GEOMETRY OF THE CORNER GROWTH MODEL

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Summary. Qualitative and quantitative results on the geodesics, Busemann functions, and competition interfaces of the explicitly solvable corner growth model through the **joint distribution of Busemann functions**.

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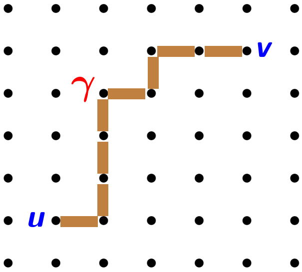
Corner growth model with exponential distribution



To each $x \in \mathbb{Z}^2$ attach random weight ω_x.

$\omega_x \sim \text{Exp}(1)$: $\mathbb{P}(\omega_x \geqslant t) = e^{-t}$ for $t \geqslant 0$.

IID random medium $\omega = (\omega_x : x \in \mathbb{Z}^2)$.

Weight of an up-right path γ is

$$W(\gamma) = \sum_{x \in \gamma} \omega_x$$

Point-to-point last-passage percolation:

$$G(u, v) = \max_{\gamma : u \to v} \sum_{x \in \gamma} \omega_x \qquad \text{for } u \leqslant v \text{ in } \mathbb{Z}^2$$

A maximizing path is called a **geodesic**.

Corner growth model with exponential distribution: limit shape

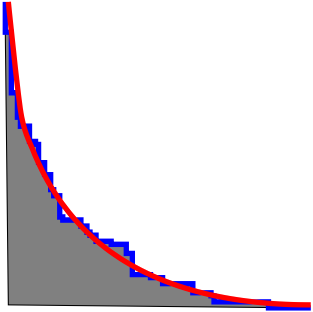
Theorem We have this law of large numbers:

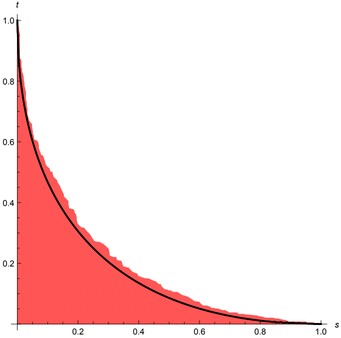
$$\lim_{n\to\infty} n^{-1} G(0, \lfloor n\boldsymbol{\xi} \rfloor) = g(\boldsymbol{\xi}) \qquad \text{a.s.} \qquad \forall\, \boldsymbol{\xi} \in \mathbb{R}_+^2$$

with explicit **shape function**

$$g(\boldsymbol{\xi}) = \left(\sqrt{\xi_1} + \sqrt{\xi_2}\,\right)^2.$$

[Rost 1981, several authors in the 1990's]





The scaled growing cluster $t^{-1}\{(m,n) : G(0,(m,n)) \leqslant t\}$ at times $t = 100$ and $t = 400$.

The curve $\sqrt{x} + \sqrt{y} = 1$ (level curve of the shape function) is the boundary of the limit shape.

[Simulations: Firas Rassoul-Agha, Elnur Emrah]

Methods for studying exponential corner growth model

• Coupling with TASEP (totally asymmetric simple exclusion process).

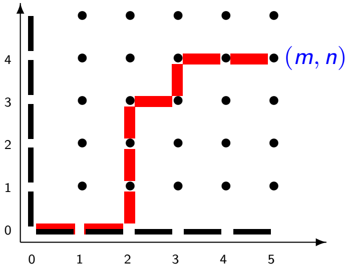
• Methods of integrable probability: combinatorics (versions of RSK), determinantal structures, Fredholm determinants, asymptotic analysis.

• Tractable stationary version.

Focus of the talk: a natural coupling of all the stationary CGMs and some consequences for the geometry of the CGM.

Let's first go over the stationary CGM.

Last-passage percolation with stationary increments



$\forall\ x \in \mathbb{N}_+^2$ attach weight $\omega_x \sim \mathrm{Exp}(1)$.

Let $0 < \boldsymbol{\alpha} < 1$.

Edge weights: $I^{\boldsymbol{\alpha}}_{ie_1} \sim \mathrm{Exp}(\boldsymbol{\alpha})$

$J^{\boldsymbol{\alpha}}_{je_2} \sim \mathrm{Exp}(1-\boldsymbol{\alpha})$

Define last-passage percolation $G^{\boldsymbol{\alpha}}$ by maximizing over paths that use both boundary weights and interior weights:

$$G^{\boldsymbol{\alpha}}_{(0,0),(m,n)} = \max_{1\leqslant k\leqslant m}\left\{\sum_{i=1}^{k} I^{\boldsymbol{\alpha}}_{ie_1} + G_{(k,1),(m,n)}\right\} \bigvee \max_{1\leqslant \ell\leqslant n}\left\{\sum_{j=1}^{\ell} J^{\boldsymbol{\alpha}}_{je_2} + G_{(1,\ell),(m,n)}\right\}$$

Benefit?

Here again the LPP process with boundary weights and interior weights:

$$G^{\alpha}_{(0,0),(m,n)} = \max_{1\leqslant k\leqslant m}\left\{\sum_{i=1}^{k} I^{\alpha}_{ie_1} + G_{(k,1),(m,n)}\right\} \bigvee \max_{1\leqslant \ell\leqslant n}\left\{\sum_{j=1}^{\ell} J^{\alpha}_{je_2} + G_{(1,\ell),(m,n)}\right\}$$

Stationary increments: $\forall x$ $\begin{cases} I^{\alpha}_x = G^{\alpha}_{0,x} - G^{\alpha}_{0,x-e_1} \sim \text{Exp}(\alpha) \\ J^{\alpha}_x = G^{\alpha}_{0,x} - G^{\alpha}_{0,x-e_2} \sim \text{Exp}(1-\alpha) \end{cases}$

Shape function immediate: $\lim_{N\to\infty} N^{-1} G^{\alpha}_{(0,0),(Ns,Nt)} = \dfrac{s}{\alpha} + \dfrac{t}{1-\alpha} \equiv g^{\alpha}(s,t).$

Next solve for the shape function g that comes from the i.i.d. weights.

Rewrite the coupling with the scaling and take the limit:

$$G^{\alpha}_{(0,0),(Ns,Nt)} = \max_{0\leqslant a\leqslant s}\left\{\sum_{i=1}^{Na} I^{\alpha}_{ie_1} + G_{(Na,1),(Ns,Nt)}\right\} \bigvee \max_{1\leqslant b\leqslant t}\left\{\sum_{j=1}^{Nb} J^{\alpha}_{je_2} + G_{(1,Nb),(Ns,Nt)}\right\}$$

$$G^{\boldsymbol{\alpha}}_{(0,0),(Ns,Nt)} = \max_{0 \leqslant a \leqslant s} \left\{ \sum_{i=1}^{Na} I^{\boldsymbol{\alpha}}_{ie_1} + G_{(Na,1),(Ns,Nt)} \right\} \bigvee \max_{1 \leqslant b \leqslant t} \left\{ \sum_{j=1}^{Nb} J^{\boldsymbol{\alpha}}_{je_2} + G_{(1,Nb),(Ns,Nt)} \right\}$$

Let $N \to \infty$. Write $\boldsymbol{\xi} = (s, t)$ and $\eta = (a, 0)$ or $(0, b)$.

$$g^{\boldsymbol{\alpha}}(\boldsymbol{\xi}) = \sup_{\eta \in \text{ boundary}} \{ g^{\boldsymbol{\alpha}}(\eta) + g(\boldsymbol{\xi} - \eta) \}$$

From this $g(\boldsymbol{\xi}) = g^{\boldsymbol{\alpha}}(\boldsymbol{\xi})$ for the unique $\boldsymbol{\alpha} = \boldsymbol{\alpha}(\boldsymbol{\xi})$ such that the geodesic for the increment-stationary LPP process $G^{\boldsymbol{\alpha}}_{0,N\boldsymbol{\xi}}$ spends $o(N)$ time on the boundary.

This specifies a one-to-one correspondence between a direction vector $\boldsymbol{\xi} = (\xi_1, 1 - \xi_1) \in (\mathbf{e}_2, \mathbf{e}_1)$ and a parameter $\boldsymbol{\alpha} \in (0, 1)$:

$$\boldsymbol{\alpha}(\boldsymbol{\xi}) = \frac{\sqrt{\xi_1}}{\sqrt{\xi_1} + \sqrt{1 - \xi_1}} \in (0, 1)$$

Remark in passing: What is the obstacle to generalization to other i.i.d. weight distributions to find explicit limit shapes?

Their stationary last-passage percolation processes exist but not sufficiently understood.

NEXT STEP: Look for a natural coupling of the entire family of stationary LPP processes $\{G^{\alpha} : 0 < \alpha < 1\}$.

WHY? Parameter α associated with directions in the quadrant, and (as we shall see) directions are associated with geodesics. Only a joint distribution can reveal path-level properties such as singularities.

HOW? Let the LPP process itself produce the coupling for us. This leads us to Busemann functions.

Like a Markov chain produces its invariant distribution by passing to a limit, the LPP process produces its stationary versions by going to a limit in different spatial directions.

Busemann function

Busemann function in direction $\boldsymbol{\xi} \in (\mathbf{e}_2, \mathbf{e}_1)$ is defined by

$$B^{\boldsymbol{\xi}}(x, y) = \lim_{n \to \infty} [G(x, v_n) - G(y, v_n)]$$

for a sequence $v_n \to \infty$ s.t. $v_n/n \to \boldsymbol{\xi}$.

For a given $\boldsymbol{\xi}$ this can be proved, almost surely, simultaneously for all sequences $v_n/n \to \boldsymbol{\xi}$.

Two proofs: (i) Techniques due to Newman et al. 1990s applied by Cator, P.A.Ferrari, Martin, Pimentel 2005–2012. (ii) More recent proof through coupling with stationary LPP processes.

$\{B^{\boldsymbol{\xi}}(x, y) : x, y \in \mathbb{Z}^2\}$ is a stationary cocycle with marginals

$$B^{\boldsymbol{\xi}}(x, x + \mathbf{e}_1) \sim \mathrm{Exp}(\boldsymbol{\alpha}) \quad \text{and} \quad B^{\boldsymbol{\xi}}(x, x + \mathbf{e}_2) \sim \mathrm{Exp}(1 - \boldsymbol{\alpha})$$

where

$$\boldsymbol{\alpha} = \boldsymbol{\alpha}(\boldsymbol{\xi}) = \frac{\sqrt{\xi_1}}{\sqrt{\xi_1} + \sqrt{1 - \xi_1}}.$$

Busemann function as a stationary LPP process

Fix any down-right path $\mathcal{Y} = \{y_k\}_{k\in\mathbb{Z}}$ on $\mathbb{Z}^2$. (Means: $y_k - y_{k-1} \in \{\mathbf{e}_1, -\mathbf{e}_2\}$.)

Then for points x below and left of $\mathcal{Y}$,

$$B(x, y_0) = \sup_{\pi:\, x\to y\,\in\,\mathcal{Y}} \Big\{ \sum_{x\,\in\,\pi\setminus\{y\}} \omega_x + B(y, y_0) \Big\}$$

where supremum over up-right paths π from x to the boundary $\mathcal{Y}$.

Proof:
$$\begin{aligned} B(x, y_0) &= \lim_{n\to\infty} \big[G(x, v_n) - G(y_0, v_n) \big] \\ &= \lim_{n\to\infty} \big[\omega_x + G(x+\mathbf{e}_1, v_n) \vee G(x+\mathbf{e}_2, v_n) - G(y_0, v_n) \big] \\ &= \omega_x + B(x+\mathbf{e}_1, y_0) \vee B(x+\mathbf{e}_2, y_0). \end{aligned}$$

Busemann process $\{B^{\zeta\pm} : \zeta \in (\mathbf{e}_2, \mathbf{e}_1)\}$

For a dense countable set of directions ξ the almost sure limits

$$B^{\xi}(x,y) = \lim_{v_n/n \to \xi} [G(x,v_n) - G(y,v_n)]$$

define the Busemann functions.

Left and right limits $\xi \to \zeta\pm$ (directions ordered from $\mathbf{e}_2$ to $\mathbf{e}_1$)

$$B^{\zeta+}(x,y) = \lim_{\xi \searrow \zeta} B^{\xi}(x,y) \quad \text{and} \quad B^{\zeta-}(x,y) = \lim_{\xi \nearrow \zeta} B^{\xi}(x,y)$$

define a process $\{B^{\zeta\pm}\}$ indexed by the full set of directions ζ.

The limits come from monotonicity (a planar feature).

For a **fixed** ζ, with probability 1, $B^{\zeta+} = B^{\zeta-}$ and

$$B^{\zeta}(x,y) = \lim_{v_n/n \to \zeta} [G(x,v_n) - G(y,v_n)]$$

From Busemann functions to semi-infinite geodesics

Recall: a (finite) geodesic is the (almost surely unique) maximizing path between two points.

A **semi-infinite geodesic** is an infinite up-right nearest-neighbor path $(x_k)_{k \geqslant 0}$ that is the geodesic between any two of its points:

$$G(x_m, x_n) = \sum_{i=m}^{n} \omega_{x_i} \quad \forall\, m < n$$

Semi-infinite geodesic $(x_k)_{k \geqslant 0}$ is $\boldsymbol{\xi}$**-directed** if $\quad \lim_{n \to \infty} \frac{x_n}{n} = \boldsymbol{\xi}$

Questions:

• Given x and $\boldsymbol{\xi}$, existence and uniqueness of $\boldsymbol{\xi}$-directed semi-infinite geodesic from x**?**

• Given x, y and $\boldsymbol{\xi}$, do the $\boldsymbol{\xi}$-directed geodesics from x and y cross**?** Coalesce**?**

From Busemann functions to semi-infinite geodesics

For a **fixed** $\boldsymbol{\xi}$, the (almost sure) answers have been known for a while:

- $\forall\, x\ \exists$ unique $\boldsymbol{\xi}$-directed semi-infinite geodesic $\pi^{\boldsymbol{x},\boldsymbol{\xi}}$.
- Coalescence: $\forall\, \boldsymbol{x}, \boldsymbol{y} \in \mathbb{Z}^2\ \exists\, \boldsymbol{z} \in \mathbb{Z}^2 : \pi^{\boldsymbol{x},\boldsymbol{\xi}} \cap \pi^{\boldsymbol{y},\boldsymbol{\xi}} = \pi^{\boldsymbol{z},\boldsymbol{\xi}}$.

(Think here of geodesics as collections of edges and points.)

[Newman et al. 1990s, P.A.Ferrari-Pimentel 2005, Coupier 2011]

These facts can also be derived from the Busemann functions and their properties. Take **existence** as an example:

Semi-infinite geodesics from local increments of Busemann functions

The unique semi-infinite geodesic $\pi = \pi^{x,\boldsymbol{\xi}}$ from x in direction $\boldsymbol{\xi}$ can be defined by following minimal local increments of the $\boldsymbol{\xi}$-Busemann function:

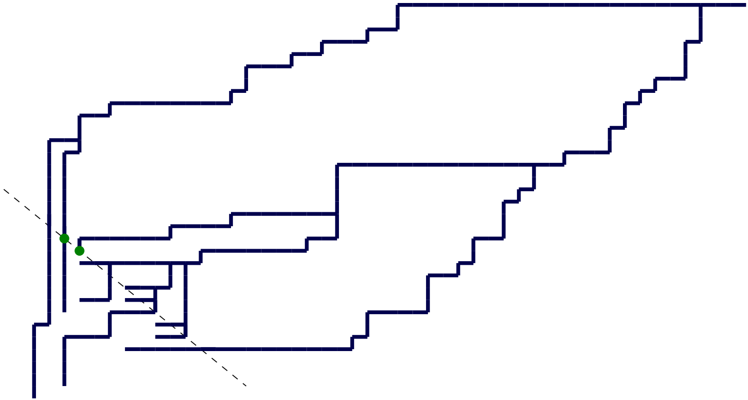
$$\pi_0 = x$$

and $$\pi_{k+1} = \begin{cases} \pi_k + \mathbf{e}_1, & \text{if} \;\; B^{\boldsymbol{\xi}}(\pi_k, \pi_k + \mathbf{e}_1) \leqslant B^{\boldsymbol{\xi}}(\pi_k, \pi_k + \mathbf{e}_2) \\ \pi_k + \mathbf{e}_2, & \text{if} \;\; B^{\boldsymbol{\xi}}(\pi_k, \pi_k + \mathbf{e}_2) < B^{\boldsymbol{\xi}}(\pi_k, \pi_k + \mathbf{e}_1). \end{cases}$$

"Proof"

$$\begin{aligned} \pi_1 = x + \mathbf{e}_2 \;\; &\text{roughly iff} \;\; G(x+\mathbf{e}_2, n\boldsymbol{\xi}) > G(x+\mathbf{e}_1, n\boldsymbol{\xi}) \\ &\iff G(x, n\boldsymbol{\xi}) - G(x+\mathbf{e}_2, n\boldsymbol{\xi}) < G(x, n\boldsymbol{\xi}) - G(x+\mathbf{e}_1, n\boldsymbol{\xi}) \\ &\text{roughly iff} \;\; B^{\boldsymbol{\xi}}(x, x+\mathbf{e}_2) < B^{\boldsymbol{\xi}}(x, x+\mathbf{e}_1) \end{aligned}$$

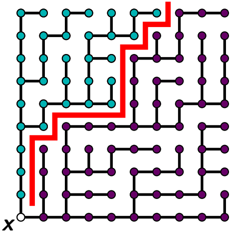
Coalescing geodesics directed to $\boldsymbol{\xi} = (\frac{2}{3}, \frac{1}{3})$



Blue paths $=$ up-right $\boldsymbol{\xi} = (\frac{2}{3}, \frac{1}{3})$-directed geodesics that cross the hyphened anti-diagonal segment. Picture shows the paths until coalescence.

[Simulation: Firas Rassoul-Agha]

Uniqueness fails in random directions!



$(\varphi_n^x)_{n\geqslant 0}$ = **competition interface** from x.

$\exists$ almost sure **random** asymptotic direction:

$$\boldsymbol{\xi}^*(x) = (\xi_1^*, 1-\xi_1^*) = \lim_{n\to\infty} \frac{\varphi_n^x}{n}$$

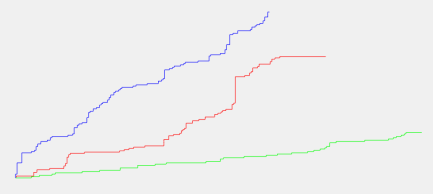
[Ferrari-Pimentel 2005]

From x $\exists$ **two** distinct $\boldsymbol{\xi}^*(x)$-directed semi-infinite geodesics. (One takes the initial $\mathbf{e}_1$ step, the other the $\mathbf{e}_2$ step.)

No point x is the source of **three** distinct semi-infinite geodesics in the same direction.

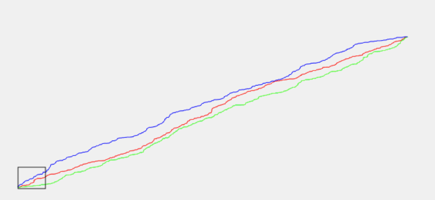
[Coupier 2011, Coupier-Heinrich 2012, with TASEP input from Amir-Angel-Valkó 2011]

Two distinct $\xi^*(x)$-directed semi-infinite geodesics on either side of the competition interface



Red competition interface, blue and green geodesics from 0 in direction $\xi^*(0)$.

Picture above is the initial 300-step part of the 5000-step picture below. The geodesics are the p2p geodesics from 0 to two points on either side of the competition interface. These p2p geodesics converge to the true semi-infinite things. [Simulations F. Rassoul-Agha]



Unifying the geodesics picture

Recall Busemann process $\{B^{\boldsymbol{\xi}\pm} : \boldsymbol{\xi} \in (\mathbf{e}_2, \mathbf{e}_1)\}$.

With this process we can define $\forall\, x \in \mathbb{Z}^2$ and $\forall\, \boldsymbol{\xi}$ and $\pm$ a semi-infinite geodesic $\pi^{x,\boldsymbol{\xi}\pm}$ by following the minimal increments of $B^{\boldsymbol{\xi}\pm}$ and by breaking ties with $\mathbf{e}_1$ for $+$ and with $\mathbf{e}_2$ for $-$.

We can characterize the simultaneous existence, uniqueness and coalescence of all geodesics.

Global geodesics picture

Theorem [Janjigian, Rassoul-Agha, S]

$\exists$ countable random set $\mathcal{V}^\omega \subset (\mathbf{e}_2, \mathbf{e}_1)$ of directions, with the following properties, all with probability 1.

• For each direction $\boldsymbol{\xi} \notin \mathcal{V}^\omega$ there is a unique geodesic from each lattice point. For a given $\boldsymbol{\xi}$ these geodesics coalesce.

• For directions $\boldsymbol{\xi} \in \mathcal{V}^\omega$, from each lattice point x there are exactly two geodesics $\pi^{x,\boldsymbol{\xi}+}$ and $\pi^{x,\boldsymbol{\xi}-}$ in direction $\boldsymbol{\xi}$ that eventually separate. Geodesics $\{\pi^{x,\boldsymbol{\xi}+} : x \in \mathbb{Z}^2\}$ form a coalescing tree, and geodesics $\{\pi^{x,\boldsymbol{\xi}-} : x \in \mathbb{Z}^2\}$ form a separate coalescing tree.

• $\mathcal{V}^\omega = \{\boldsymbol{\xi}^*(x) : x \in \mathbb{Z}^2\}$, the collection of asymptotic directions of the competition interfaces at all x, at a fixed ω.

• $\mathcal{V}^\omega = \{\boldsymbol{\xi} : \exists x, y \in \mathbb{Z}^2 : B^{\boldsymbol{\xi}+}(x, y) \neq B^{\boldsymbol{\xi}-}(x, y)\}$, the set of discontinuities of Busemann functions.

• There are no other semi-infinite geodesics except the trivial ones $x + k\mathbf{e}_i$, $k \geqslant 0$.

Global geodesics picture

Proof of the global geodesics theorem comes from a combination of earlier facts with properties of the Busemann process $\{B^{\boldsymbol{\xi}\pm} : \boldsymbol{\xi} \in (\mathbf{e}_2, \mathbf{e}_1)\}$.

First the distribution of the process $\boldsymbol{\xi} \mapsto B^{\boldsymbol{\xi}\pm}(x, x + \mathbf{e}_1)$ on a fixed horizontal edge $(x, x + \mathbf{e}_1)$.

Busemann process on an edge, indexed by directions $\boldsymbol{\xi}$

Parametrize directions $\boldsymbol{\xi} = (\xi_1, 1-\xi_1) \in (\mathbf{e}_2, \mathbf{e}_1)$ with $\alpha \in (0,1)$:

$$\alpha(\boldsymbol{\xi}) = \frac{\sqrt{\xi_1}}{\sqrt{\xi_1} + \sqrt{1-\xi_1}}$$

Define a marked point process X on $(0,1]$:

• On $(0,1)$, $N =$ Poisson point process with intensity measure $r^{-1}dr$, and $N\{1\} = 1$.

• To each point $r \in N$ attach an independent variable $Z_r \sim \text{Exp}(r)$.

• Set $X(\alpha) = \sum_{r \in N(\alpha,1]} Z_r$ for $0 < \alpha \leqslant 1$. $X(\alpha) \sim \text{Exp}(\alpha)$.

Theorem [Fan-S]

On a fixed horizontal edge $(x, x+\mathbf{e}_1)$,

$$\{B^{\boldsymbol{\xi}+}(x, x+\mathbf{e}_1) : \boldsymbol{\xi} \in (\mathbf{e}_2, \mathbf{e}_1)\} \overset{d}{=} \{X(\alpha(\boldsymbol{\xi})) : \boldsymbol{\xi} \in (\mathbf{e}_2, \mathbf{e}_1)\}.$$

Geometric significance of jumps of $\xi \mapsto B^{\xi\pm}_{x,x+\mathbf{e}_1}$

Recall: for a countable dense set of directions ζ, geodesics $\{\pi^{x,\zeta}\}_{x\in\mathbb{Z}^2}$ coalesce a.s.

If $\mathbf{z}^{\zeta}(x,y)$ = coalescence point of geodesics $\pi^{x,\zeta}$ and $\pi^{y,\zeta}$
then $B^{\zeta}(x,y) = G(x,\mathbf{z}^{\zeta}(x,y)) - G(y,\mathbf{z}^{\zeta}(x,y))$.

If $B^{\zeta}(x,y)$ is constant for $\zeta \in (\eta',\eta'')$ $\zeta \in (\xi,\eta)$ then $\mathbf{z}^{\zeta}(x,y)$ cannot jump.

Let $\zeta \searrow \xi$. Geodesics converge: $\pi^{x,\zeta} \to \pi^{x,\xi+}$, $\pi^{y,\zeta} \to \pi^{y,\xi+}$ and $B^{\xi+}(x,y) = G(x,\mathbf{z}^{\xi+}(x,y)) - G(y,\mathbf{z}^{\xi+}(x,y))$.

We conclude that $\xi+$ geodesics coalesce, as claimed in the global geodesics theorem.

Furthermore, the coalescence point $\xi \mapsto \mathbf{z}^{\xi\pm}(x,x+\mathbf{e}_1)$ jumps at the locations of an inhomogeneous Poisson process.

Distribution of increments on the x-axis of the lattice

Let $\boldsymbol{\zeta}, \boldsymbol{\eta} \in (\mathbf{e}_2, \mathbf{e}_1)$ again satisfy $\zeta_1 < \eta_1$.

$$\Delta_k = B^{\boldsymbol{\zeta}}(k\mathbf{e}_1, (k+1)\mathbf{e}_1) - B^{\boldsymbol{\eta}}(k\mathbf{e}_1, (k+1)\mathbf{e}_1) \geqslant 0$$

Distribution of process $\{\Delta_k\}_{k\in\mathbb{Z}}$ **?**

Define 2-sided RW $\quad S_n = \begin{cases} \sum_{i=1}^n Y_i, & n > 0 \\ 0, & n = 0 \\ -\sum_{i=n+1}^0 Y_i, & n < 0. \end{cases}$

with steps $Y_i \sim \text{Exp}(\alpha(\boldsymbol{\zeta})) - \text{Exp}(\alpha(\boldsymbol{\eta}))$. $E(Y_i) > 0$.

Theorem $\quad \{\Delta_k\}_{k\in\mathbb{Z}} \overset{d}{=} \{(\inf_{m>k} S_m - S_k)^+\}_{k\in\mathbb{Z}}$

Finding the joint distribution of $\{B^{\boldsymbol{\xi}} : \boldsymbol{\xi} \in (\mathbf{e}_2, \mathbf{e}_1)\}$

$\forall$ level $t \in \mathbb{Z}$ define bi-infinite sequences

$$\overline{\omega}_t = (\omega_{(k,t)})_{k\in\mathbb{Z}} \quad \text{and} \quad \overline{B}_t^{\boldsymbol{\xi},\mathbf{e}_1} = (B^{\boldsymbol{\xi}}_{(k,t),(k+1,t)})_{k\in\mathbb{Z}}$$

$\exists$ mapping D from a subset of $\mathbb{R}_+^{\mathbb{Z}} \times \mathbb{R}_+^{\mathbb{Z}}$ into $\mathbb{R}_+^{\mathbb{Z}}$ such that

$$\overline{B}_t^{\boldsymbol{\xi},\mathbf{e}_1} = D\big(\overline{B}_{t+1}^{\boldsymbol{\xi},\mathbf{e}_1}, \overline{\omega}_t\big) \quad \forall \boldsymbol{\xi} \text{ and } t \in \mathbb{Z}.$$

Definition of $\widetilde{I} = D(I, \omega)$: with G satisfying $I_k = G_k - G_{k+1}$, let

$$\widetilde{G}_k = \sup_{m:\, m\geqslant k} \Big\{ G_m + \sum_{i=k}^{m} \omega_i \Big\}, \quad \widetilde{I}_k = \widetilde{G}_k - \widetilde{G}_{k+1}.$$

$\widetilde{I}$ is the departure process of a FIFO queue with arrivals I and services ω with time running right to left on $\mathbb{Z}$.

Level-by-level evolution of the Busemann function

For $\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_n \in (\mathbf{e}_2, \mathbf{e}_1)$, the n-tuple of sequences evolves as a Markov chain backwards in the time parameter t via the mapping

$$\left(\overline{B}_t^{\boldsymbol{\xi}_1, \mathbf{e}_1}, \ldots, \overline{B}_t^{\boldsymbol{\xi}_n, \mathbf{e}_1} \right) = \Big(D\big(\overline{B}_{t+1}^{\boldsymbol{\xi}_1, \mathbf{e}_1}, \overline{\omega}_t \big), \ldots, D\big(\overline{B}_{t+1}^{\boldsymbol{\xi}_n, \mathbf{e}_1}, \overline{\omega}_t \big) \Big)$$

Theorem [Fan-S] Given $(\rho_1, \ldots, \rho_n) \in (1, \infty)^n$, the Markov chain above has a unique invariant distribution ergodic under spatial translation and with mean

$$\big(\mathbf{E} B^{\boldsymbol{\xi}_1}_{(k,t),(k+1,t)}, \ldots, \mathbf{E} B^{\boldsymbol{\xi}_n}_{(k,t),(k+1,t)} \big) = (\rho_1, \ldots, \rho_n).$$

Description of the invariant distribution

Let $D^{(n)}(\boldsymbol{\zeta}^1, \zeta^2, \ldots, \zeta^n)$ = departure process from sending arrival process $\boldsymbol{\zeta}^1$ successively through service processes $\zeta^2, \ldots, \zeta^n$.

Let $I^1, I^2, \ldots, I^n$ be independent sequences of i.i.d. exponentials with $I^i_k \sim \text{Exp}(\alpha(\boldsymbol{\xi}_i))$. Define sequences $\boldsymbol{\eta}^1, \ldots, \boldsymbol{\eta}^n$ by

$$\boldsymbol{\eta}^1 = I^1, \qquad \boldsymbol{\eta}^2 = D(I^2, I^1), \qquad \boldsymbol{\eta}^3 = D^{(3)}(I^3, I^2, I^1), \ldots,$$
$$\boldsymbol{\eta}^n = D^{(n)}(I^n, I^{n-1}, \ldots, I^1).$$

Well-defined if $\boldsymbol{\xi}_1 \cdot \mathbf{e}_1 > \cdots > \boldsymbol{\xi}_n \cdot \mathbf{e}_1$ so that $\alpha(\boldsymbol{\xi}_1) > \cdots > \alpha(\boldsymbol{\xi}_n)$.

Then the invariant distribution is given by

$$\big(\,\overline{B}_t^{\boldsymbol{\xi}_1, \mathbf{e}_1}, \ldots, \overline{B}_t^{\boldsymbol{\xi}_n, \mathbf{e}_1}\big) \overset{d}{=} (\boldsymbol{\eta}^1, \ldots, \boldsymbol{\eta}^n)$$

[Ferrari-Martin 2006-2009 on invariant distributions of multiclass TASEP pointed the way. Existence of invariant distributions for multiclass TASEP go back to Liggett 1976.]