GEOMETRY OF THE CORNER GROWTH MODEL

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Summary. Qualitative and quantitative results on the geodesics, Busemann functions, and competition interfaces of the explicitly solvable corner growth model through the joint distribution of Busemann functions.

Collaborators: Louis Fan (Indiana), Firas Rassoul-Agha and Chris Janjigian (Utah).
Corner growth model with exponential distribution

To each $x \in \mathbb{Z}^2$ attach random weight $\omega_x$.

$\omega_x \sim \text{Exp}(1): \mathbb{P}(\omega_x \geq t) = e^{-t}$ for $t \geq 0$.

IID random medium $\omega = (\omega_x : x \in \mathbb{Z}^2)$.

Weight of an up-right path $\gamma$ is

$$W(\gamma) = \sum_{x \in \gamma} \omega_x$$

Point-to-point last-passage percolation:

$$G(u, v) = \max_{\gamma : u \rightarrow v} \sum_{x \in \gamma} \omega_x \quad \text{for } u \leq v \text{ in } \mathbb{Z}^2$$

A maximizing path is called a geodesic.
Corner growth model with exponential distribution: limit shape

**Theorem**  We have this law of large numbers:

\[
\lim_{n \to \infty} n^{-1} G(0, [n\xi]) = g(\xi) \quad \text{a.s.} \quad \forall \xi \in \mathbb{R}_+^2
\]

with explicit *shape function*

\[
g(\xi) = \left( \sqrt{\xi_1} + \sqrt{\xi_2} \right)^2.
\]

[Rost 1981, several authors in the 1990’s]
The scaled growing cluster $t^{-1}\{(m, n) : G(0, (m, n)) \leq t\}$ at times $t = 100$ and $t = 400$.

The curve $\sqrt{x} + \sqrt{y} = 1$ (level curve of the shape function) is the boundary of the limit shape.

[Simulations: Firas Rassoul-Agha, Elnur Emrah]
Methods for studying exponential corner growth model

- Coupling with TASEP (totally asymmetric simple exclusion process).

- Methods of integrable probability: combinatorics (versions of RSK), determinantal structures, Fredholm determinants, asymptotic analysis.

- Tractable stationary version.

Focus of the talk: a natural coupling of all the stationary CGMs and some consequences for the geometry of the CGM.

Let’s first go over the stationary CGM.
Last-passage percolation with stationary increments

∀ x ∈ ℕ⁺ attach weight ωₓ ∼ Exp(1).

Let 0 < α < 1.

Edge weights: 

\[ I_{ie_1}^\alpha \sim \text{Exp}(\alpha) \]
\[ J_{je_2}^\alpha \sim \text{Exp}(1 - \alpha) \]

Define last-passage percolation \( G^\alpha \) by maximizing over paths that use both boundary weights and interior weights:

\[
G_{(0,0),(m,n)}^\alpha = \max_{1 \leq k \leq m} \left\{ \sum_{i=1}^{k} I_{ie_1}^\alpha + G_{(k,1),(m,n)} \right\} \lor \max_{1 \leq \ell \leq n} \left\{ \sum_{j=1}^{\ell} J_{je_2}^\alpha + G_{(1,\ell),(m,n)} \right\}
\]
Here again the LPP process with boundary weights and interior weights:

\[
G^\alpha(0,0),(m,n) = \max_{1 \leq k \leq m} \left\{ \sum_{i=1}^{k} l^\alpha_{ie_1} + G(k,1),(m,n) \right\} \vee \max_{1 \leq \ell \leq n} \left\{ \sum_{j=1}^{\ell} J^\alpha_{je_2} + G(1,\ell),(m,n) \right\}
\]

Stationary increments: \( \forall x \left\{ l^\alpha_x = G^\alpha_0, x - G^\alpha_0, x-e_1 \sim \text{Exp}(\alpha) \right\} \)

\( J^\alpha_x = G^\alpha_0, x - G^\alpha_0, x-e_2 \sim \text{Exp}(1-\alpha) \)

Shape function immediate: \( \lim_{N \to \infty} N^{-1} G^\alpha(0,0),(N_s,N_t) = \frac{s}{\alpha} + \frac{t}{1-\alpha} \equiv g^\alpha(s, t) \).

Next solve for the shape function \( g \) that comes from the i.i.d. weights.

Rewrite the coupling with the scaling and take the limit:

\[
G^\alpha(0,0),(N_s,N_t) = \max_{0 \leq a \leq s} \left\{ \sum_{i=1}^{Na} l^\alpha_{ie_1} + G(Na,1),(N_s,N_t) \right\} \vee \max_{1 \leq b \leq t} \left\{ \sum_{j=1}^{Nb} J^\alpha_{je_2} + G(1,Nb),(N_s,N_t) \right\}
\]
\[ G_{a0, (Ns, Nt)} = \max_{0 \leq a \leq s} \left\{ \sum_{i=1}^{Na} J_{ie_1} + G_{(Na, 1), (Ns, Nt)} \right\} \vee \max_{1 \leq b \leq t} \left\{ \sum_{j=1}^{Nb} J_{je_2} + G_{(1, Nb), (Ns, Nt)} \right\} \]

Let \( N \to \infty \). Write \( \xi = (s, t) \) and \( \eta = (a, 0) \) or \((0, b)\).

\[ g^\alpha(\xi) = \sup_{\eta \in \text{boundary}} \{ g^\alpha(\eta) + g(\xi - \eta) \} \]

From this \( g(\xi) = g^\alpha(\xi) \) for the unique \( \alpha = \alpha(\xi) \) such that the geodesic for the increment-stationary LPP process \( G_{a, N\xi} \) spends \( o(N) \) time on the boundary.

This specifies a one-to-one correspondence between a direction vector \( \xi = (\xi_1, 1 - \xi_1) \in (e_2, e_1) \) and a parameter \( \alpha \in (0, 1) \):

\[ \alpha(\xi) = \frac{\sqrt{\xi_1}}{\sqrt{\xi_1} + \sqrt{1 - \xi_1}} \in (0, 1) \]
Remark in passing: What is the obstacle to generalization to other i.i.d. weight distributions to find explicit limit shapes?

Their stationary last-passage percolation processes exist but not sufficiently understood.
**NEXT STEP:** Look for a natural coupling of the entire family of stationary LPP processes \( \{ G^\alpha : 0 < \alpha < 1 \} \).

**WHY?** Parameter \( \alpha \) associated with directions in the quadrant, and (as we shall see) directions are associated with geodesics. Only a joint distribution can reveal path-level properties such as singularities.

**HOW?** Let the LPP process itself produce the coupling for us. This leads us to Busemann functions.

Like a Markov chain produces its invariant distribution by passing to a limit, the LPP process produces its stationary versions by going to a limit in different spatial directions.
Busemann function

Busemann function in direction $\xi \in (e_2, e_1)$ is defined by

$$B^\xi(x, y) = \lim_{n \to \infty} [G(x, v_n) - G(y, v_n)]$$

for a sequence $v_n \to \infty$ s.t. $v_n/n \to \xi$.

For a given $\xi$ this can be proved, almost surely, simultaneously for all sequences $v_n/n \to \xi$.


$\{B^\xi(x, y) : x, y \in \mathbb{Z}^2\}$ is a stationary cocycle with marginals

$$B^\xi(x, x + e_1) \sim \text{Exp}(\alpha) \quad \text{and} \quad B^\xi(x, x + e_2) \sim \text{Exp}(1 - \alpha)$$

where

$$\alpha = \alpha(\xi) = \frac{\sqrt{\xi_1}}{\sqrt{\xi_1} + \sqrt{1 - \xi_1}}.$$
Busemann function as a stationary LPP process

Fix any down-right path $\mathcal{Y} = \{y_k\}_{k \in \mathbb{Z}}$ on $\mathbb{Z}^2$. (Means: $y_k - y_{k-1} \in \{e_1, -e_2\}$.)

Then for points $x$ below and left of $\mathcal{Y}$,

$$B(x, y_0) = \sup_{\pi: x \to y \in \mathcal{Y}} \left\{ \sum_{x \in \pi \setminus \{y\}} \omega_x + B(y, y_0) \right\}$$

where supremum over up-right paths $\pi$ from $x$ to the boundary $\mathcal{Y}$.

**Proof:** \[ B(x, y_0) = \lim_{n \to \infty} [G(x, v_n) - G(y_0, v_n)] \]

\[ = \lim_{n \to \infty} \left[ \omega_x + G(x + e_1, v_n) \vee G(x + e_2, v_n) - G(y_0, v_n) \right] \]

\[ = \omega_x + B(x + e_1, y_0) \vee B(x + e_2, y_0). \]
Busemann process $\{B^{\zeta^\pm} : \zeta \in (e_2, e_1)\}$

For a dense countable set of directions $\xi$ the almost sure limits

$$B^{\xi}(x, y) = \lim_{v_n/n \to \xi} [G(x, v_n) - G(y, v_n)]$$

define the Busemann functions.

Left and right limits $\xi \to \zeta^\pm$ (directions ordered from $e_2$ to $e_1$)

$$B^{\zeta^+}(x, y) = \lim_{\xi \searrow \zeta} B^{\xi}(x, y) \quad \text{and} \quad B^{\zeta^-}(x, y) = \lim_{\xi \nearrow \zeta} B^{\xi}(x, y)$$

define a process $\{B^{\zeta^\pm}\}$ indexed by the full set of directions $\zeta$.

The limits come from monotonicity (a planar feature).

For a fixed $\zeta$, with probability 1, $B^{\zeta^+} = B^{\zeta^-}$ and

$$B^{\zeta}(x, y) = \lim_{v_n/n \to \zeta} [G(x, v_n) - G(y, v_n)]$$
From Busemann functions to semi-infinite geodesics

Recall: a (finite) geodesic is the (almost surely unique) maximizing path between two points.

A semi-infinite geodesic is an infinite up-right nearest-neighbor path \((x_k)_{k \geq 0}\) that is the geodesic between any two of its points:

\[
G(x_m, x_n) = \sum_{i=m}^{n} \omega_{x_i} \quad \forall \ m < n
\]

Semi-infinite geodesic \((x_k)_{k \geq 0}\) is \(\xi\)-directed if

\[
\lim_{n \to \infty} \frac{x_n}{n} = \xi
\]

Questions:

- Given \(x\) and \(\xi\), existence and uniqueness of \(\xi\)-directed semi-infinite geodesic from \(x\)?
- Given \(x\), \(y\) and \(\xi\), do the \(\xi\)-directed geodesics from \(x\) and \(y\) cross? Coalesce?
For a fixed $\xi$, the (almost sure) answers have been known for a while:

- $\forall x \exists \text{ unique } \xi$-directed semi-infinite geodesic $\pi^x, \xi$.

- Coalescence: $\forall x, y \in \mathbb{Z}^2 \exists z \in \mathbb{Z}^2 : \pi^x, \xi \cap \pi^y, \xi = \pi^z, \xi$.

(Think here of geodesics as collections of edges and points.)


These facts can also be derived from the Busemann functions and their properties. Take **existence** as an example:
Semi-infinite geodesics from local increments of Busemann functions

The unique semi-infinite geodesic $\pi = \pi^x,\xi$ from $x$ in direction $\xi$ can be defined by following minimal local increments of the $\xi$-Busemann function:

$$
\pi_0 = x
$$

and

$$
\pi_{k+1} = \begin{cases} 
\pi_k + e_1, & \text{if } B^\xi(\pi_k, \pi_k + e_1) \leq B^\xi(\pi_k, \pi_k + e_2) \\
\pi_k + e_2, & \text{if } B^\xi(\pi_k, \pi_k + e_2) < B^\xi(\pi_k, \pi_k + e_1).
\end{cases}
$$

"Proof"

$\pi_1 = x + e_2$ roughly iff $G(x + e_2, n\xi) > G(x + e_1, n\xi)$

$$
\iff G(x, n\xi) - G(x + e_2, n\xi) < G(x, n\xi) - G(x + e_1, n\xi)
$$

roughly iff $B^\xi(x, x + e_2) < B^\xi(x, x + e_1)$
Coalescing geodesics directed to $\xi = (\frac{2}{3}, \frac{1}{3})$

Blue paths = up-right $\xi = (\frac{2}{3}, \frac{1}{3})$-directed geodesics that cross the hyphened anti-diagonal segment. Picture shows the paths until coalescence.

[Simulation: Firas Rassoul-Agha]
Uniqueness fails in random directions!

\[(\varphi_n^x)_{n \geq 0} = \text{competition interface} \text{ from } x.\]

\[\exists \text{ almost sure random asymptotic direction:}\]

\[\xi^*(x) = (\xi_1^*, 1 - \xi_1^*) = \lim_{n \to \infty} \frac{\varphi_n^x}{n}\]

[Fierrari-Pimentel 2005]

From \(x\) \exists two distinct \(\xi^*(x)\)-directed semi-infinite geodesics. (One takes the initial \(e_1\) step, the other the \(e_2\) step.)

No point \(x\) is the source of three distinct semi-infinite geodesics in the same direction.

[Coupier 2011, Coupier-Heinrich 2012, with TASEP input from Amir-Angel-Valkó 2011]
Two distinct $\xi^*(x)$-directed semi-infinite geodesics on either side of the competition interface

Red competition interface, blue and green geodesics from 0 in direction $\xi^*(0)$.

Picture above is the initial 300-step part of the 5000-step picture below. The geodesics are the p2p geodesics from 0 to two points on either side of the competition interface. These p2p geodesics converge to the true semi-infinite things. [Simulations F. Rassoul-Agha]
Recall Busemann process \( \{ B^{\xi \pm} : \xi \in (e_2, e_1) \} \).

With this process we can define \( \forall x \in \mathbb{Z}^2 \) and \( \forall \xi \) and \( \pm \) a semi-infinite geodesic \( \pi^{x, \xi \pm} \) by following the minimal increments of \( B^{\xi \pm} \) and by breaking ties with \( e_1 \) for + and with \( e_2 \) for −.

We can characterize the simultaneous existence, uniqueness and coalescence of all geodesics.
Global geodesics picture

**Theorem** [Janjigian, Rassoul-Agha, S]

∃ countable random set $\mathcal{V}^{\omega} \subset (e_2, e_1)$ of directions, with the following properties, all with probability 1.

- For each direction $\xi \notin \mathcal{V}^{\omega}$ there is a unique geodesic from each lattice point. For a given $\xi$ these geodesics coalesce.

- For directions $\xi \in \mathcal{V}^{\omega}$, from each lattice point $x$ there are exactly two geodesics $\pi^{x,\xi^+}$ and $\pi^{x,\xi^-}$ in direction $\xi$ that eventually separate. Geodesics $\{\pi^{x,\xi^+} : x \in \mathbb{Z}^2\}$ form a coalescing tree, and geodesics $\{\pi^{x,\xi^-} : x \in \mathbb{Z}^2\}$ form a separate coalescing tree.

- $\mathcal{V}^{\omega} = \{\xi^*(x) : x \in \mathbb{Z}^2\}$, the collection of asymptotic directions of the competition interfaces at all $x$, at a fixed $\omega$.

- $\mathcal{V}^{\omega} = \{\xi : \exists x, y \in \mathbb{Z}^2 : B^{\xi^+}(x, y) \neq B^{\xi^-}(x, y)\}$, the set of discontinuities of Busemann functions.

- There are no other semi-infinite geodesics except the trivial ones $x + k e_i$, $k \geq 0$. 

Global geodesics picture

Proof of the global geodesics theorem comes from a combination of earlier facts with properties of the Busemann process \( \{ B^{\xi_{\pm}} : \xi \in (e_2, e_1) \} \).

First the distribution of the process \( \xi \mapsto B^{\xi_{\pm}}(x, x + e_1) \) on a fixed horizontal edge \((x, x + e_1)\).
Busemann process on an edge, indexed by directions $\xi$

Parametrize directions $\xi = (\xi_1, 1 - \xi_1) \in (e_2, e_1)$ with $\alpha \in (0, 1)$:

$$\alpha(\xi) = \frac{\sqrt{\xi_1}}{\sqrt{\xi_1} + \sqrt{1 - \xi_1}}$$

Define a marked point process $X$ on $(0, 1]$:

- On $(0, 1)$, $N$ = Poisson point process with intensity measure $r^{-1} \, dr$, and $N\{1\} = 1$.
- To each point $r \in N$ attach an independent variable $Z_r \sim \text{Exp}(r)$.
- Set $X(\alpha) = \sum_{r \in N(\alpha,1]} Z_r$ for $0 < \alpha \leq 1$. $X(\alpha) \sim \text{Exp}(\alpha)$.

**Theorem** [Fan-S]

On a fixed horizontal edge $(x, x + e_1)$,

$$\{B^{\xi^+}(x, x + e_1) : \xi \in (e_2, e_1)\} \overset{d}{=} \{X(\alpha(\xi)) : \xi \in (e_2, e_1)\}.$$
Geometric significance of jumps of $\xi \mapsto B_{x,x+e_1}^{\xi\pm}$

Recall: for a countable dense set of directions $\zeta$, geodesics $\{\pi^{x,\zeta}\}_{x \in \mathbb{Z}^2}$ coalesce a.s.

If $z^{\zeta}(x, y) =$ coalescence point of geodesics $\pi^{x,\zeta}$ and $\pi^{y,\zeta}$ then $B^{\zeta}(x, y) = G(x, z^{\zeta}(x, y)) - G(y, z^{\zeta}(x, y))$.

If $B^{\zeta}(x, y)$ is constant for $\zeta \in (\eta', \eta'')$ $\zeta \in (\xi, \eta)$ then $z^{\zeta}(x, y)$ cannot jump.

Let $\zeta \searrow \xi$. Geodesics converge: $\pi^{x,\zeta} \to \pi^{x,\xi^+}$, $\pi^{y,\zeta} \to \pi^{y,\xi^+}$ and $B^{\xi^+}(x, y) = G(x, z^{\xi^+}(x, y)) - G(y, z^{\xi^+}(x, y))$.

We conclude that $\xi^+$ geodesics coalesce, as claimed in the global geodesics theorem.

Furthermore, the coalescence point $\xi \mapsto z^{\xi\pm}(x, x + e_1)$ jumps at the locations of an inhomogeneous Poisson process.
Distribution of increments on the $x$-axis of the lattice

Let $\zeta, \eta \in (e_2, e_1)$ again satisfy $\zeta_1 < \eta_1$.

$$\Delta_k = B^\zeta(ke_1, (k+1)e_1) - B^\eta(ke_1, (k+1)e_1) \geq 0$$

Distribution of process $\{\Delta_k\}_{k \in \mathbb{Z}}$?

Define 2-sided RW

$$S_n = \begin{cases} 
\sum_{i=1}^{n} Y_i, & n > 0 \\
0, & n = 0 \\
-\sum_{i=n+1}^{0} Y_i, & n < 0.
\end{cases}$$

with steps $Y_i \sim \text{Exp}(\alpha(\zeta)) - \text{Exp}(\alpha(\eta))$. $E(Y_i) > 0$.

**Theorem**

$$\{\Delta_k\}_{k \in \mathbb{Z}} \overset{d}{=} \{\left(\inf_{m>k} S_m - S_k\right)^+\}_{k \in \mathbb{Z}}$$
Finding the joint distribution of \( \{ B^\xi : \xi \in (e_2, e_1) \} \)

\( \forall \) level \( t \in \mathbb{Z} \) define bi-infinite sequences

\[ \bar{\omega}_t = (\omega(k,t))_{k \in \mathbb{Z}} \quad \text{and} \quad \bar{B}^{\xi,e_1}_t = (B^{\xi}_{(k,t),(k+1,t)})_{k \in \mathbb{Z}} \]

\( \exists \) mapping \( D \) from a subset of \( \mathbb{R}^\mathbb{Z}_+ \times \mathbb{R}^\mathbb{Z}_+ \) into \( \mathbb{R}^\mathbb{Z}_+ \) such that

\[ \bar{B}^{\xi,e_1}_t = D(\bar{B}^{\xi,e_1}_{t+1}, \bar{\omega}_t) \quad \forall \xi \text{ and } t \in \mathbb{Z}. \]

**Definition** of \( \tilde{I} = D(I, \omega) \): with \( G \) satisfying \( I_k = G_k - G_{k+1} \), let

\[ \tilde{G}_k = \sup_{m: m \geq k} \left\{ G_m + \sum_{i=k}^m \omega_i \right\}, \quad \tilde{I}_k = \tilde{G}_k - \tilde{G}_{k+1}. \]

\( \tilde{I} \) is the departure process of a FIFO queue with arrivals \( I \) and services \( \omega \) with time running right to left on \( \mathbb{Z} \).
Level-by-level evolution of the Busemann function

For $\xi_1, \ldots, \xi_n \in (e_2, e_1)$, the $n$-tuple of sequences evolves as a Markov chain backwards in the time parameter $t$ via the mapping

$$(\overline{B}_{t}^{\xi_1, e_1}, \ldots, \overline{B}_{t}^{\xi_n, e_1}) = \left(D\left(\overline{B}_{t+1}^{\xi_1, e_1}, \omega_t\right), \ldots, D\left(\overline{B}_{t+1}^{\xi_n, e_1}, \omega_t\right)\right)$$

**Theorem** [Fan-S] Given $(\rho_1, \ldots, \rho_n) \in (1, \infty)^n$, the Markov chain above has a unique invariant distribution ergodic under spatial translation and with mean

$$(EB^{\xi_1}_{(k,t), (k+1,t)}, \ldots, EB^{\xi_n}_{(k,t), (k+1,t)}) = (\rho_1, \ldots, \rho_n).$$
Description of the invariant distribution

Let \( D^{(n)}(\zeta^1, \zeta^2, \ldots, \zeta^n) \) = departure process from sending arrival process \( \zeta^1 \) successively through service processes \( \zeta^2, \ldots, \zeta^n \).

Let \( I^1, I^2, \ldots, I^n \) be independent sequences of i.i.d. exponentials with \( I^i_k \sim \text{Exp}(\alpha(\xi_i)) \). Define sequences \( \eta^1, \ldots, \eta^n \) by

\[
\eta^1 = I^1, \quad \eta^2 = D(I^2, I^1), \quad \eta^3 = D^{(3)}(I^3, I^2, I^1), \ldots, \\
\eta^n = D^{(n)}(I^n, I^{n-1}, \ldots, I^1).
\]

Well-defined if \( \xi_1 \cdot e_1 > \cdots > \xi_n \cdot e_1 \) so that \( \alpha(\xi_1) > \cdots > \alpha(\xi_n) \).

Then the invariant distribution is given by

\[
\left( \overline{B}^{\xi_1, e_1}_t, \ldots, \overline{B}^{\xi_n, e_1}_t \right) \overset{d}{=} \left( \eta^1, \ldots, \eta^n \right)
\]