Yang–Mills for probabilists

Sourav Chatterjee

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- They are formally probability measures on spaces of connections on certain principal bundles.
- They have lattice analogs, known as lattice gauge theories, that are rigorously defined probabilistic models.
- Euclidean Yang–Mills theories are scaling limits of lattice gauge theories (probability-theoretic open problem).

Constructive field theory

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- The plan there is to first define Euclidean Yang–Mills theories as probability measures on appropriate spaces of generalized functions; then show that these probability measures satisfy certain axioms (the Osterwalder–Schrader axioms); this would then imply that the theory can be 'quantized' to obtain the desired quantum Yang–Mills theories.

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- Let $\mathfrak{g}$ be the Lie algebra of G.
- Then $\mathfrak{g}$ is a subspace of the space of all $N \times N$ skew-Hermitian matrices.

Connections and curvature

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- This means that at each x, $F(x)$ is an $n \times n$ array of skew-Hermitian matrices of order N, whose $(j,k)^{\mathrm{th}}$ entry is the matrix

$$F_{jk}(x) = \frac{\partial A_k}{\partial x_j} - \frac{\partial A_j}{\partial x_k} + [A_j(x), A_k(x)].$$

The Yang–Mills action

- Let $\mathcal{A}$ be the space of all smooth G connection forms on $\mathbb{R}^n$. The Yang–Mills action on this space is the function

$$S_{\mathrm{YM}}(A) := -\int_{\mathbb{R}^n} \mathrm{Tr}(F \wedge *F),$$

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- Explicitly, this is

$$S_{\mathrm{YM}}(A) = -\int_{\mathbb{R}^n} \sum_{j,k=1}^{n} \mathrm{Tr}(F_{jk}(x)^2)dx.$$

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  - Z is the normalizing constant that makes this a probability measure.

Problem

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- While it has been possible to give rigorous meanings to similar descriptions of Brownian motion and various quantum field theories in dimensions two and three, 4D Euclidean Yang–Mills theories have so far remained largely intractable.

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- The uniqueness (or non-uniqueness) is in general unknown for lattice gauge theories in dimension four when β is large.

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- Suppose that the B_i are skew-Hermitian. Then all terms within the exponential are skew-Hermitian and therefore have purely imaginary traces.
- Thus, if the entries of $B_1, \ldots, B_m$ are of order ϵ and if the entries of $B_1 + \cdots + B_m$ are of order ϵ^2, then

$$\begin{aligned}&\mathrm{Re}(\mathrm{Tr}(I - e^{B_1}\cdots e^{B_m}))\\ &= -\frac{1}{2}\mathrm{Tr}\biggl[\biggl(\sum_{a=1}^m B_a + \frac{1}{2}\sum_{1\le a<b\le m}[B_a, B_b]\biggr)^2\biggr] + O(\epsilon^5).\end{aligned}$$

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- Then

$$\begin{aligned} U_p &= U(x_1,x_2)U(x_2,x_3)U(x_3,x_4)U(x_4,x_1) \\ &= e^{\epsilon A_j(x_1)} e^{\epsilon A_k(x_2)} e^{-\epsilon A_j(x_4)} e^{-\epsilon A_k(x_1)}. \end{aligned}$$

Wilson's heuristic, continued

- Writing

$$A_k(x_2) = A_k(x + \epsilon e_j) = A_k(x) + \epsilon \frac{\partial A_k}{\partial x_j} + O(\epsilon^2)$$

and using a similar Taylor expansion for $A_j(x_4)$, we get

$$A_j(x_1) + A_k(x_2) - A_j(x_4) - A_k(x_1) = \epsilon \left(\frac{\partial A_k}{\partial x_j} - \frac{\partial A_j}{\partial x_k} \right) + O(\epsilon^2).$$

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- Thus,

$$\begin{aligned}
&\mathrm{Re}(\mathrm{Tr}(I - U_p)) \\
&= -\frac{1}{2}\epsilon^4\,\mathrm{Tr}\biggl[\biggl(\frac{\partial A_k}{\partial x_j} - \frac{\partial A_j}{\partial x_k} + [A_j(x), A_k(x)]\biggr)^2\biggr] + O(\epsilon^5) \\
&= -\frac{1}{2}\epsilon^4\,\mathrm{Tr}(F_{jk}(x)^2) + O(\epsilon^5).
\end{aligned}$$

Wilson's heuristic, continued

- This gives the formal approximation

$$
\begin{aligned}
S(U) &= \sum_p \mathrm{Re}(\mathrm{Tr}(I - U_p)) \\
&\approx -\frac{1}{4} \sum_{x \in \epsilon \mathbb{Z}^n} \sum_{j,k=1}^n \epsilon^4 \,\mathrm{Tr}(F_{jk}(x)^2) \\
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- The above heuristic was used by Wilson to justify the approximation of Euclidean Yang–Mills theory by lattice gauge theory, scaling the inverse coupling strength β like ϵ^{4-n} as the lattice spacing $\epsilon \to 0$.

Scaling in dimension four

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- But there are doubts about this belief and the question remains an open mathematical problem.

Wilson loop variables

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- Do not worry! Lattice definition coming soon.

Quark confinement

- In quantum chromodynamics, the potential between a static quark and antiquark separated by distance R is given by the formula

$$V(R) = -\lim_{T\to\infty} \frac{1}{T} \log\langle W_{\gamma_{R,T}}\rangle,$$

where $\gamma_{R,T}$ is the boundary of a rectangle of length T and breadth R, and $\langle\cdot\rangle$ denotes expectation with respect to $SU(3)$ Yang–Mills theory in dimension four.

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- If $V(R)$ grows to infinity as $R\to\infty$, the quark-antiquark pair cannot separate beyond a fixed distance.
- This is the phenomenon of quark confinement, observed in experiments but currently lacking a satisfactory theoretical understanding.

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- If the area law holds, then the quantity

$$\lim_{R\to\infty} \frac{V(R)}{R}$$

has physical significance. It is called the 'string tension' of the continuum theory, and represents the energy density per unit length in the theory.

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- Then, as $\epsilon \to 0$, the discrete Wilson loop variable W_{γ_ϵ} approaches the continuous Wilson loop variable W_γ.

The area law problem on lattices

Open problem (Area law)

Take any compact non-Abelian Lie group $G \subseteq U(N)$ for some $N \geq 2$ and consider any infinite volume limit of 4D lattice gauge theory with gauge group G at inverse coupling strength β. Let $\gamma_{R,T}$ be a rectangular loop of breadth R and length T in the lattice. Prove that

$$|\langle W_{\gamma_{R,T}} \rangle| \leq C(\beta) e^{-c(\beta)RT},$$

where $C(\beta)$ and $c(\beta)$ are positive constants that depend only on the inverse coupling strength β and the gauge group.

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- Disproof at large β for 4D $U(1)$ theory by Guth (1980) and Fröhlich & Spencer (1982).

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- Any lattice gauge theory contains information of an associated class of elementary particles called 'glueballs' or 'gluon-balls'.
- The number ξ represents the reciprocal of the mass of the lightest glueball in the theory.

Critical point

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- It is believed that in dimension four, many of the non-Abelian lattice models of interest have $\beta_c = \infty$.

One version of the mass gap problem

Open problem (Mass gap)

Take any compact non-Abelian Lie group $G \subseteq U(N)$ for some $N \geq 2$ and consider 4D lattice gauge theory with gauge group G at inverse coupling strength β. For each $x \in \mathbb{R}^4$, let p_x be the plaquette that is closest to x. Let $f_\beta(x)$ denote the correlation between W_{p_0} and W_{p_x}. Show that for any $\beta > 0$, there exists some $\xi(\beta) \in (0, \infty)$ such that

$$\lim_{|x| \to \infty} \frac{\log f_\beta(x)}{|x|} = -\frac{1}{\xi(\beta)}.$$

Moreover, prove that

$$\lim_{\beta \to \infty} \xi(\beta) = \infty.$$

Continuum limit via Wilson loops

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- As $\beta \to \beta_c$, we would like to show that the lattice spacing ϵ can be taken to 0 in such a way that if γ_ϵ is any sequence of lattice loops converging to a loop γ in $\mathbb{R}^n$, then $\langle W_{\gamma_\epsilon} \rangle$ converges to a nontrivial limit after some appropriate renormalization.

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- Recall that β_c is believed to be ∞ for 4D non-Abelian theories.

One possible formulation

Open problem (Continuum limit)

Take any compact non-Abelian Lie group $G \subseteq U(N)$ for some $N \geq 2$ and consider 4D lattice gauge theory with gauge group G at inverse coupling strength β. Let $\gamma_{R,T}$ denote a rectangular loop of length T and breadth R. Prove that as $\beta \to \infty$, there are sequences $\epsilon = \epsilon(\beta) \to 0$ and $c = c(\beta) \to \infty$, and a nonzero constant d, such that for any R and T,

$$\log\langle W_{\gamma_{R/\epsilon, T/\epsilon}}\rangle = -c(R+T) - dRT + o(1).$$

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Since this has not been proved, it is not clear to me whether the renormalization term $c(R+T)$ is indeed necessary. It seems possible that the limit holds without the renormalization term.

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- This proves a version of gauge-string duality.
- The expansion was proved only at small β (strong coupling). It is an open question whether such an expansion holds at large β.
- In 2D, the terms were explicitly evaluated by Basu & Ganguly (2016) using combinatorial techniques.

The master loop equation

The following is a generalization of what are called Makeenko–Migdal equations or master loop equations. They hold at all β, and give the starting point for the proof of the $1/N$ expansion and gauge-string duality.

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Theorem (C., 2015)

Consider $SO(N)$ LGT on $\mathbb{Z}^n$. For a collection of loops $s = (\ell_1, \ldots, \ell_m)$, define

$$\phi(s) := \frac{\langle W_{\ell_1} W_{\ell_2} \cdots W_{\ell_m} \rangle}{N^m}.$$

Let $|s|$ be the total number of edges in s. Then

$$\begin{aligned}(N-1)|s|\phi(s) &= \sum_{s' \in \mathbb{T}^-(s)} \phi(s') - \sum_{s' \in \mathbb{T}^+(s)} \phi(s') + N \sum_{s' \in \mathbb{S}^-(s)} \phi(s') \\ &\quad - N \sum_{s' \in \mathbb{S}^+(s)} \phi(s') + \frac{1}{N} \sum_{s' \in \mathbb{M}^-(s)} \phi(s') - \frac{1}{N} \sum_{s' \in \mathbb{M}^+(s)} \phi(s') \\ &\quad + N\beta \sum_{s' \in \mathbb{D}^-(s)} \phi(s') - N\beta \sum_{s' \in \mathbb{D}^+(s)} \phi(s'),\end{aligned}$$

where $\mathbb{T}^\pm$, $\mathbb{S}^\pm$, $\mathbb{M}^\pm$ and $\mathbb{D}^\pm$ are certain operations that produce new collections of loops from old.

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Theorem (C., 2018)

Consider 4D $\mathbb{Z}_2$ lattice gauge theory. Let $\{\gamma_n\}_{n\geq 1}$ be a sequence of self-avoiding loops and $\beta_n \to \infty$ such that $|\gamma_n|e^{-12\beta_n}$ converges to a limit $\theta \in (0, \infty)$, where $|\gamma_n|$ is the length of γ_n. Then

$$\lim_{n\to\infty} \langle W_{\gamma_n} \rangle_{\beta_n} = e^{-2\theta},$$

provided that the proportion of corner edges in γ_n tends to zero. (Corner edge: an edge that shares a plaquette with another edge.)

Final remarks

- The result from the previous slide shows that for 4D $\mathbb{Z}_2$ theory, the coupling constant β needs to to scale like $-\frac{1}{12}\log\epsilon$, where ϵ is the lattice spacing, to obtain a nontrivial limit of Wilson loop expectations. As mentioned earlier, such logarithmic scaling is conjectured for 4D non-Abelian theories in the physics literature.

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- The preprint also has an extensive review of the mathematical literature on these topics, which I did not cover in this talk.

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- The result from the previous slide shows that for 4D $\mathbb{Z}_2$ theory, the coupling constant β needs to to scale like $-\frac{1}{12}\log \epsilon$, where ϵ is the lattice spacing, to obtain a nontrivial limit of Wilson loop expectations. As mentioned earlier, such logarithmic scaling is conjectured for 4D non-Abelian theories in the physics literature.
- I have a very recent preprint on arXiv, with the same title as this talk, that contains most of this talk in greater detail (except the last theorem).
- The preprint also has an extensive review of the mathematical literature on these topics, which I did not cover in this talk.
- Special thanks to David Brydges, Erhard Seiler and Steve Shenker for teaching me most of what I know about Yang–Mills theories, lattice gauge theories and quantum field theories.