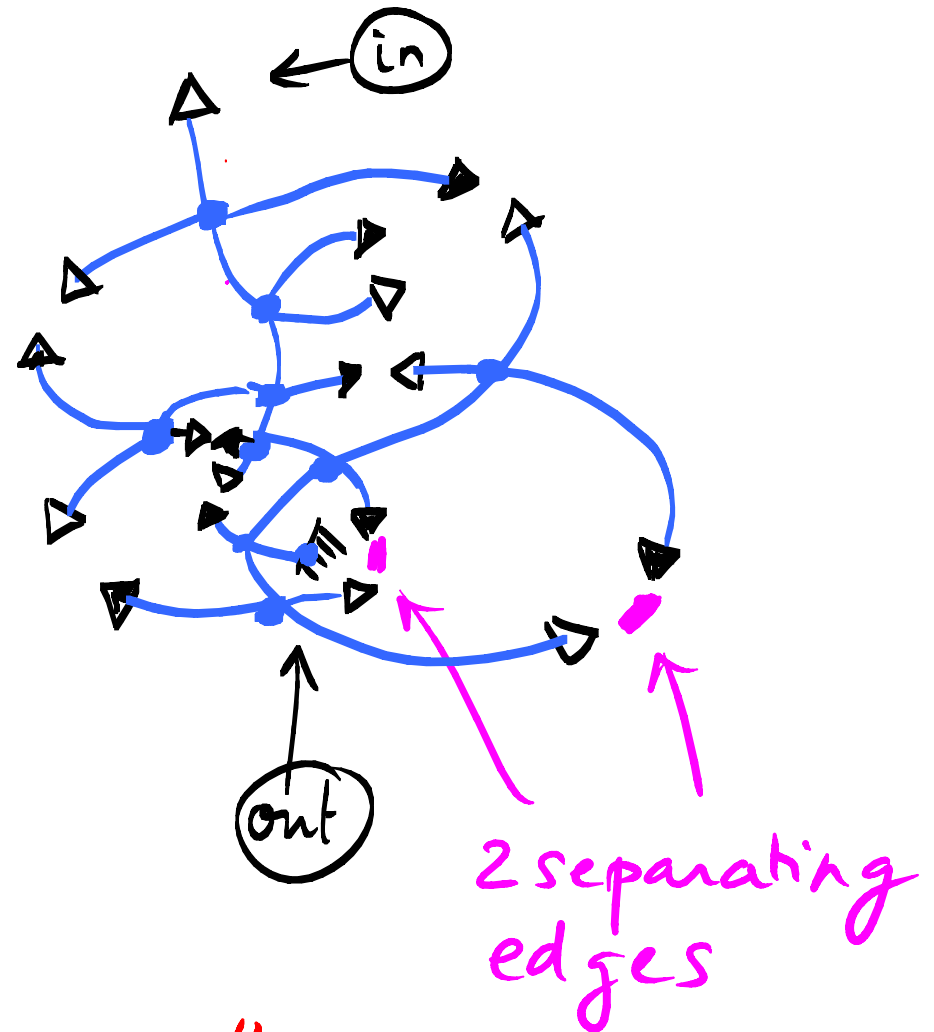
2 separating edges

TWO "EXCESS BLACK LEAVES" ON THE LEFT

GEODESIC DISTANCE:

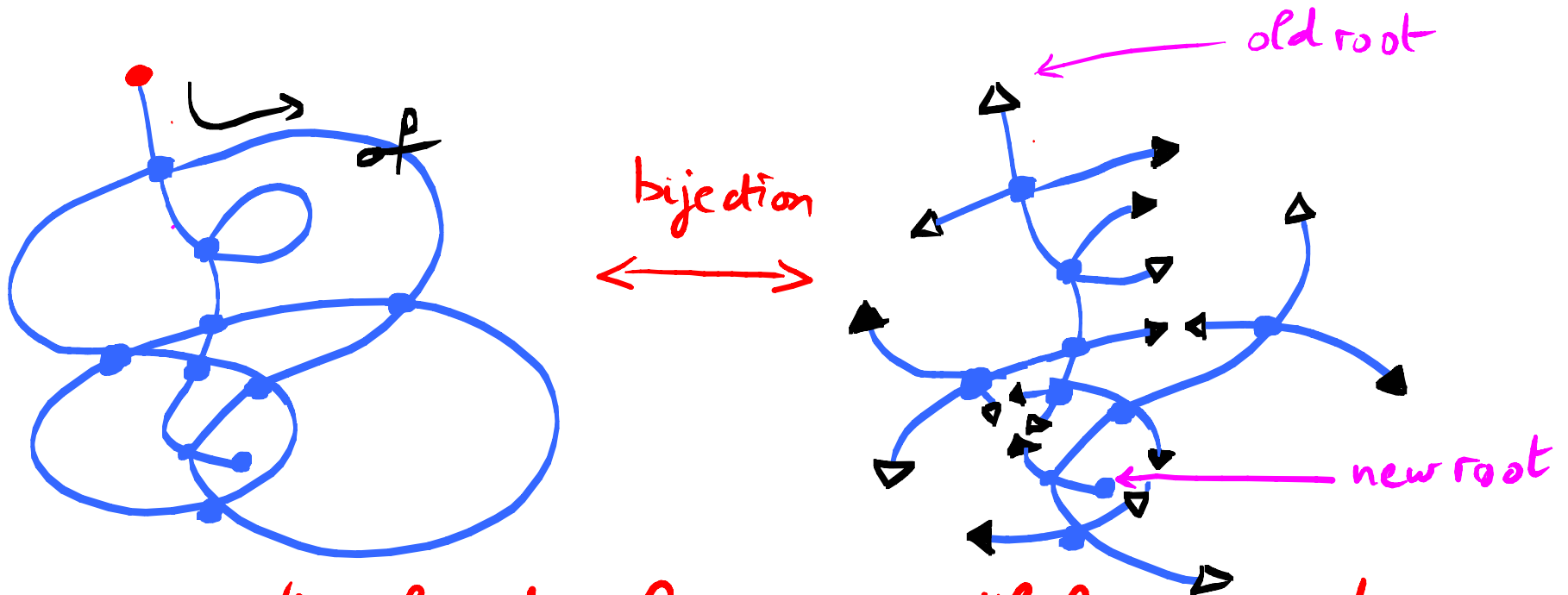


$d=2$



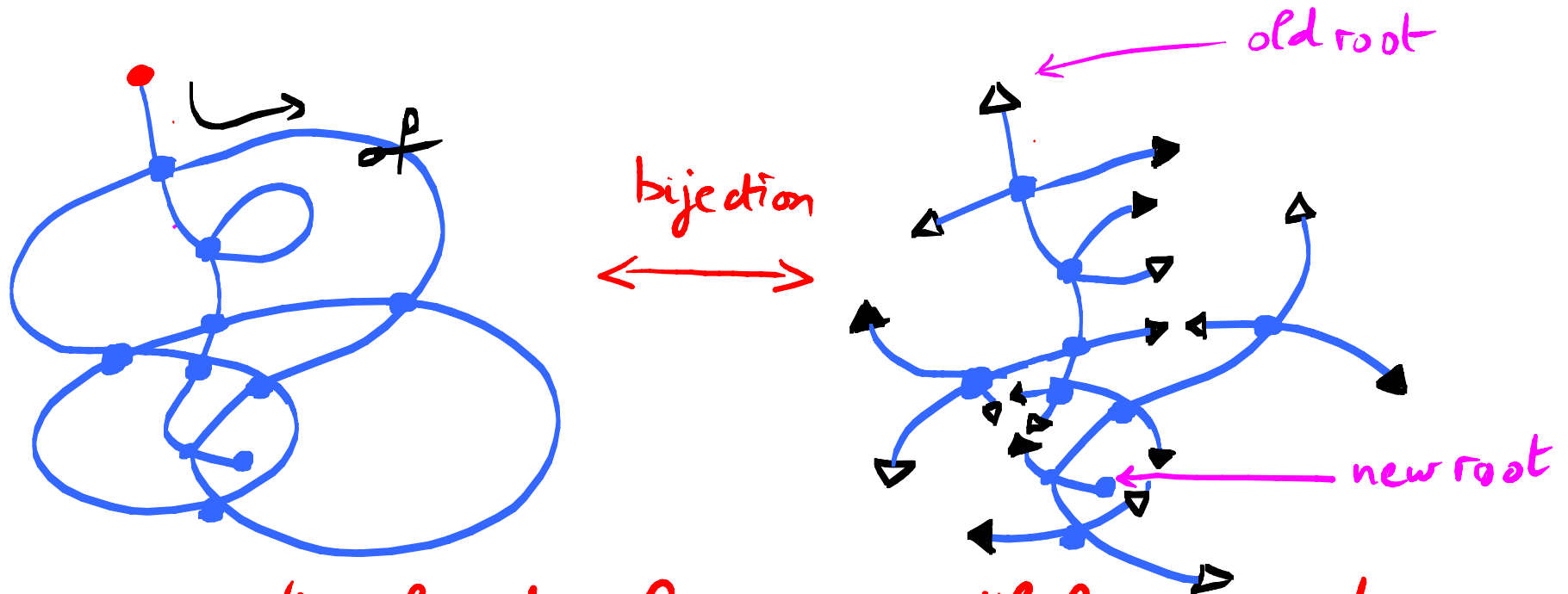
TWO "EXCESS BLACK LEAVES" ON THE LEFT

MAP - TREE BIJECTION

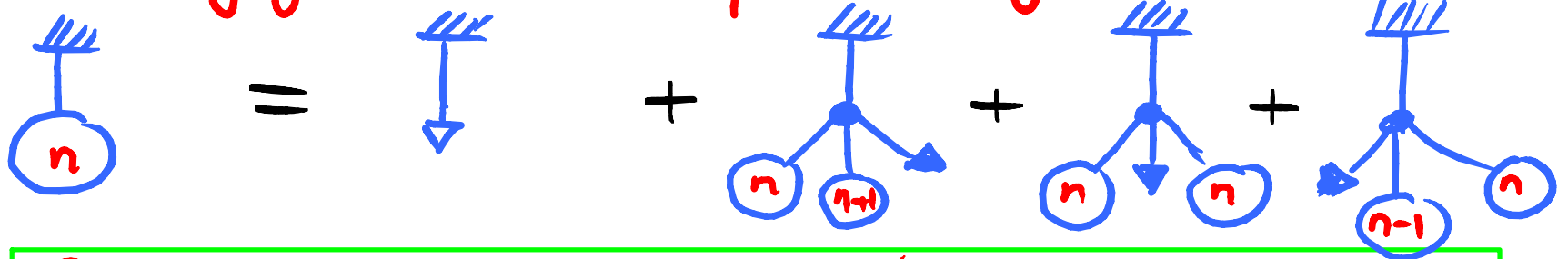


$R_n =$ generating function for maps with legs at distance $\leq n$

MAP - TREE BIJECTION (4-valent case)



R_n = generating function for maps with legs at distance $\leq n$



$$R_n = 1 + g R_n (R_{n+1} + R_n + R_{n-1})$$

INTEGRABILITY

$$0 = R_n - 1 - g R_n (R_{n+1} + R_n + R_{n-1}) \quad (*) \quad (R_{-1}=0, R_{\infty}=R)$$

• This is a discrete integrable equation ($n = \text{time}$)

$$\exists \varphi(x, y) = xy(1 - g(x+y)) - x - y \text{ such that}$$
$$\varphi(R_n, R_{n+1}) - \varphi(R_{n-1}, R_n) = (R_{n+1} - R_{n-1})x \quad (*)$$

INTEGRABILITY

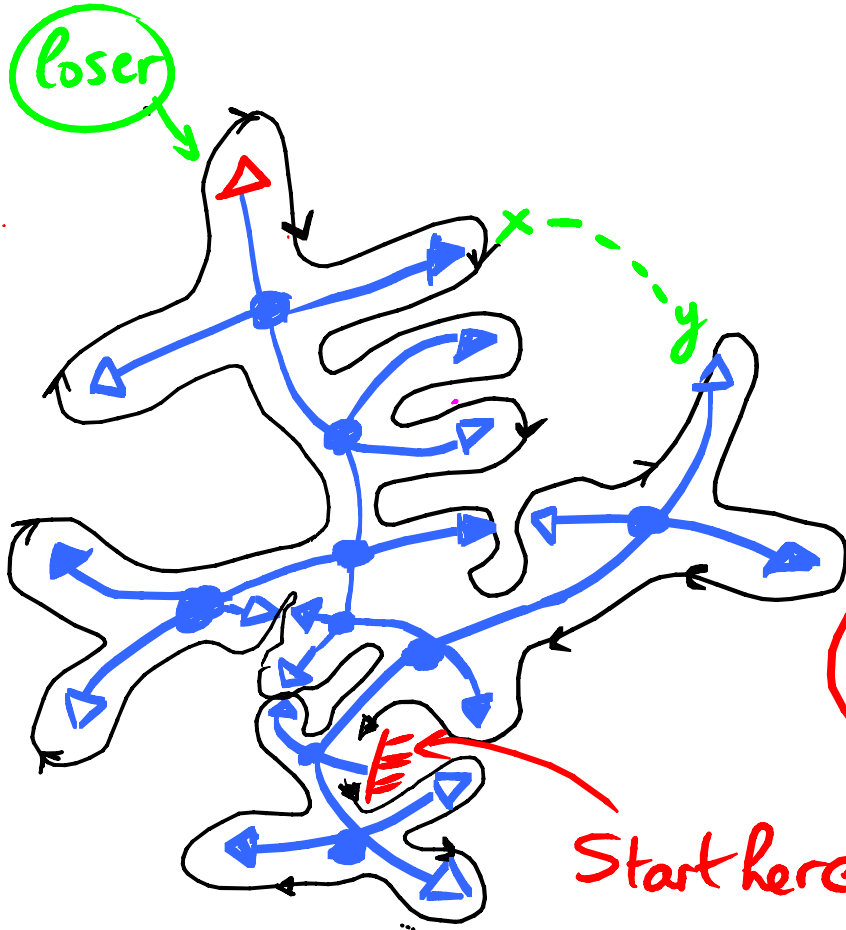
$$0 = R_n - 1 - g R_n (R_{n+1} + R_n + R_{n-1}) \quad (*) \quad (R_{-1}=0, R_{\infty}=R)$$

• This is a discrete integrable equation ($n = \text{time}$)

$$\exists \varphi(x, y) = xy(1 - g(x+y)) - x - y \text{ such that}$$
$$\varphi(R_n, R_{n+1}) - \varphi(R_{n-1}, R_n) = (R_{n+1} - R_{n-1}) \times (*) = 0!$$

Discrete 1st integral of motion

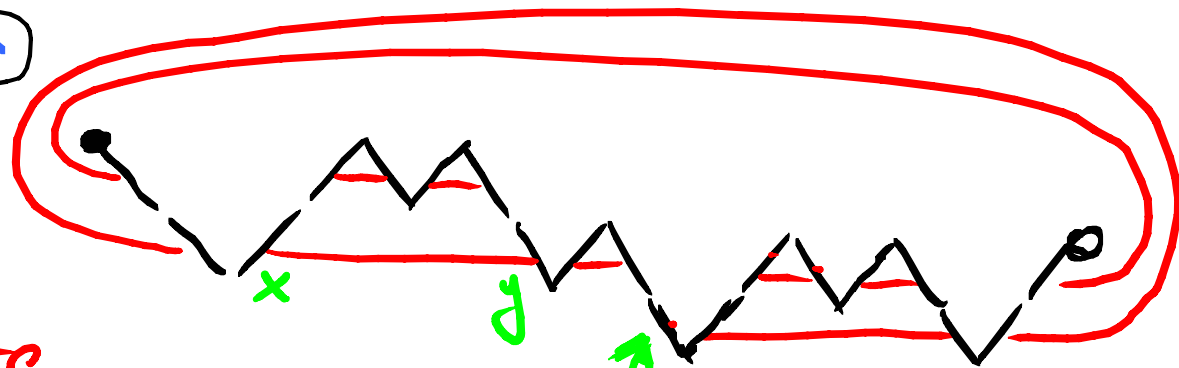
CONSERVATION LAW



$d=2$ rounds

CONTOUR WALK

leaf = $\uparrow \leftrightarrow \searrow$ down step
bud = $\downarrow \leftrightarrow \swarrow$ up step



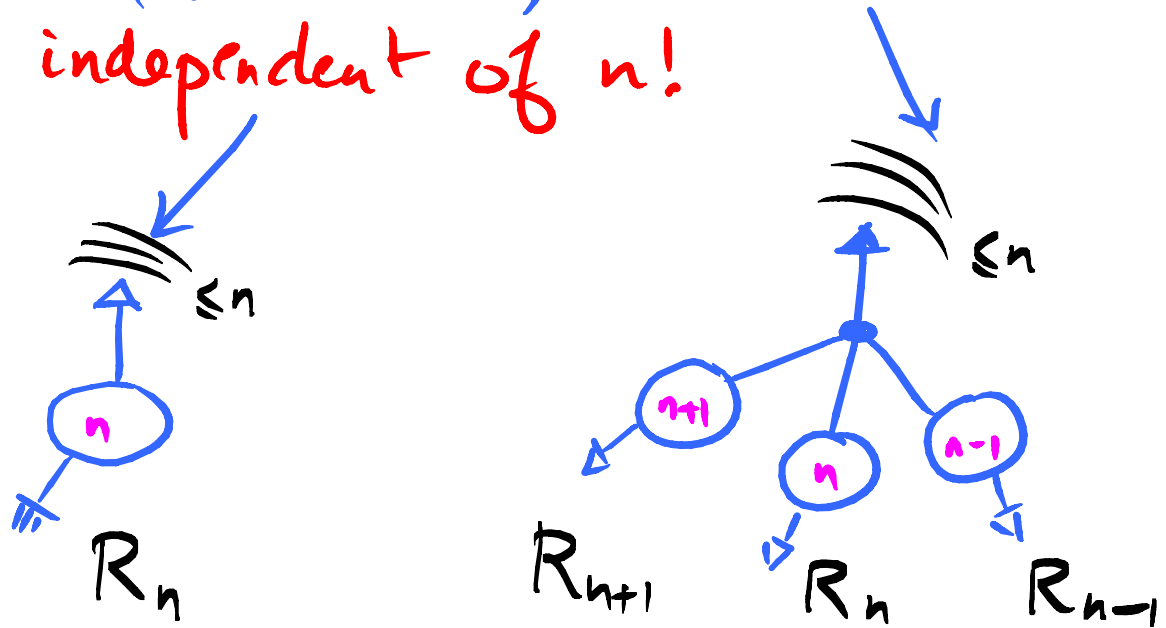
musical chair loser
= in

depth $\in \{0, 1, 2\}$

CONSERVATION LAW

Depth of a leaf = $d - \# \text{rand} + 1$; $d = \text{distance (legs)}$

Conservation law: $\#(\downarrow \text{depth} \leq n) - \#(\downarrow \text{depth} \leq n) = 1$
is independent of n !



$$R_n - g R_{n+1} R_n R_{n-1} = R_0 \quad (\text{indep of } n).$$

equivalent to $\psi(R_n, R_m) = \text{const modulo } (\mathbb{Z})$

SOLUTION

use $R_\infty = R$ and expand $R_n = R(1 - f_n)$

f_n "small" $\rightarrow 0$ as $n \rightarrow \infty$. Linearize eqn around R :

$$R(1 - f_n) - 1 - g R^2(1 - f_n)(3 - f_{n+1} - f_n - f_{n-1}) = 0$$

$$-R f_n + g R^2(4 f_n + f_{n+1} + f_{n-1}) = 0$$

look for $f_n \sim x^n$ then

$$x + \frac{1}{x} + 4 = \frac{1}{gR}$$

• write higher orders:

$$R_n = R(1 - f_n^{(1)} - f_n^{(2)} - \dots)$$

$$d_1 x^n \quad d_2 x^{2n} \quad \dots$$

recursion relations

$\rightarrow$ The series sums nicely...

INTEGRABILITY & SOLUTION

$$0 = R_n - 1 - g R_n (R_{n+1} + R_n + R_{n-1}) (*) \quad (R_{-1}=0, R_{\infty}=R)$$

- Exact soliton solution [DF, Bouttier, Guillet 05-07]

$$R_n = R \frac{1-x^{n+1}}{1-x^{n+2}} \frac{1-x^{n+4}}{1-x^{n+3}}$$

$$R = 1 + 3g R^2$$

$$x + \frac{1}{x} + 4 = \frac{1}{gR}, \quad |x| < 1.$$

- $d_F = 4$
- scaling functions
- continuum limit

[Le Gall, Micromont, Bouttier, Guillet, ...]

- other boundary conditions

$u_n =$ elliptic function of n $x =$ angle variable

FRACTAL DIMENSION

$$\frac{R_n | g^N}{R_0 | g^N} = \# \{ \text{vertices in a ball of radius } n \text{ in maps w/ } N \text{ vertices} \}$$

$$\sim \frac{3}{56} n^4 \quad (N \rightarrow \infty) \quad d_F = 4$$

Proof: $R_n = R \frac{1-x^{n+1}}{1-x^{n+2}} \frac{1-x^{n+4}}{1-x^{n+3}}$

set $v = gR \Rightarrow R = \frac{1}{1-3v}$; $x = \frac{1-4v - \sqrt{(1-2v)(1-6v)}}{2v}$.

$$R_n | g^N = \oint \frac{dg}{2\pi i g^{N+1}} R_n(g) \quad \text{use variable } v, \text{ do saddle pt}$$

around critical value $v_c = \frac{1}{6}$.

$$v = v_c (1 + i \sqrt{\frac{1}{N}})$$

SCALING

near critical maps (4 valent case) $g = g_c (1 - \varepsilon^4)$ $g_c = \frac{1}{12}$

$$\Rightarrow \begin{cases} R = 2(1 - \varepsilon^2) \\ X = 1 - \sqrt{6} \varepsilon \end{cases} \quad X^n \sim e^{-n\sqrt{6}\varepsilon}$$

has a nice limit if $r = n\varepsilon$ finite (= renormalized geodesic distance). ($d_F = 4$)

$\mathcal{F}(r)$ = partition function for continuous random surfaces with 2 marked pts at distance $\geq r$

$$= \lim_{\varepsilon \rightarrow 0} \frac{R - R_n}{\varepsilon^2 R} = \frac{3}{\sinh^2(\sqrt{\frac{3}{2}} r)}$$

$$\mathcal{G}(r) = \text{idem at distance } r = -\frac{\partial \mathcal{F}}{\partial r} = 3\sqrt{6} \frac{\cosh(\sqrt{\frac{3}{2}} r)}{\sinh(\sqrt{\frac{3}{2}} r)^3}$$

- Alternatively, start from: $R_n = 1 + g R_n (R_{n+1} + R_n + R_{n-1})$
 and insert $R_n = R (1 - \varepsilon^2 \bar{F}(r = n\varepsilon))$, expand in ε .
 order 4 in ε :
 $(= R_c (1 - \varepsilon^2 (1 + \bar{F})))$

$$\bar{F}''(r) - 3 \bar{F}(r)^2 - 6 \bar{F}(r) = 0$$

$$\lim_{r \rightarrow 0} \bar{F}(r) = \infty$$

$$\lim_{r \rightarrow \infty} \bar{F}(r) = 0$$

- set $u = 1 + \bar{F}$ then

$$u^2 - \frac{u''}{3} = 1$$

(Painlevé I)

SCALING

- $R_n/g^N \sim \frac{12^N}{N^{3/2}} \int_0^{\infty} d\zeta \zeta^2 e^{-\zeta^2} \mathcal{U}(r\sqrt{-i\zeta})$

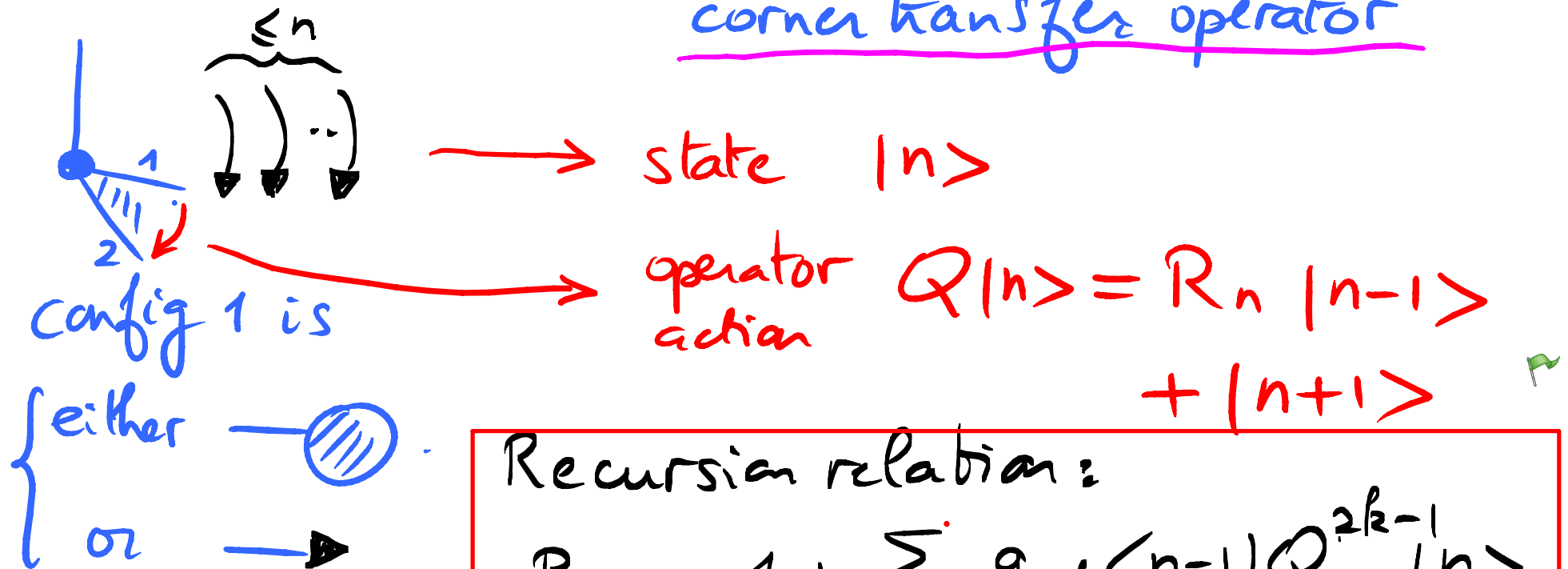
$$\Rightarrow P(r) = \int_0^{\infty} d\zeta \zeta^2 e^{-\zeta^2} \mathcal{U}(r\sqrt{-i\zeta})$$

is the probability that $\text{dist} \geq r$ in random surfaces with 2 marked points.

Generalization to arbitrary even valences

Describe the local environment of a vertex in a blossom-tree via :

corner transfer operator



Recursion relation:

$$R_n = 1 + \sum_{k \geq 1} g_{2k} \langle n-1 | Q^{2k-1} | n \rangle$$

PATH INTERPRETATION OF Q :

$$Q|n\rangle = |n+1\rangle + R_n|n-1\rangle$$

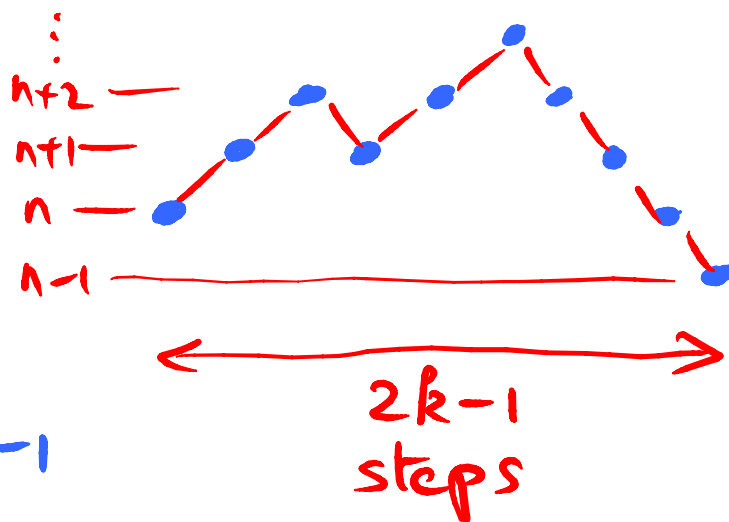
$$\langle n| \text{ dual basis} : \langle m|p\rangle = \delta_{m,p}$$

$$\langle n-1| Q^{2k-1} |n\rangle$$

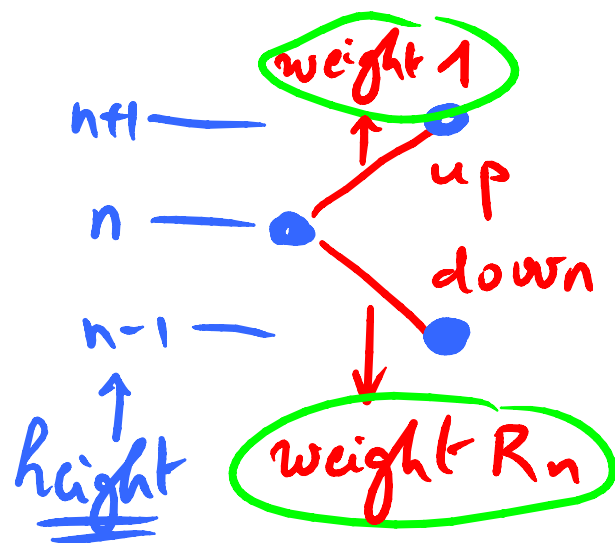
$$= \sum$$

paths

$n \rightarrow n-1$

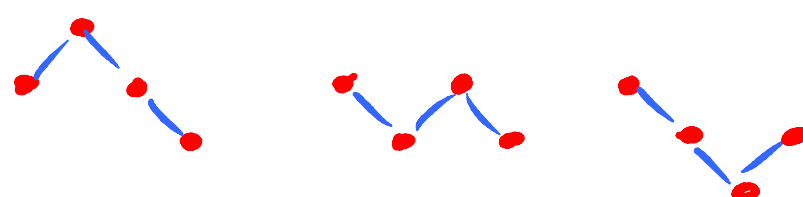


$$= \text{Pol}(R_n, R_{n\pm 1}, R_{n\pm 2}, \dots)$$



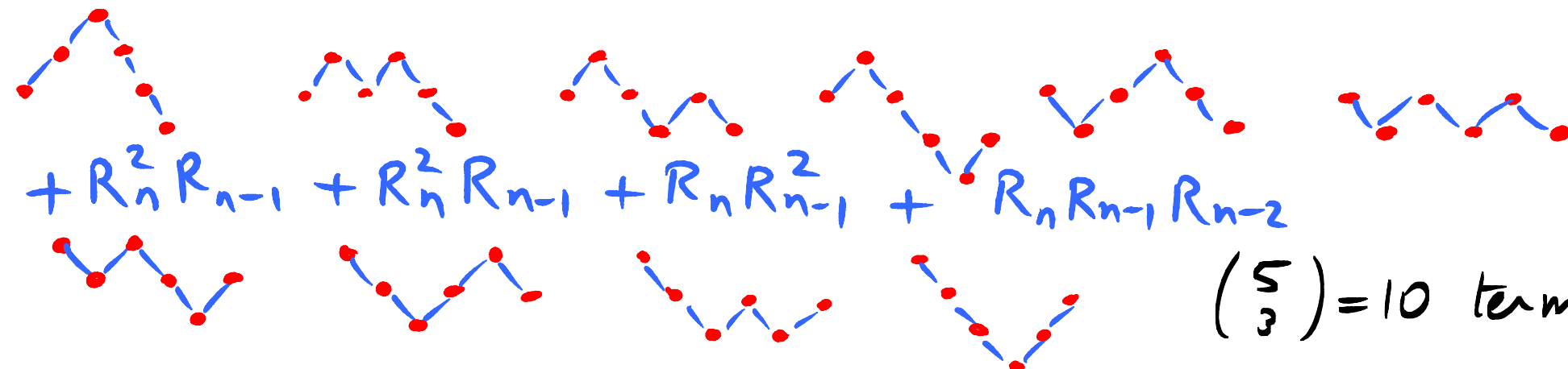
Examples

$k=2$ (4 valent) : $\langle n-1 | Q^3 | n \rangle = R_{n+1} R_n + R_n^2 + R_n R_{n-1}$



$k=3$ (6 valent) : $\langle n-1 | Q^5 | n \rangle =$

$$R_{n+2} R_{n+1} R_n + R_n^2 R_{n-1} + R_n R_{n-1}^2 + R_{n+1} R_n R_{n-1} + R_n^2 R_{n+1} + R_n^3$$



$$+ R_n^2 R_{n-1} + R_n^2 R_{n-1} + R_n R_{n-1}^2 + R_n R_{n-1} R_{n-2}$$

$\binom{5}{3} = 10$ terms

$$R_n = 1 + \sum_{m \geq k \geq 2} g_{2k} \langle n-1 | Q^{2k-1} | n \rangle$$

(THM) This is integrable

$$Q|n\rangle = |n+1\rangle + R_n |n-1\rangle$$

$$\langle m | n \rangle = \delta_{m,n}$$

(THM) This has the following exact solution
(m) - soliton solution

$$R_n = R \frac{u_n u_{n+3}}{u_{n+1} u_{n+2}}$$

$$u_n = \det_{1 \leq i, j \leq n} \left(\delta_{ij} - \lambda_i x_j^{n+2i-2} \right)$$

discrete
[n soliton
τ-function]

$$|X_i| < 1; (x_1, x_2, \dots, x_m, \frac{1}{x_1}, \frac{1}{x_2}, \dots, \frac{1}{x_m})$$

= roots of the linearized characteristic equation (obtained by expanding

the equation for $R_h = R(1 - \lambda x^n)$ at first order in λ).

- Access to fractal properties by taking a critical continuum limit
- Scaling 2pt function obeys Painlevé eqn.

MULTICRITICAL 2pt FUNCTION

- fine-tune γ_{ik} to get to $\gamma_{sk} = -\frac{1}{m+1}$
- m of the x_i tend to 1 in the multicritical limit

$$F(r) = -2 \frac{\partial^2}{\partial r^2} \log W(\{\sinh(a_i r)\}_{1 \leq i \leq m})$$

↑
Wronskian

↑
determined by γ_{ik} .

$$G(r) = -F'(r) \quad \left(\begin{array}{l} \text{2pt function of} \\ \text{multicritical random surface} \\ \text{with 2 marked pts at dist } r \end{array} \right)$$

obeys higher diff^{al} eqn of the KdV hierarchy

- Differential equation for $u = 1 + \bar{x}$

$$R_{m+1}[u] = R_{m+1}[1] \quad \text{where}$$

$$\text{Res}\left((d^2 - u)^{m+\frac{1}{2}}\right) = R_{m+1}[u].$$

Ex: • $u'' - 3u^2 = -3 \quad (m=1)$

• $u^{(4)} - 10u u'' - 5(u')^2 + 10u^3 = 10 \quad (m=2)$

Obtained by writing $[P, Q] = 0$

$$Q = d^2 - u; \quad P = d^{2m+1} + a_1 d^{2m-1} + \dots + a_{2m}$$

INTEGRABLE BRANCHING PROCESSES

Re-consider the eqn : $R_n = 1 + g R_n (R_{n+1} + R_n + R_{n-1})$
 and write it as $R_n = \frac{1}{1 - g(R_{n+1} + R_n + R_{n-1})}$

- R_n = g.f of vertex-labeled trees with root label n
 and the "sicilian mama" rule = children stay close enough
 $\text{child}(n) \in \{n, n \pm 1\}$. $n = \text{position/embedding}$
- $R_{-1} = 0$; $\lim_{n \rightarrow \infty} R_n = R$

Branching process : $\begin{cases} (1) \text{ arbitrary planar tree} \\ (2) \text{ embedding in } \mathbb{Z} \end{cases}$

- genealogical tree
- $\text{proba}(k \text{ children}) = (1-p) p^k \quad k=0,1,2,\dots$
- diffusion of position: proba $\frac{1}{3}$ of $(n+1, n \text{ or } n-1)$ if parent at n
- $E_n(t)$ = Proba of extinction of a family in time $\leq t$, w top at n

$$E_n(t) = \frac{1-p}{1 - \frac{p}{3}(E_{n+1}(t-1) + E_n(t-1) + E_{n-1}(t-1))} ; E_n(0)=0$$

$$R_n = \frac{E_n(\infty)}{1-p} \rightarrow \text{proba of extinction of a family sitting at } n \text{ at the first generation, and staying at } n \geq 0 \text{ (presence of a wall).}$$

$$g = \frac{p(1-p)}{3}$$

• Note: $S_n = 1 - (1-p)R_n$ = probability of survival

• g_c is attained at $p = p_c = \frac{1}{2}$ (GW tree).

• 1 wall $n \geq 0 \Rightarrow$

$$\left\{ \begin{array}{l} S_n = 1 - \frac{1 - |2p-1|}{2p} \frac{1-x^{n+1}}{1-x^{n+2}} \frac{1-x^{n+4}}{1-x^{n+3}} \\ x = \frac{1 - \sqrt{|2p-1|}}{1 + \sqrt{|2p-1|}} \end{array} \right.$$

(EXACT SOLUTION)

no wall:

$n \rightarrow \infty \quad S_n \rightarrow S(p) = \frac{1-2p - |1-2p|}{2p} = 0$ if $p < \frac{1}{2}$ or average # of children $= 1 = p/(1-p)$
 (GW tree almost surely finite). $2p$

ESCAPE FROM A SEGMENT

- Solutions of $R_n = 1 + g R_n (R_{n+1} + R_n + R_{n-1})$
with $R_{-1} = 0$ & $R_{L+1} = 0$; $S_n = 1 - (1-p) R_n$.

Elliptic exact solution:

$$R_n = R \frac{u_{n+1}}{u_{n+2}} \frac{u_{n+4}}{u_{n+3}}$$

$$u_n = \Theta_1(n\alpha) \quad (\text{Jacobi theta fn})$$

$$x = e^{2i\pi\alpha}$$

$$\alpha = \frac{1}{L+5}$$

$$\begin{cases} R = 4 \frac{\Theta_1(\alpha)\Theta_1(2\alpha)}{\Theta_1'(0)\Theta_1(3\alpha)} \left(\frac{\Theta_1'(\alpha)}{\Theta_1'(0)} - \frac{1}{2} \frac{\Theta_1'(2\alpha)}{\Theta_1'(0)} \right) \\ g = \frac{\Theta_1'(0)\Theta_1(3\alpha)}{16\Theta_1(\alpha)^2\Theta_1(2\alpha)} \left(\frac{\Theta_1'(\alpha)}{\Theta_1'(0)} - \frac{1}{2} \frac{\Theta_1'(2\alpha)}{\Theta_1'(0)} \right)^2 \end{cases}$$

$$R_n \sim 2(1 - \varepsilon^2 \mathcal{U}(r))$$

BOUNDED MAPS ...

$$\mathcal{U}(r) = 2 \mathcal{P}\left(\sqrt{\frac{3}{2}} r\right) \quad (\text{Weierstrass}).$$

CONCLUSION

2D Quantum Gravity
(Decorated random surfaces)

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2D Quantum Gravity
(Decorated random surfaces)



Exact solution
(matrix model)

CONCLUSION

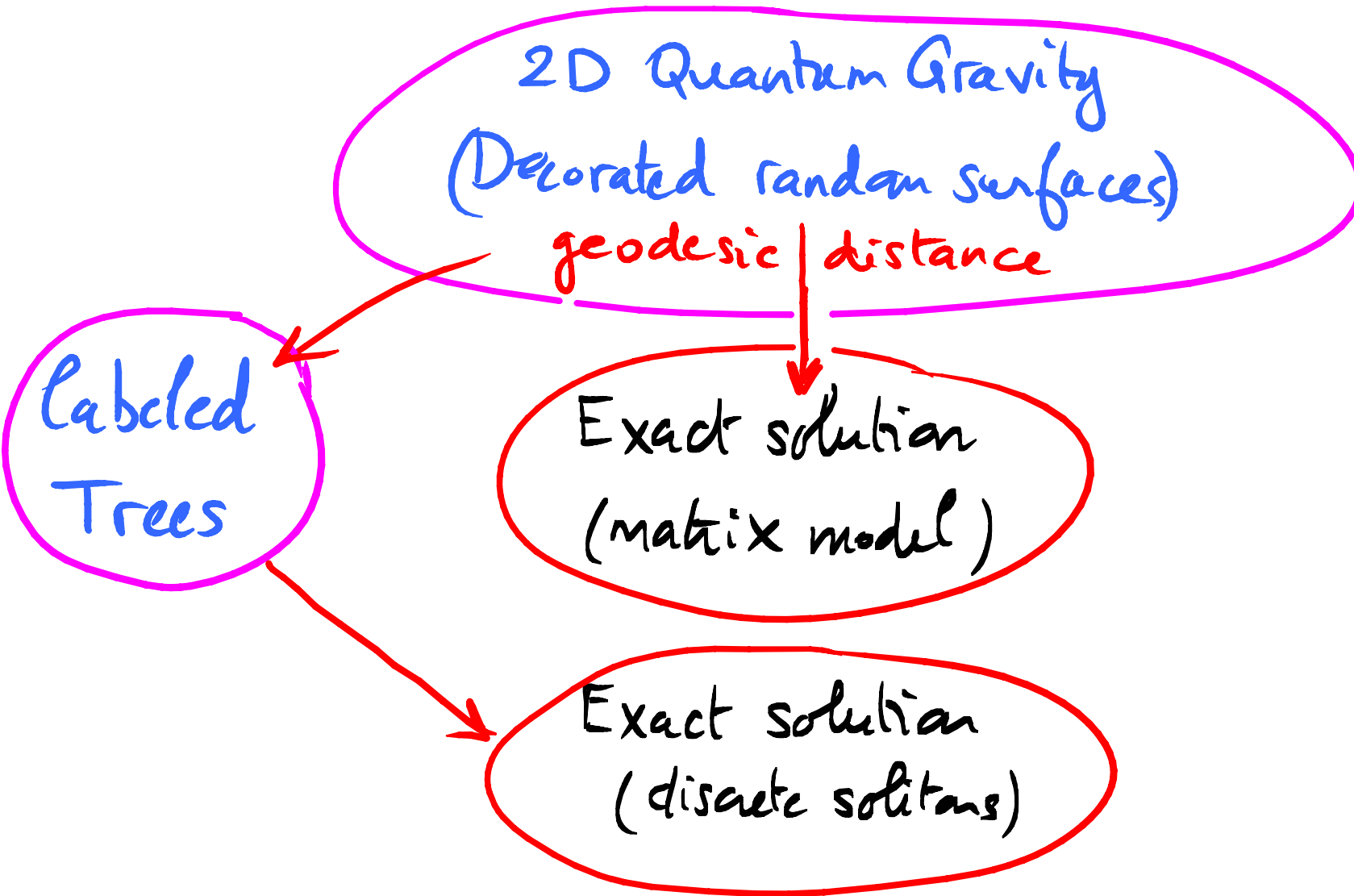
2D Quantum Gravity
(Decorated random surfaces)

geodesic distance

Labeled
Trees

Exact solution
(matrix model)

Exact solution
(discrete solitons)



CONCLUSION

2D Quantum Gravity
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Branching
Processes

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2D Quantum Gravity
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geodesic distance

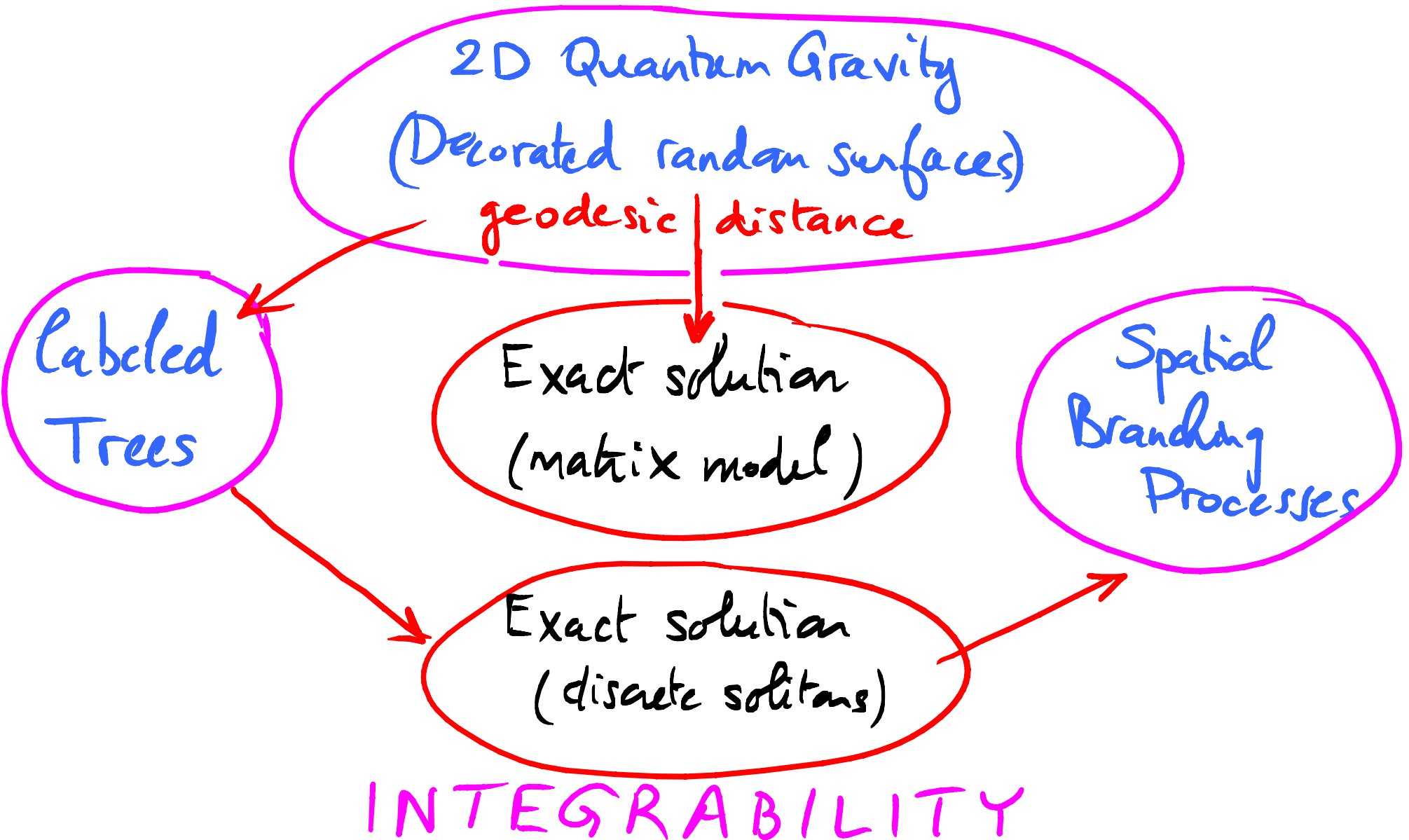
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INTEGRABILITY



Question:

Relation between geodesic distance
and genus expansion via orthogonal polynomials
almost same equation

Trees $[P, Q] = 0$ $Q = d^2 U(r)$
 $\uparrow$ geodesic distance

Matrix Model $[P, Q] = 1$ $Q = d^2 - u(x)$
↖ topological expⁿ parameter