Conformal blocks in nonrational CFTs with $c \leq 1$

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Based on various joint works with

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Recent developments in Constructive Field Theory, Columbia University
• Belavin-Polyakov-Zamolodchikov (BPZ) PDEs in 2D conformal field theory
• “physical solutions”: find single-valued ones?
• basis for solution space: “conformal blocks”
Plan

- Belavin-Polyakov-Zamolodchikov (BPZ) PDEs in 2D conformal field theory
- “physical solutions”: find single-valued ones?
- basis for solution space: “conformal blocks”
- How to solve the PDEs?
  - Coulomb gas formalism
  - action of the quantum group $\mathcal{U}_q(\mathfrak{sl}_2)$ on the solution space
  - can use representation theory to analyze solution space
  - currently works in the nonrational case
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- Belavin-Polyakov-Zamolodchikov (BPZ) PDEs in 2D conformal field theory
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  - Coulomb gas formalism
  - action of the quantum group $\mathcal{U}_q(\mathfrak{sl}_2)$ on the solution space
  - can use representation theory to analyze solution space
  - currently works in the nonrational case
- explicit formulas?
  - integral formulas (Coulomb gas)
  - algebraic formulas when $c = 1$ and $c = -2$
Introduction:

BPZ

Partial Differential Equations
consider a 2D QFT with “fields” $\phi(z)$

**Impose conformal symmetry:**
- fields $\phi(z)$ carry action of *Virasoro algebra* $\mathfrak{Vir}$
  - $\mathfrak{Vir}$: Lie algebra generated by $(L_n)_{n \in \mathbb{N}}$ and central element $C$
    
    $$[L_n, L_m] = (n - m)L_{n+m} + \frac{C}{12} n(n^2 - 1) \delta_{n+m,0}, \quad [C, L_n] = 0$$

- central element $C$ acts as a scalar
  - = central charge $c = \frac{1}{2\kappa}(3\kappa - 8)(6 - \kappa)$ (with $\kappa \geq 0$)

[ Initiated by Belavin, Polyakov, Zamolodchikov (1984) ]
at fixed $z_i$, to each generator $L_{-m}$ one associates a differential operator $\mathcal{L}_{-m}^{(z_i)}$ that acts on correlations of the fields:

$$\langle \phi_1(z_1) \cdots (L_{-m}\phi_i(z_i)) \cdots \phi_d(z_d) \rangle = \mathcal{L}_{-m}^{(z_i)}\langle \phi_1(z_1) \cdots \phi_d(z_d) \rangle$$

where $\mathcal{L}_{-m}^{(z_i)} = -\sum_{j \neq i} \left( \frac{1}{(z_j - z_i)^{m-1}} \frac{\partial}{\partial z_j} + \frac{(1 - m) h_{\phi_j}}{(z_j - z_i)^m} \right)$

$h_\phi$ are conformal weights of the fields $\phi(z)$: transformation rule

$$\langle \phi(z) \cdots \rangle = (\partial f(z))^{h_\phi} \langle \phi(f(z)) \cdots \rangle$$

for conformal maps $f$
What is the \( \mathfrak{vir} \)-action on a field \( \phi(z) \)?

- field \( \phi(z) \) generates a \( \mathfrak{vir} \)-module \( M_\phi = \mathfrak{vir} \cdot \phi(z) \)
- \( M_\phi \) is isomorphic to a quotient of some Verma module \( V \):
  \[ M_\phi \cong V/N \]
What is the Vir -action on a field \( \phi(z) \)?

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Is this quotient the whole Verma module?
Singular vectors

What is the Vir-action on a field $\phi(z)$?

- Field $\phi(z)$ generates a Vir-module $M_\phi = \text{Vir}.\phi(z)$
- $M_\phi$ is isomorphic to a quotient of some Verma module $V$:
  $$M_\phi \cong V/N$$

Is this quotient the whole Verma module?

- In “degenerate CFTs”, $N$ is non-trivial (containing “null fields”)
- Elements generating $N$ are called singular vectors:
  $$\mathcal{P}(L_m : m \in \mathbb{N}).\phi(z) \in N$$

where $\mathcal{P}$ is a polynomial in the Virasoro generators
singular vector $\mathcal{P}(L_{-m}: m \in \mathbb{N}).\phi(z) \in N$ in the quotient $M_\phi \cong V/N$ gives rise to a PDE with $\mathcal{D}^{(z)} = \mathcal{P}(L_{-m}^{(z)}: m \in \mathbb{N})$

$$\mathcal{D}^{(z)} \langle \phi(z)\phi_1(z_1)\cdots\phi_d(z_d) \rangle = 0$$

where $L_{-m}^{(z_i)} = -\sum_{j \neq i} \left( \frac{1}{(z_j - z_i)^{m-1}} \frac{\partial}{\partial z_j} + \frac{(1 - m) h_{\phi_j}}{(z_j - z_i)^m} \right)$
singular vector $\mathcal{P}(L_{-m} : m \in \mathbb{N}).\phi(z) \in N$ in the quotient $M_\phi \cong V/N$ gives rise to a PDE with $\mathcal{D}^{(z)} = \mathcal{P}(\mathcal{L}^{(z)}_{-m} : m \in \mathbb{N})$

$$\mathcal{D}^{(z)} \langle \phi(z)\phi_1(z_1) \cdots \phi_d(z_d) \rangle = 0$$

where $\mathcal{L}^{(z_i)}_{-m} = -\sum_{j \neq i} \left( \frac{1}{(z_j - z_i)^{m-1}} \frac{\partial}{\partial z_j} + \frac{(1 - m) h_{\phi_j}}{(z_j - z_i)^m} \right)$

Benoit, Saint-Aubin (1988): explicit formulas for singular vectors when conformal weights are of type

$$\frac{s(2s + 4 - \kappa)}{2\kappa} = h_{1,s+1} \text{ or } h_{s+1,1} \quad s \geq 0$$

Recall: $c = \frac{1}{2\kappa}(3\kappa - 8)(6 - \kappa)$
Benoit, Saint-Aubin (1988): explicit formulas for singular vectors when conformal weights are of type

\[ \frac{s(2s + 4 - \kappa)}{2\kappa} \left( = h_{1,s+1} \text{ or } h_{s+1,1} \right) \]

example: \( h_{1,2} = \frac{6-\kappa}{2\kappa} \), field \( \phi_{1,2} \) (or \( \phi_{2,1} \)):

\[ \left( L_{-2} - \frac{3}{2(2h_{1,2} + 1)(L_{-1})^2} \right) \phi_{1,2}(z) \]

(with translation invariance) gives rise to the PDE

\[
\left\{ \frac{\kappa}{2} \frac{\partial^2}{\partial z^2} + \sum_{i=1}^{d} \left( \frac{2}{z_i - z} \frac{\partial}{\partial z_i} - \frac{2h_{1,2}}{(z_i - z)^2} \right) \right\} \langle \phi_{1,2}(z)\phi_1(z_1)\cdots\phi_d(z_d) \rangle = 0
\]
Benoit & Saint-Aubin partial differential equations

- Benoit, Saint-Aubin (1988): explicit formulas for singular vectors when conformal weights are of type

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\]

- in general: homogeneous PDE operators of degree \( s + 1 \)

\[
\mathcal{D}_{s+1}^{(z)} = \sum_{n_1+\ldots+n_j=s} c_{n_1,\ldots,n_j}(s, \kappa) \times \mathcal{L}_{-n_1}^{(z)} \cdots \mathcal{L}_{-n_j}^{(z)}
\]
We seek solutions $F_\varsigma(z; \bar{z})$ to the PDE system

\[ D^{(z_j)}_{s_j+1} F_\varsigma(z; \bar{z}) = 0, \quad D^{(\bar{z}_j)}_{s_j+1} F_\varsigma(z; \bar{z}) = 0, \quad \text{for all } j = 1, \ldots, d \]

for variables $z = (z_1, z_2, \ldots, z_d)$ and c.c. $\bar{z} = (\bar{z}_1, \bar{z}_2, \ldots, \bar{z}_d)$
We seek solutions $F_\varsigma(z; \bar{z})$ to the PDE system

$$D_{s_{j+1}}^{(z_j)} F_\varsigma(z; \bar{z}) = 0, \quad D_{s_{j+1}}^{(\bar{z}_j)} F_\varsigma(z; \bar{z}) = 0,$$

for all $j = 1, \ldots, d$

for variables $z = (z_1, z_2, \ldots, z_d)$ and c.c. $\bar{z} = (\bar{z}_1, \bar{z}_2, \ldots, \bar{z}_d)$

That is, we seek correlation functions

$$\langle \phi_{s_1}(z_1; \bar{z}_1) \cdots \phi_{s_d}(z_d; \bar{z}_d) \rangle = F_\varsigma(z; \bar{z})$$

where

- $\varsigma = (s_1, \ldots, s_d)$
- $\phi_s$ have conformal weights of type $\theta_s = \frac{s(2s+4-\kappa)}{2\kappa}$
  (i.e. $h_{1,s+1}$ or $h_{s+1,1}$ in Kac table)
We seek solutions $F_\varsigma(z; \bar{z})$ to the PDE system

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for variables $z = (z_1, z_2, \ldots, z_d)$ and c.c. $\bar{z}$

That is, we seek correlation functions

$$\langle \phi_{s_1}(z_1; \bar{z}_1) \cdots \phi_{s_d}(z_d; \bar{z}_d) \rangle = F_\varsigma(z; \bar{z}) = \sum_{k, \ell} F^{(k)}(z_1, \ldots, z_d) \ F^{(\ell)}(\bar{z}_1, \ldots, \bar{z}_d)$$

where

- $\varsigma = (s_1, \ldots, s_d)$
- $\phi_s$ have conformal weights of type $\theta_s = \frac{s(2s+4-\kappa)}{2\kappa}$ (i.e. $h_{1,s+1}$ or $h_{s+1,1}$ in Kac table)
Question:
∃? solutions
QUESTION:

∃? SINGLE-VALUED SOLUTIONS
Benoit & Saint-Aubin PDE system:

\[
\mathcal{D}_{s_j+1}^{(z_j)} F_\varsigma(z; \bar{z}) = 0, \quad \mathcal{D}_{s_j+1}^{(\bar{z}_j)} F_\varsigma(z; \bar{z}) = 0, \quad \text{for all } j = 1, \ldots, d
\]

**covariance**: for all conformal maps \( f \),

\[
F_\varsigma(f(z), f(\bar{z})) = \left( \prod_{i=1}^{n} (\partial f(z_i) \bar{\partial} f(\bar{z}_i))^{-\theta_{si}} \right) F_\varsigma(z, \bar{z})
\]

\( F_\varsigma \) is defined for \( \{(z_1, \ldots, z_d) \in \mathbb{C}^d \mid z_i \neq z_j \text{ for } i \neq j\} \)
Correlation functions

- Benoit & Saint-Aubin PDE system:

\[ D_{s_j + 1}^{(z_j)} F_\varsigma(z; \bar{z}) = 0, \quad D_{s_j + 1}^{(\bar{z}_j)} F_\varsigma(z; \bar{z}) = 0, \quad \text{for all } j = 1, \ldots, d \]

- covariance: for all conformal maps \( f \),

\[ F_\varsigma(f(z), f(\bar{z})) = \left( \prod_{i=1}^{n} (\partial f(z_i) \bar{\partial} f(\bar{z}_i))^{-\theta_{s_i}} \right) F_\varsigma(z, \bar{z}) \]

- \( F_\varsigma \) is defined for \( \{(z_1, \ldots, z_d) \in \mathbb{C}^d \mid z_i \neq z_j \text{ for } i \neq j \} \)

**Theorem** [Flores, P. (2018+)]

Suppose \( \kappa \in (0, 8) \setminus \mathbb{Q} \) (so \( c = \frac{1}{2\kappa}(3\kappa - 8)(6 - \kappa) \leq 1 \) irrational).

There exists a unique single-valued, conformally covariant solution \( F_\varsigma(z, \bar{z}) \) to the Benoit & Saint-Aubin PDEs.

(Uniqueness holds up to normalization in a natural solution space.)
correlation function $F_\varsigma$ single-valued in
$\mathcal{W}_d = \{(z_1, \ldots, z_d) \in \mathbb{C}^d \mid z_i \neq z_j \text{ for } i \neq j\}$

braid group $\mathcal{B}_{r_d}$ is the fundamental group of $\mathcal{W}_d$

invariance: $\sigma_j . F_\varsigma(z, \bar{z}) = F_\varsigma(z, \bar{z}) \quad \forall j = 1, 2, \ldots, d - 1$

where $\sigma_j$ are generators of the braid group $\mathcal{B}_{r_d}$

holomorphic

antiholomorphic
Theorem [ Flores, P. (2018+) ]

Suppose \( \kappa \in (0, 8) \setminus \mathbb{Q} \) (so \( c = \frac{1}{2 \kappa}(3\kappa - 8)(6 - \kappa) \leq 1 \) irrational).

There exists a unique collection of smooth functions \( F_\varsigma(z, \bar{z}) \) with \( \varsigma = (s_1, \ldots, s_d) \) satisfying \( F_{(1)} = 1 \) and properties

- **PDE system:**
  \[
  D^{(z_j)}_{s_j+1} F_\varsigma(z; \bar{z}) = 0, \quad D^{(\bar{z}_j)}_{s_j+1} F_\varsigma(z; \bar{z}) = 0, \quad \forall j = 1, \ldots, d
  \]

- **covariance:** for all conformal maps \( f \),
  \[
  F_\varsigma(f(z), f(\bar{z})) = \left( \prod_{i=1}^{n} (\partial f(z_i) \bar{\partial} f(\bar{z}_i))^{-\theta_{s_i}} \right) F_\varsigma(z, \bar{z})
  \]

- **monodromy invariance:** \( \sigma_j F = F \) for all \( j = 1, \ldots, d - 1 \)

- **power law growth:** there exist \( C > 0 \) and \( p > 0 \) s.t.
  \[
  | F_\varsigma | \leq C \times \prod_{1 \leq i < j \leq d} \max \left( (z_j - z_i)^p, (z_j - z_i)^{-p} \right)
  \]
Corollary: Asymptotics / OPE:

$$F_s(z; \bar{z}) \sim z_i, z_{i+1} \to \zeta \sum_s C_{s_i,s_{i+1}}^s \times (z_{i+1} - z_i)^{-\theta_{s_i} - \theta_{s_{i+1}} + \theta_s} \times \text{c.c.}$$

$$\times F_{(s_1, \ldots, s_d)}(\ldots, z_{i-1}, \zeta, z_{i+2}, \ldots; \text{c.c.})$$

where $s$ belongs to a finite index set and $C_{r,t}^s$ are structure constants
Further properties

Corollary: Asymptotics / OPE:

\[ F_S(z; \bar{z}) \sim \sum_{s} C^s_{s_i,s_i+1} (z_{i+1} - z_i)^{-\theta_{s_i} - \theta_{s_i+1} + \theta_{s}} \times \text{c.c.} \]

\[ \times F_{(s_1,...,s,...,s_d)}(\ldots, z_{i-1}, \zeta, z_{i+2}, \ldots ; \text{c.c.}) \]

where \( s \) belongs to a finite index set and \( C^s_{r,t} \) are structure constants:

\[ C^s_{r,t} := \frac{(B^s_{r,t})^2 B^0_{s,s}}{B^0_{r,r} B^0_{t,t}} \frac{([s]!)^2 \sqrt{[r+1][s+1][t+1][\frac{r+t-s}{2}]}!}{\left[\frac{r+s+t}{2} + 1\right]! \left[\frac{s+r-t}{2}\right]! \left[\frac{t+s-r}{2}\right]!} \]

\[ [n]! = [1][2] \cdots [n-1][n] \text{ and } [n] = \frac{\sin(4\pi n/\kappa)}{\sin(4\pi/\kappa)} \text{ and } \]

\[ B^s_{r,t} := \frac{1}{\left(\frac{r+t-s}{2}\right)!} \prod_{i=1}^{\frac{r+t-s}{2}} \frac{\Gamma(1 - \frac{4}{\kappa}(r - i + 1)) \Gamma(1 - \frac{4}{\kappa}(t - i + 1)) \Gamma(1 + \frac{4}{\kappa} i)}{\Gamma(2 - \frac{4}{\kappa}\left(\frac{r+s+t}{2} - i + 2\right)) \Gamma(1 + \frac{4}{\kappa})} \]

Compare with the work of Dotsenko & Fateev in the 1980s: they calculated the 4-point function and structure constants

- **uniqueness:** representation theory + knowledge of the solution space of the PDEs (I’ll give some idea later... )
Construction: conformal blocks


- uniqueness: representation theory + knowledge of the solution space of the PDEs (I’ll give some idea later...)
- construction: we can write

\[ F(z; \bar{z}) = \sum_{\varrho} U_{\varrho}(z) U_{\varrho}(\bar{z}) \]

- \(U_{\varrho}\) are conformal blocks, which can be written as contour integrals à la Feigin & Fuchs / Dotsenko & Fateev
**Construction: conformal blocks**

**Ref.** In preparation; see the introduction section 1D in [arXiv:1801.10003] Flores, P. *Standard modules, Jones-Wenzl projectors, and the valenced Temperley-Lieb algebra.*

- **uniqueness:** representation theory + knowledge of the solution space of the PDEs (I’ll give some idea later...)
- **construction:** we can write

\[ F(z; \bar{z}) = \sum_{\varrho} \mathcal{U}_\varrho(z) \mathcal{U}_\varrho(\bar{z}) \]

- \( \mathcal{U}_\varrho \) are *conformal blocks*, which can be written as contour integrals à la Feigin & Fuchs / Dotsenko & Fateev
- Example: when \( \varsigma = (1,1,\ldots,1) \)

\[ \mathcal{U}_\alpha(z) = \prod_{i<j} (z_i - z_j)^{2/\kappa} \int_{\Gamma_\alpha} \prod_r \prod_j (w_r - z_j)^{-4/\kappa} \prod_{r<s} (w_r - w_s)^{8/\kappa} dw, \]

where \( \Gamma_\alpha \) are certain integration surfaces

Theorem [Karrila, Kytölä, P. (2017)] case \( \zeta = (1, 1, \ldots, 1) \)

Suppose \( \kappa \in (0, 8) \setminus \mathbb{Q} \) (so \( c = \frac{1}{2\kappa} (3\kappa - 8)(6 - \kappa) \leq 1 \) irrational).

There exists a unique collection of smooth functions \( \mathcal{U}_\alpha(x) \)
(with \( x = (x_1 < \cdots < x_{2N}) \) and \( \alpha \) planar pair partitions of \( 2N \) points)
satisfying \( \mathcal{U}_0 = 1 \) and properties

- **PDE system:** \( \forall \, 1 \leq j \leq 2N, \)
  \[
  \left\{ \frac{\kappa \, \partial^2}{2 \, \partial x_j^2} + \sum_{i \neq j} \left( \frac{2}{x_i - x_j} \frac{\partial}{\partial x_i} - \frac{6/\kappa - 1}{(x_i - x_j)^2} \right) \right\} \mathcal{U}_\alpha(x_1, \ldots, x_{2N}) = 0
  \]

- **covariance:** for all (admissible) Möbius maps \( \mu : \mathbb{H} \to \mathbb{H} \)
  \[
  \mathcal{U}(\mu(x_1), \ldots, \mu(x_{2N})) = \prod_j \mu'(x_j)^{\frac{\kappa - 6}{2\kappa}} \times \mathcal{U}_\alpha(x_1, \ldots, x_{2N})
  \]

- **specific asymptotics** (related to fusion)

- **power law growth:** there exist \( C > 0 \) and \( p > 0 \) s.t.
  \[
  | \mathcal{U}_\alpha | \leq C \times \prod_{i < j} \max \left( (x_j - x_i)^p, (x_j - x_i)^{-p} \right)
  \]
Suppose $\kappa \in (0, 8) \setminus \mathbb{Q}$ (so $c = \frac{1}{2\kappa}(3\kappa - 8)(6 - \kappa) \leq 1$ irrational).

There exists a unique collection of smooth functions $U_\alpha(x)$ (with $x = (x_1 < \cdots < x_{2N})$ and $\alpha$ planar pair partitions of $2N$ points) satisfying $U_0 = 1$ and properties

- PDE system
- covariance
- asymptotics
- power law growth

Furthermore:

- $U_\alpha(x_1, \ldots, x_{2N})$ form basis for the solution space
- $U_\alpha$ give a formula for the unique single-valued (monodromy invariant) correlation function when $\varsigma = (1, 1, \ldots, 1)$

$$F_{(1,1,\ldots,1)}(z; \bar{z}) = \sum_\alpha U_\alpha(z_1, \ldots, z_{2N}) U_\alpha(\bar{z}_1, \ldots, \bar{z}_{2N})$$
What’s next?

1. Rational examples
2. Ideas for proof
Loop-erased random walk \((c = -2)\)

Gaussian free field \((c = 1)\)
Level line of Gaussian free field: $\text{SLE}_4$ ($c = 1$)

GFF with boundary data $\pm \lambda$:
- take boundary values
  - $-\lambda$ on the left,
  - $+\lambda$ on the right
- zero level line:
  $\text{SLE}_\kappa$ with $\kappa = 4$
  [Schramm & Sheffield (2003)]
Conformal blocks for GFF \( (c = 1) \)

\[ U_{\alpha}(x_1, \ldots, x_{2N}) = \prod_{1 \leq i < j \leq 2N} (x_j - x_i)^{1/2} \vartheta_{\alpha}(i, j), \]

where \( \vartheta_{\alpha}(i, j) = \begin{cases} +1 & \text{if } i, j \in \{a_1, a_2, \ldots, a_N\} \text{ or } i, j \in \{b_1, b_2, \ldots, b_N\} \\ -1 & \text{otherwise} \end{cases} \)

Associate boundary data to “walks” i.e. planar pairings
\( \alpha = \{\{a_1, b_1\}, \ldots, \{a_N, b_N\}\} \)

where \( \{a_1, \ldots, b_N\} = \{1, \ldots, 2N\}, \)
\( a_1 < a_2 < \cdots < a_N, \)
and \( a_j < b_j \) for all \( j \in \{1, \ldots, N\}. \)
Conformal blocks for GFF (\(c = 1\))

\[ \mathcal{U}_\alpha(x_1, \ldots, x_{2N}) = \prod_{1 \leq i < j \leq 2N} (x_j - x_i)^{\frac{1}{2} \vartheta_\alpha(i, j)}, \]

where \(\vartheta_\alpha(i, j) := \begin{cases} +1 & \text{if } i, j \in \{a_1, a_2, \ldots, a_N\} \text{ or } i, j \in \{b_1, b_2, \ldots, b_N\} \\ -1 & \text{otherwise} \end{cases} \)

Associate boundary data to “walks” i.e. planar pairings

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where \(\{a_1, \ldots, b_N\} = \{1, \ldots, 2N\},\)

\(a_1 < a_2 < \cdots < a_N,\)

and \(a_j < b_j\) for all \(j \in \{1, \ldots, N\}\)
Level lines form some connectivity and, for any $\beta$

$$\mathbb{P}[\text{connectivity } = \beta] = \frac{\sum_{\gamma} c_{\beta \gamma} \mathcal{U}_\gamma(x_1, \ldots, x_{2N})}{\mathcal{U}_\alpha(x_1, \ldots, x_{2N})}$$

where the coefficients $c_{\beta \gamma}$ are explicit (combinatorial)
Loop-erased random walk ($c = -2$)

Gaussian free field ($c = 1$)
Realize walks as branches in a wired uniform spanning tree:

Connectivities encoded to planar pairings
\[ \alpha = \{(a_1, b_1), \ldots, (a_N, b_N)\} \]
where \( \{a_1, \ldots, b_N\} = \{1, \ldots, 2N\}, \)
a_1 < a_2 < \cdots < a_N,
and \( a_j < b_j \) for all \( j \in \{1, \ldots, N\} \)

The conformal blocks for \( c = -2 \) (so \( \kappa = 2 \)) are combinatorial determinants \( \text{á la} \) Fomin, involving Poisson kernels:

\[
U_\alpha(x_1, \ldots, x_{2N}) = \det \left( (x_{a_i} - x_{b_j})^{-2} \right)_{i,j=1}^N
\]

General method:

Hidden quantum group

Symmetry


Kytölä, P., Conformally covariant boundary correlation functions with a quantum group.
Theorem [Flores, P. (2018+)]

Suppose $\kappa \in (0, 8) \setminus \mathbb{Q}$ (so $c = \frac{1}{2\kappa}(3\kappa - 8)(6 - \kappa) \leq 1$ irrational).

There exists a unique collection of smooth functions $F_\varsigma(z, \bar{z})$ with $\varsigma = (s_1, \ldots, s_d)$ satisfying $F_{(1)} = 1$ and properties

- **PDE system:**
  $$D_{s_j} F_\varsigma(z; \bar{z}) = 0, \quad \forall j = 1, \ldots, d$$

- **covariance:** for all conformal maps $f$,
  $$F_\varsigma(f(z), f(\bar{z})) = \left(\prod_{i=1}^{n} \left(\frac{\partial f(z_i)}{\partial f(\bar{z}_i)}\right)^{-\theta_{s_i}}\right) F_\varsigma(z, \bar{z})$$

- **monodromy invariance:** $\sigma_j F = F$ for all $j = 1, \ldots, d - 1$

- **power law growth:** there exist $C > 0$ and $p > 0$ s.t.
  $$| F_\varsigma | \leq C \times \prod_{1 \leq i < j \leq d} \max \left((z_j - z_i)^p, (z_j - z_i)^{-p}\right)$$
How to find solutions of PDEs

\[ \mathcal{D}_{s_j+1}^{(z_j)} F = 0 \quad \text{for all } j = 1, \ldots, d \]

Write solution in integral form & use Stokes theorem:

\[
F(z_1, \ldots, z_d) = \int_{\Gamma} f(z_1, \ldots, z_d; w_1, \ldots, w_\ell) \, dw_1 \cdots dw_\ell
\]
How to find solutions of PDEs

\[ \mathcal{D}_{s_{j}+1}^{(z_j)} F = 0 \quad \text{for all } j = 1, \ldots, d \]

Write solution in integral form & use Stokes theorem:

\[
F(z_1, \ldots, z_d) = \int_{\Gamma} f(z_1, \ldots, z_d; w_1, \ldots, w_\ell) \, dw_1 \cdots dw_\ell
\]

- find \( f \) s.t. for all \( j \)

\[
\mathcal{D}_{s_{j}+1}^{(z_j)} f = \text{“total derivative”}
\]

\[ \Rightarrow \text{ then } \mathcal{D}_{s_{j}+1}^{(z_j)} f \, dw = d\eta \text{ is exact } \ell \text{-form} \]
How to find solutions of PDEs

\[ \mathcal{D}_{s_{j+1}}^{(z_j)} F = 0 \quad \text{for all } j = 1, \ldots, d \]

Write solution in integral form & use Stokes theorem:

\[ F(z_1, \ldots, z_d) = \int_{\Gamma} f(z_1, \ldots, z_d; w_1, \ldots, w_\ell) \, dw_1 \cdots dw_\ell \]

- find \( f \) s.t. for all \( j \)
  \[ \mathcal{D}_{s_{j+1}}^{(z_j)} f = \text{“total derivative”} \]

  \[ \Rightarrow \quad \text{then } \mathcal{D}_{s_{j+1}}^{(z_j)} f \, dw = d\eta \text{ is exact } \ell\text{-form} \]

- find **closed** \( \ell\)-surface \( \Gamma \): \( \partial \Gamma = \emptyset \)

  \[ \Rightarrow \quad \text{then } \mathcal{D}_{s_{j+1}}^{(z_j)} F = \int_{\Gamma} \mathcal{D}_{s_{j+1}}^{(z_j)} f \, dw = \int_{\Gamma} d\eta = \int_{\partial \Gamma} \eta = \int_{\emptyset} \eta = 0 \]
How to find solutions of PDEs

\[ \mathcal{D}_{s_j+1}^{(z_j)} F = 0 \quad \text{for all } j = 1, \ldots, d \]

Write solution in integral form & use Stokes theorem:

\[ F(z_1, \ldots, z_d) = \int_{\Gamma} f(z_1, \ldots, z_d; w_1, \ldots, w_\ell) \, dw_1 \cdots dw_\ell \]

Task: find \( f \) and \( \Gamma \) such that:

- \( \mathcal{D}_{s_j+1}^{(z_j)} f = \) “total derivative”
- \( \partial \Gamma = \emptyset \quad \Rightarrow \quad \mathcal{D}_{s_j+1}^{(z_j)} F = 0 \)

1. How to find \( f \)? [Feigin-Fuchs / Dotsenko-Fateev '84, “Coulomb gas”]:

\[ f = \prod_{i<j} (z_j - z_i)^{2\alpha_i \alpha_j} \prod_{i,r} (w_r - z_i)^{2\alpha_i \alpha_r} \prod_{r<s} (w_s - w_r)^{2\alpha_r \alpha_\ell} \]

\[ \alpha_i = \frac{s_i}{\sqrt{k}} \text{ and } \alpha_\ell = -\frac{2}{\sqrt{k}} \]
How to find solutions of PDEs

\[ \mathcal{D}^{(z_j)}_{s_j+1} F = 0 \quad \text{for all } j = 1, \ldots, d \]

Write solution in integral form & use Stokes theorem:

\[ F(z_1, \ldots, z_d) = \int_{\Gamma} f(z_1, \ldots, z_d; w_1, \ldots, w_\ell) \, dw_1 \cdots dw_\ell \]

**Task:** find \( f \) and \( \Gamma \) such that:

- \( \mathcal{D}^{(z_j)}_{s_j+1} f = \text{"total derivative"} \)
- \( \partial \Gamma = \emptyset \quad \Rightarrow \quad \mathcal{D}^{(z_j)}_{s_j+1} F = 0 \)

1. **How to find** \( f \)? [Feigin-Fuchs / Dotsenko-Fateev ’84, “Coulomb gas”]
2. **How to choose closed** \( \Gamma \) **s.t. also get**
   - conformal covariance
   - monodromy invariance?
How to find solutions of PDEs

\[ D_{s_j+1}^{(z_j)} F = 0 \quad \text{for all } j = 1, \ldots, d \]

Write solution in integral form & use Stokes theorem:

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Task: find \( f \) and \( \Gamma \) such that:

- \( D_{s_j+1}^{(z_j)} f = \) “total derivative”
- \( \partial \Gamma = \emptyset \quad \Rightarrow \quad D_{s_j+1}^{(z_j)} F = 0 \)

1. How to find \( f \)? [Feigin-Fuchs / Dotsenko-Fateev ’84, “Coulomb gas”]
2. How to find \( \Gamma \)?

Idea: Use action of quantum group on integration surfaces \( \Gamma \)!
Spin chain – Coulomb gas correspondence


Suppose \( \kappa \in (0, 8) \setminus \mathbb{Q} \) (so \( c = \frac{1}{2\kappa}(3\kappa - 8)(6 - \kappa) \leq 1 \) irrational).

We have an embedding of \( H_\varsigma := \{v \in M_\varsigma \mid E.v = 0, K.v = v\} \) onto the solution space

\[
\mathcal{F} : H_\varsigma \hookrightarrow \{ \text{solutions} \int_\Gamma f(z; w)dw \text{ to BSA PDEs} \}
\]

- \( E, F, K \) generators of \( \mathcal{U}_q(\mathfrak{sl}_2) \) ( = “quantum” \( \mathfrak{sl}_2(\mathbb{C}) \) )
- \( M_\varsigma = M_{(s_1)} \otimes \cdots \otimes M_{(s_d)} \), where \( M_{(s)} \) is the irreducible type one \( \mathcal{U}_q(\mathfrak{sl}_2) \)-module of dimension \( s + 1 \)

Spin chain – Coulomb gas correspondence


Suppose \( \kappa \in (0, 8) \setminus \mathbb{Q} \) (so \( c = \frac{1}{2\kappa}(3\kappa - 8)(6 - \kappa) \leq 1 \) irrational).

We have an embedding of \( H_\varsigma := \{ v \in M_\varsigma \mid E.v = 0, K.v = v \} \) onto the solution space

\[
F: H_\varsigma \hookrightarrow \left\{ \text{solutions } \int_{\Gamma} f(z; w)dw \text{ to BSA PDEs} \right\}
\]

- \( E, F, K \) generators of \( \mathcal{U}_q(\mathfrak{sl}_2) \) ( = “quantum” \( \mathfrak{sl}_2(\mathbb{C}) \) )
- \( M_\varsigma = M_{(s_1)} \otimes \cdots \otimes M_{(s_d)} \), where \( M_{(s)} \) is the irreducible type one \( \mathcal{U}_q(\mathfrak{sl}_2) \)-module of dimension \( s + 1 \)
- when \( \varsigma = (1, 1, \ldots, 1) \), then the map \( F \) is surjective onto
  \[
  \left\{ \begin{array}{l}
  \text{conformally covariant solutions to} \\
  \text{the 2nd order BPZ PDE system } D_2^{(z_i)} F = 0 \ \forall \ i \\
  \text{with at most power-law growth}
  \end{array} \right\}
  \]

Quantum group $\mathcal{U}_q(\mathfrak{sl}_2) = \text{“quantum } \mathfrak{sl}_2(\mathbb{C})\text{”}$

- generators $E, F, K, K^{-1}$
- deformation parameter $q = e^{4\pi i/\kappa} \notin e^{\pi i \mathbb{Q}}$
- relations
  \[
  KK^{-1} = 1 = K^{-1}K \\
  KE = q^2EK, \quad KF = q^{-2}FK \\
  EF - FE = \frac{K - K^{-1}}{q - q^{-1}}
  \]
- vectors $v \in M(s_1) \otimes \cdots \otimes M(s_d)$ $\longleftrightarrow$ functions $\mathcal{F}[v](z_1, \ldots, z_d)$

Theorem [ parts 1-3: Kytölä, P. (JEMS 2018); part 4: “folklore” ]

1. highest weight vectors ($E.v = 0$) $\mathcal{F}$ solutions of PDEs
2. weight ($K.v = q^\lambda v$) $\mathcal{F}$ conformal transformation properties
3. projections onto subrepresentations $\mathcal{F}$ asymptotics
4. braiding $\mathcal{F}$ monodromy
hidden quantum group: $\mathcal{U}_q(\mathfrak{sl}_2) \quad (= \text{"quantum" } \mathfrak{sl}_2(\mathbb{C}))$

vectors $v \in M_{(s_1)} \otimes \cdots \otimes M_{(s_d)} \leftrightarrow \text{ functions } \mathcal{F}[v](z_1, \ldots, z_d)$

basis vectors $e_{\ell_1} \otimes \cdots \otimes e_{\ell_d} \leftrightarrow \text{ integral functions } \int_{\Gamma_{\ell_1, \ldots, \ell_d}} f \, dw$

$$\mathcal{F}[v](z) = \int_{\Gamma(v)} f(z; w) \, dw$$
hidden quantum group: \( \mathcal{U}_q(\mathfrak{sl}_2) \) ( = “quantum” \( \mathfrak{sl}_2(\mathbb{C}) \) )

vectors \( v \in M(s_1) \otimes \cdots \otimes M(s_d) \) \( \longleftrightarrow \) functions \( \mathcal{F}[v](z_1, \ldots, z_d) \)

basis vectors \( e_{\ell_1} \otimes \cdots \otimes e_{\ell_d} \) \( \mathcal{F} \) \( \longleftrightarrow \) integral functions \( \int_{\Gamma_{\ell_1, \ldots, \ell_d}} f \, dw \)

\[
\mathcal{F}[v](z) = \int_{\Gamma(v)} f(z; w) dw
\]

---

**Theorem** [ parts 1-3: Kytölä, P. (JEMS 2018); part 4: “folklore” ]

1. highest weight vectors ( \( E.v = 0 \) ) \( \mathcal{F} \) solutions of PDEs
2. weight ( \( K.v = q^\lambda v \) ) \( \mathcal{F} \) conformal transformation properties
3. projections onto subrepresentations \( \mathcal{F} \) asymptotics
4. braiding \( \mathcal{F} \) monodromy
Irreducible representations of $\mathcal{U}_q(\mathfrak{sl}_2)$

- $M_{(s)} = \text{span}\{e_0, \ldots, e_s\}$
- $e_0$ is highest weight vector: $E.e_0 = 0$ and $K.e_0 = q^s e_0$
- $e_0$ generates $M_{(s)}$: $e_k = F^k e_0$
- Associate contours to $e_k$:

\[ e_k = z \]

- $F$ adds a contour:

\[ F. = z \]

- $E$ removes a contour:

\[ E. = [k]_q [d-k]_q \times \]
using coproduct $\Delta : U_q(\mathfrak{sl}_2) \to U_q(\mathfrak{sl}_2) \otimes U_q(\mathfrak{sl}_2)$ can define action on tensor products of representations:

$U_q(\mathfrak{sl}_2) \curvearrowright M(s_1) \otimes M(s_2)$

$K.(w_1 \otimes w_2) := \Delta(K)(w_1 \otimes w_2) = K.w_1 \otimes K.w_2$

$E.(w_1 \otimes w_2) := \Delta(E)(w_1 \otimes w_2) = 1.w_1 \otimes E.w_2 + E.w_1 \otimes K.w_2$

$F.(w_1 \otimes w_2) := \Delta(F)(w_1 \otimes w_2) = F.w_1 \otimes 1.w_2 + K^{-1}.w_1 \otimes F.w_2$
Calculating monodromy = braiding

vectors \( v \in \mathcal{M}_{(s_1)} \otimes \cdots \otimes \mathcal{M}_{(s_d)} \) \( \leftrightarrow \) functions \( \mathcal{F}[v](z_1, \ldots, z_d) \)

basis vectors \( e_{\ell_1} \otimes \cdots \otimes e_{\ell_d} \) \( \leftrightarrow \) integral functions \( \int_{\Gamma_{\ell_1,\ldots,\ell_d}} f \, dw \)

- consider correlation function

\[
F(z_1, \ldots, z_d) = \mathcal{F}[v](z) = \int_{\Gamma} f(z; w) dw
\]

\[
f = \prod_{i<j}(z_j - z_i)^{2\alpha_i\alpha_j} \prod_{i,r}(w_r - z_i)^{2\alpha-\alpha_i} \prod_{r<s}(w_s - w_r)^{2\alpha-\alpha-}
\]

- monodromy can be calculated by contour deformation method:

\[
\sigma \rightarrow e^{\pi i(\cdots)} \times
\]
Conclusion: \( \exists \) ! single-valued solution

**Theorem** [Flores, P. (2018+)]

Suppose \( \kappa \in (0, 8) \setminus \mathbb{Q} \) (so \( c = \frac{1}{2\kappa} (3\kappa - 8)(6 - \kappa) \leq 1 \) irrational).

There exists a unique single-valued, conformally covariant solution \( F_\varsigma(z, \bar{z}) \) to the Benoit & Saint-Aubin PDEs.
Conclusion: ∃ ! single-valued solution

Theorem [ Flores, P. (2018+) ]

Suppose $\kappa \in (0, 8) \setminus \mathbb{Q}$ (so $c = \frac{1}{2\kappa}(3\kappa - 8)(6 - \kappa) \leq 1$ irrational).

There exists a unique single-valued, conformally covariant solution $F_\varsigma(z, \bar{z})$ to the Benoit & Saint-Aubin PDEs.

Proposition [ Flores, P (2018+) ]

There exists a unique braiding invariant vector $v \in H_\varsigma \otimes \overline{H}_\varsigma$.

Idea:

- observe that
  \[
  \{ \text{braiding invariant vectors in } H_\varsigma \otimes \overline{H}_\varsigma \} \cong \text{Hom}_{Br_n}(H_\varsigma, H_\varsigma)
  \]
- use representation theory to prove that $\dim \text{Hom}_{Br_n}(H_\varsigma, H_\varsigma) = 1$
- use the “Spin chain – Coulomb gas correspondence” bijection $\mathcal{F}: H_\varsigma \otimes \overline{H}_\varsigma \leftrightarrow \{ \text{solution space} \}$ to conclude with the theorem